Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Differential Geometry

Homework 4

due on Tuesday, October 16

1. In $\mathbb{R}^2$ consider the vector fields $X = \frac{\partial}{\partial x}$ and $Y = x\,\frac{\partial}{\partial y}$. Compute the Lie bracket $[X,Y]$ in two distinct ways:

   (i) using the definition;

   (ii) using the flows of X and Y, as in Proposition 10.3 of the Lecture Notes.

2. Let $X, Y \in \mathfrak{X}(M)$ be complete vector fields with flows ϕ_X^t and ϕ_Y^s. Show that:

   (a) given a diffeomorphism $\Phi : M \to M$, we have $\Phi_* X = X$ *iff* $\Phi \circ \phi_X^t = \phi_X^t \circ \Phi \ \forall t$.

   (b) $\phi_Y^s \circ \phi_X^t = \phi_X^t \circ \phi_Y^s$ for all t and s *iff* $[X,Y] = 0$.

3. Let D_1 and D_2 denote the one and two dimensional distributions on $\mathbb{R}^3$ defined by

$$D_1 = \langle X \rangle \quad \text{e} \quad D_2 = \langle X, Y \rangle,$$

   where X, Y are the smooth vector fields on $\mathbb{R}^3$ defined by

$$X = \sin y \frac{\partial}{\partial x} + 8y^3 \frac{\partial}{\partial y} + \frac{\partial}{\partial z};$$

$$Y = \cos y \frac{\partial}{\partial x} + \sin(x+z) \frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

   (a) Is there a one dimensional foliation of $\mathbb{R}^3$ tangent to D_1?

   (b) Is there a two dimensional foliation of $\mathbb{R}^3$ tangent to D_2?