Differential Geometry

Homework 14 - Connections in Principal Bundles

Let $\xi = (\pi, E, M)$ be a vector bundle of rank r. Given a local frame $\{s_1, \ldots, s_r\}$ in a trivializing open set $U \subset M$, we can parametrize all the bases $\{v_1, \ldots, v_r\}$ of the fibers E_p over points $p \in U$ by

$$\phi(v_1, \ldots, v_r) = (p, B) \in U \times GL(r),$$

where $B \in GL(r)$ is the matrix of change of basis from $s(p) = \{s_1(p), \ldots, s_r(p)\}$ to $\{v_1, \ldots, v_r\}$. Hence the set $F(E)$ of all bases of E is a differentiable manifold with a natural projection $\pi : F(E) \to M$ and trivializing charts $\phi : \pi^{-1}(U) \to U \times GL(r)$. The triple $\Xi = (\pi, F(E), M)$ is called the **frame bundle** of ξ; note that the local frame $\{s_1, \ldots, s_r\}$ can be seen as a local section $s : U \to F(E)$. Since the fibers $F(E)_p$ are isomorphic to the Lie group $GL(r)$, Ξ is a **principal bundle**.

1. Show that the matrix $B' \in GL(r)$ which represents a basis of E_p in another local frame s' is related with the matrix $B \in GL(r)$ which represents the same basis in s by $B' = S^{-1}B$, where $S : U \to GL(r)$ is the matrix of change of basis from s to s'. Show that then ξ and Ξ are determined by the same cocycle.

2. In $\pi^{-1}(U)$ define the matrix of 1-forms

$$\tilde{\omega} = B^{-1}\left(\pi^*\omega\right)B + B^{-1}dB,$$

where ω is the matrix of the 1-forms of the connection in ξ associated to the local frame s. Show that $\tilde{\omega}$ is independent of the choice of frame s, and therefore is globally defined in $F(E)$ (Hint: use HW12, problem 2). Show also that $\omega = s^*\tilde{\omega}$.

3. In $\pi^{-1}(U)$ define the matrix of 2-forms

$$\tilde{\Omega} = B^{-1}\left(\pi^*\Omega\right)B,$$

where Ω is the matrix of the forms of curvature associated to the local frame s. Show that

$$\tilde{\Omega} = d\tilde{\omega} + \tilde{\omega} \wedge \tilde{\omega} = d\tilde{\omega} + \frac{1}{2}[\tilde{\omega}, \tilde{\omega}]$$

so $\tilde{\Omega}$ is independent of the choice of frame s, and therefore is globally defined in $F(E)$. Show also that $\Omega = s^*\tilde{\Omega}$. Note: $[\eta, \alpha] := \eta \wedge \alpha - (-1)^{\deg\alpha \deg\eta}\, \alpha \wedge \eta$.

4. Show that

$$d\tilde{\Omega} + \tilde{\omega} \wedge \tilde{\Omega} - \tilde{\Omega} \wedge \tilde{\omega} = 0 \Leftrightarrow d\tilde{\Omega} + [\tilde{\omega}, \tilde{\Omega}] = 0.$$

5. Show that the kernel of the 1-forms of the matrix $\tilde{\omega}$ define a distribution H of dimension d, where $\dim M = d$, in $F(E)$, and that the curves in $F(E)$ tangent to H correspond to frames that are obtained by parallel transport along the projection of the curve in M. Show

also that H is integrable *iff* the connection ∇ is flat.
Hint: Note that a vector field $X \in \mathfrak{X}(F(E))$ in local coordinates in $U \times GL(r)$ is given by

$$X = \sum_{i=1}^{d} v^i \frac{\partial}{\partial x^i} + \sum_{a,b=1}^{r} V^{ab} \frac{\partial}{\partial B_{ab}} \quad \text{where } v^i \text{ and } V^{ab} \text{ are smooth functions.}$$

6. Show that $GL(r)$ acts on the right on $F(E)$, that the action is free and that $F(E)/GL(r) = M$. Show also that if $\Psi(S) : F(E) \to F(E)$ is the action of $S \in GL(r)$ then $\Psi(S)^*\tilde{\omega} = S^{-1}\tilde{\omega}S = \mathrm{Ad}(S^{-1})\tilde{\omega}$, and if $\psi(A) \in \mathfrak{X}(F(E))$ is the infinitesimal action of $X \in \mathfrak{gl}(r)$ then $\tilde{\omega}(\psi(A)) = A$.

7. Any vector $X \in T_uF(E)$ can be decomposed in a unique way as the sum $X = X^h + X^v$, where $X^h \in H_u$ and $X^v \in \ker d_u\pi$. If $\theta \in \Omega^k(F(E), \mathfrak{gl}(r))$, define the **covariant exterior derivative** $D\theta \in \Omega^{k+1}(F(E), \mathfrak{gl}(r))$ by

$$D\theta(X_1, \ldots, X_{k+1}) = d\theta(X_1^h, \ldots, X_{k+1}^h).$$

Show that

$$D\tilde{\omega} = \tilde{\Omega}$$

and

$$D\tilde{\Omega} = 0.$$

Remark: Note that if ξ is endowed with a fibre metric which is compatible with the connection, the structure group of the frame bundle $F(E)$ can be reduced to $O(r)$. If it is also orientable it can be further reduced to $SO(r)$. In both cases a similar construction of the principal connection holds. Similarly, if ξ is an orientable complex vector bundle endowed with a hermitian fibre metric compatible with the connection, the structure group of the frame bundle can be reduced to $SU(r)$ and the principal connection can be constructed in the same way.