Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Differential Geometry

Exam - January 9, 2018

Duration: 3 hours
Justify your answers carefully.

(2 val.) 1. Let $f : N \to M$ denote a smooth map between smooth manifolds, and let ω denote a smooth n-form on M where $n = \dim N$. Show that if $f^*\omega$ does not vanish on N then $f : N \to M$ is an immersion.

2. Let G be a Lie group.

(2 val.) (a) Show that if X is a left invariant vector field then the flow of X satisfies $\phi^t_X = R_{\exp(tX)}$, where R denotes right multiplication.

(1 val.) (b) Show that if G is an abelian Lie group then its Lie algebra is abelian (i.e., $[v, w] = 0\ \forall v, w \in \mathfrak{g}$).

(3 val.) 3. Recall that a symplectic manifold is a pair (M, ω), where M is a differentiable manifold and $\omega \in \Omega^2(M)$ is a closed 2-form and non–degenerate (the map $T_pM \to T^*_pM$ given by $v \mapsto i_v\omega$ is an isomorphism of vector spaces for all $p \in M$).

For (M, ω) a symplectic manifold, a vector field is called *symplectic* if $\mathcal{L}_X\omega = 0$, and *Hamiltonian* if there exists a function H on M such that

$$i_X\omega = dH.$$

Show that all symplectic vector fields on M are Hamiltonian *iff* $H^1(M) = 0$.

(3 val.) 4. Let α and β be 1-forms such that $(\alpha \wedge \beta)_p \neq 0,\ \forall p \in M$ and $\alpha \wedge \beta$ is closed. Show that the distribution Σ generated by α and β ($\Sigma = \ker \alpha \cap \ker \beta$) is integrable.

(4 val.) 5. Let $X = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 1\}$, $Y = \{(x, 0, 0) \in \mathbb{R}^3 : -1 < x < 1\}$ and $Z = \{(0, 1/2, 0)\}$. Use the Mayer-Vietoris sequence to compute the cohomology of $M = X - (Y \cup Z)$, that is, compute $H^k(M)$, with $k = 0, 1, 2, \ldots$.

6. Recall that there exists a complex line bundle over $\mathbb{S}^2 \cong \mathbb{C} \cup \{\infty\}$ with a connection ∇ defined in $\mathbb{S}^2 \backslash \{\infty\} \cong \mathbb{C}$ by the 1-form

$$\omega = \frac{1}{2} \frac{z d\bar{z} - \bar{z} dz}{1 + |z|^2}.$$

Let $c : [0, 2\pi] \to \mathbb{S}^2$ be the path $c(t) = e^{it}$. Show that:

(1 val.) (a) The induced connection in $c^*\xi$ is defined by the form

$$c^*\omega = -\frac{i}{2} dt.$$

(2 val.) (b) The holonomy homomorphism along c is $H_1(c) = -\mathrm{id}$.

(2 val.) (c) The connection ∇ is not flat.