Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Differential Geometry

Homework 5

due on Tuesday, October 24

1. Let G and H be Lie groups and $\Phi : G \to H$ a Lie group homomorphism. Show that:
   (a) if $(d\Phi)_e$ is an isomorphism then Φ is a local diffeomorphism;
   (b) if Φ is a surjective local diffeomorphism then Φ is a covering map.

2. (a) Show that $\mathbb{R} \cdot SU(2)$ is a 4–dimensional real linear subspace of $\mathcal{M}_{2\times 2}(\mathbb{C})$, closed under matrix multiplication, with basis
$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$
satisfying $i^2 = j^2 = k^2 = ijk = -1$. Therefore this space can be identified with the **quaternions**. Show that $SU(2)$ can be identified with the quaternions of Euclidean norm equal to 1, and is therefore diffeomorphic to S^3.
   (b) Show that the Lie algebra of $SU(2)$ consists of the skew–hermitian matrices of trace zero
$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} i\alpha & \beta \\ -\overline{\beta} & -i\alpha \end{pmatrix} : \alpha \in \mathbb{R}, \beta \in \mathbb{C} \right\}$$
and therefore can be identified with $\mathbb{R}^3$.
   (c) Show that if $n \in \mathbb{R}^3$ is a unit vector, which we identify with a quaternion with zero real part, then
$$\exp\left(\frac{n\theta}{2}\right) = 1 \cos\left(\frac{\theta}{2}\right) + n \sin\left(\frac{\theta}{2}\right)$$
is also a unit quaternion.
   (d) Again, identifying $\mathbb{R}^3$ with quaternions with zero real part, show that the map $\mathrm{Ad}_{n,\theta} : \mathbb{R}^3 \to \mathbb{R}^3$ given by
$$v \mapsto \exp\left(\frac{n\theta}{2}\right) \cdot v \cdot \exp\left(-\frac{n\theta}{2}\right)$$
is a rotation by an angle θ about the axis defined by n.
   (e) Show that there exists a surjective homomorphism $\Phi : SU(2) \to SO(3)$, and use it to conclude that $SU(2)$ is the universal covering space of $SO(3)$.

It might me useful to recall that the product of two quaternions is given by
$$(a + \mathbf{u})(b + \mathbf{v}) = ab - \mathbf{u} \cdot \mathbf{v} + a\mathbf{v} + b\mathbf{u} + \mathbf{u} \times \mathbf{v}$$
where a, b are the real parts and $\mathbf{u}, \mathbf{v}$ are the vector parts of the quaternions $(a + \mathbf{u})$ and $(b + \mathbf{v})$.