

Differential Geometry

Homework 12

due on Wednesday, December 14

1. Let $\xi = (\pi, E, M)$ be a vector bundle of rank r equipped with a connection ∇ . Let $\{s_1, \dots, s_r\}$ and $\{s'_1, \dots, s'_r\}$ be two local frames in a trivializing open set $U \subset M$. Denote by $A = (a_i^j) \in C^\infty(U, GL(r))$ the matrix of change of basis so that $s'_i = \sum_j a_i^j s_j$. Show that:

- (a) The matrices ω and ω' of the corresponding connection 1-forms are related by

$$\omega' = A^{-1}\omega A + A^{-1}dA.$$

- (b) The matrices Ω and Ω' of the corresponding curvature 2-forms are related by

$$\Omega' = A^{-1}\Omega A.$$

2. Consider the **complex** line bundle (hence a real vector bundle of rank 2) $\xi = (\pi, E, \mathbb{CP}^1)$, where $E = \mathbb{CP}^2 \setminus \{[0, 0, 1]\}$, $\mathbb{CP}^1 \simeq \mathbb{S}^2$ and $\pi([z, w, t]) = [z, w]$, defined by the trivializing charts (U_1, ϕ_1) and (U_2, ϕ_2) , where

$$\begin{aligned} U_1 &= \{[z, 1] \in \mathbb{CP}^1 : z \in \mathbb{C}\}, & \phi_1([z, 1, t]) &= ([z, 1], t); \\ U_2 &= \{[1, w] \in \mathbb{CP}^1 : w \in \mathbb{C}\}, & \phi_2([1, w, t]) &= ([1, w], t). \end{aligned}$$

Show that:

- (a) There exists a cocycle $g_{21} : U_1 \cap U_2 \rightarrow \mathbb{C} \setminus \{0\}$ for this vector bundle given by

$$g_{21}([z, 1]) = \frac{\bar{z}}{|z|}.$$

- (b) The 1-forms with complex values ω and η given in the coordinates z and w by

$$\omega = \frac{1}{2} \frac{z d\bar{z} - \bar{z} dz}{1 + |z|^2} \quad \text{and} \quad \eta = \frac{1}{2} \frac{w d\bar{w} - \bar{w} dw}{1 + |w|^2}$$

define for this cocycle one connection ∇ in ξ .

- (c) The curvature form is given in U_1 by

$$\Omega = \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2}.$$