Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Differential Geometry

Homework 13

due on Tuesday, December 18

1. Consider the **complex** line bundle (hence a real vector bundle of rank 2) $\xi = (\pi, E, \mathbb{CP}^1)$, where $E = \mathbb{CP}^2 \backslash \{[0,0,1]\}$, $\mathbb{CP}^1 \simeq \mathbb{S}^2$ and $\pi([z,w,t]) = [z,w]$, defined by the trivializing charts (U_1, ϕ_1) and (U_2, ϕ_2), where

$$U_1 = \{[z,1] \in \mathbb{CP}^1 \, : \, z \in \mathbb{C}\}, \quad \phi_1([z,1,t]) = ([z,1],t);$$
$$U_2 = \{[1,w] \in \mathbb{CP}^1 \, : \, w \in \mathbb{C}\}, \quad \phi_2([1,w,t]) = ([1,w],t).$$

Show that:

(a) There exists a cocycle $g_{21} : U_1 \cap U_2 \to \mathbb{C}\backslash\{0\}$ for this vector bundle given by

$$g_{21}([z,1]) = \frac{\bar{z}}{|z|}.$$

(b) The 1-forms with complex values ω and η given in the coordinates z and w by

$$\omega = \frac{1}{2}\frac{zd\bar{z} - \bar{z}dz}{1+|z|^2} \quad \text{and} \quad \eta = \frac{1}{2}\frac{wd\bar{w} - \bar{w}dw}{1+|w|^2}$$

define for this cocycle one connection ∇ in ξ.

(c) The curvature form is given in U_1 by

$$\Omega = \frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}.$$

Let $c : [0, 2\pi] \to \mathbb{S}^2$ be the path $c(t) = e^{it}$. Show that:

(d) The induced connection in $c^*\xi$ is defined by the form

$$c^*\omega = -\frac{i}{2}dt.$$

(e) The holonomy homomorphism along c is $H_1(c) = -\text{id}$.

(f) The connection ∇ is not flat.