Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Differential Geometry

Homework 2

due on Tuesday, October 2

1. (a) Consider the real valued function $f(x,y,z) = (2-\sqrt{x^2+y^2})^2 + z^2$ on $\mathbb{R}^3 \setminus \{(0,0,z)\}$. Show that 1 is a regular value of f. Identify the manifold $M = f^{-1}(1)$.

   (b) Show that the manifold M of Problem 1.(a) is transverse to the surface

   $$N = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2 = 4\}.$$

   Identify the manifold $M \cap N$.

   (c) Show that the manifold M of Problem 1.(a) is *not* transverse to the plane

   $$N = \{(x,y,z) \in \mathbb{R}^3 \mid x = 1\}.$$

   Is $M \cap N$ a manifold?

2. (a) Let M be a smooth manifold of dimension m, $p \in M$ and $f_1, \ldots, f_k$, with $k < m$ real valued smooth functions defined in a neighbourhood of p such that their differentials $d_p f_i$ are linearly independent in $T_p^* M$. Show that there is a neighbourhood U of p and functions g_j, $1 \leq j \leq m-k$ such that $(U, f_1, \ldots, f_k, g_1, \ldots, g_{m-k})$ is a coordinate system for M around p.

   Hint: use the Inverse Function Theorem.

   (b) Let $\Phi : M \to N$ be a smooth map between smooth manifolds and let $\mathcal{F} = \{L_\alpha\}_{\alpha \in A}$ be a foliation of N, such that Φ **is transversal to** $\mathcal{F}$, i.e., Φ is transversal to every leaf L of $\mathcal{F}$:

   $$d_p\Phi(T_pM) + T_{\Phi(p)}L = T_{\Phi(p)}N, \quad \forall p \in M.$$

   Show that the pull-back

   $$\Phi^*(\mathcal{F}) := \{\text{connected components of } \Phi^{-1}(L_\alpha)\}_{\alpha \in A}$$

   is a foliation of M and $\operatorname{codim} \Phi^*(\mathcal{F}) = \operatorname{codim} \mathcal{F}$.

   Hint: use Problem 2.(a) to define a foliated coordinate chart of M.