Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Differential Geometry

Homework 1

due on Tuesday, September 25

1. Consider the 3-sphere $\mathbb{S}^3$ as the set of unit quaternions

$$\{x + \mathbf{i}y + \mathbf{j}z + \mathbf{k}w \mid x^2 + y^2 + z^2 + w^2 = 1\}.$$

Let $\phi : U \to \mathbb{R}^3$ be defined by $\phi(x + iy + jz + kw) = (y, z, w) \in \mathbb{R}^3$ where

$$U = \{x + \mathbf{i}y + \mathbf{j}z + \mathbf{k}w \in \mathbb{S}^3 \mid x > 0\}.$$

Consider ϕ as a chart. For each $q \in \mathbb{S}^3$, define a chart ψ_q by $\psi_q(p) = \phi(q^{-1}p)$. Show that this set of charts is an atlas for a smooth structure on $\mathbb{S}^3$.

It might me useful to recall that the product of two quaternions is given by

$$(a + \mathbf{u})(b + \mathbf{v}) = ab - \mathbf{u} \cdot \mathbf{v} + a\mathbf{v} + b\mathbf{u} + \mathbf{u} \times \mathbf{v}$$

where a, b are the real parts and $\mathbf{u}, \mathbf{v}$ are the vector parts of the quaternions $(a + \mathbf{u})$ and $(b + \mathbf{v})$.

2. The real projective plane $\mathbb{P}^2$ can be considered as $\mathbb{S}^2/\sim$, where $\mathbb{S}^2$ is the 2-sphere, and the relation $\sim$ is $(x, y, z) \sim (-x, -y, -z)$. The map $\phi : \mathbb{R}^3 \to \mathbb{R}^6$ defined by

$$\phi(x, y, z) = (x^2, y^2, z^2, xy, xz, yz)$$

defines a map $\mathbb{S}^2 \to \mathbb{R}^6$ which is invariant under the relation $\sim$, and therefore defines a map $\Phi : \mathbb{P}^2 \to \mathbb{R}^6$.

   (a) Show that Φ is an immersion, i.e. that Φ has rank 2 at every point of $\mathbb{P}^2$.

   (b) Show that Φ is injective.

   (c) Suppose Φ is composed with projection onto $\mathbb{R}^5$ by suppressing one of the first three coordinates. Is the resulting map still an immersion? Is it injective?