Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Differential Geometry

Homework 12

due on Tuesday, December 12

1. Let $\xi = (\pi, E, M)$ be a vector bundle of rank r equipped with a connection ∇. Let $\{s_1, \dots, s_r\}$ and $\{s'_1, \dots, s'_r\}$ be two local frames in a trivializing open set $U \subset M$. Denote by $A = (a_i^j) \in C^\infty(U, \mathrm{GL}(r))$ the matrix of change of basis so that $s'_i = \sum_j a_i^j s_j$. Show that:

   (a) The matrices ω and ω' of the corresponding connection 1-forms are related by
   $$\omega' = A^{-1}\omega A + A^{-1}dA.$$

   (b) The matrices Ω and Ω' of the corresponding curvature 2-forms are related by
   $$\Omega' = A^{-1}\Omega A.$$

2. Consider the **complex** line bundle (hence a real vector bundle of rank 2) $\xi = (\pi, E, \mathbb{CP}^1)$, where $E = \mathbb{CP}^2 \backslash \{[0,0,1]\}$, $\mathbb{CP}^1 \simeq \mathbb{S}^2$ and $\pi([z,w,t]) = [z,w]$, defined by the trivializing charts (U_1, ϕ_1) and (U_2, ϕ_2), where
   $$U_1 = \{[z,1] \in \mathbb{CP}^1 \, : \, z \in \mathbb{C}\}, \quad \phi_1([z,1,t]) = ([z,1],t);$$
   $$U_2 = \{[1,w] \in \mathbb{CP}^1 \, : \, w \in \mathbb{C}\}, \quad \phi_2([1,w,t]) = ([1,w],t).$$

   Show that:

   (a) There exists a cocycle $g_{21} : U_1 \cap U_2 \to \mathbb{C}\backslash\{0\}$ for this vector bundle given by
   $$g_{21}([z,1]) = \frac{\bar{z}}{|z|}.$$

   (b) The 1-forms with complex values ω and η given in the coordinates z and w by
   $$\omega = \frac{1}{2}\frac{z d\bar{z} - \bar{z} dz}{1+|z|^2} \quad \text{and} \quad \eta = \frac{1}{2}\frac{w d\bar{w} - \bar{w} dw}{1+|w|^2}$$
   define for this cocycle one connection ∇ in ξ.

   (c) The curvature form is given in U_1 by
   $$\Omega = \frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}.$$