Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Differential Geometry

Homework 1

due on Tuesday, September 26

1. The **real projective d-dimensional space** is the set

$$\mathbb{P}^d = \{L \subset \mathbb{R}^{d+1} : L \text{ is a straight line through the origin}\}.$$

We can think of $\mathbb{P}^d$ as the quotient space $\mathbb{R}^{d+1}\backslash\{0\}/\sim$ where $\sim$ is the equivalence relation:

$$(x^0, \dots, x^d) \sim (y^0, \dots, y^d) \textit{ if and only if } (x^0, \dots, x^d) = \lambda(y^0, \dots, y^d),$$

for some $\lambda \in \mathbb{R}\backslash\{0\}$. Taking the quotient topology on $\mathbb{P}^d$, it becomes a topological manifold of dimension d: if we denote by $[x^0 : \dots : x^d]$ the equivalence class of $(x^0, \dots, x^d) \in \mathbb{R}^{d+1}$, then for each $\alpha = 0, \dots, d$ we have the coordinate system (U_α, ϕ_α) where:

$$U_\alpha = \{[x^0 : \dots : x^d] : x^\alpha \neq 0\},$$
$$\phi_\alpha : U_\alpha \to \mathbb{R}^d, \quad [x^0 : \dots : x^d] \mapsto \left(\frac{x^0}{x_\alpha}, \dots, \frac{x^{\alpha-1}}{x_\alpha}, \frac{x^{\alpha+1}}{x_\alpha}, \dots, \frac{x^d}{x_\alpha}\right).$$

Compute the transition functions for the atlas of the projective space $\mathbb{P}^d$ and show they are smooth. Show also that $\mathbb{P}^1$ is diffeomorphic to $\mathbb{S}^1$.

2. Let (x, y, z) be the Cartesian coordinates in $\mathbb{R}^3$ and consider the usual spherical coordinates (r, θ, φ) defined by

$$\begin{cases} x = r \sin\theta \cos\varphi \\ y = r \sin\theta \operatorname{sen}\varphi \\ z = r\cos\theta \end{cases}$$

Compute:

(a) The components of the tangent vectors to $\mathbb{R}^3$ $\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \varphi}$ in Cartesian coordinates.

(b) The components of the tangent vectors to $\mathbb{R}^3$ $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ in spherical coordinates.

3. Let $\Phi : \mathbb{P}^2 \to \mathbb{R}^3$ be the map defined by

$$\Phi([x : y : z]) = \frac{1}{x^2 + y^2 + z^2}(yz, xz, xy).$$

Show that Φ is smooth and show that it only fails to be an immersion at 6 points.

4. Consider a submanifold (N, Φ) of M with N compact. Show that (N, Φ) is an embedded submanifold.