Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Differential Geometry

Exam - January 31, 2018

Duration: 3 hours
Justify your answers carefully.

(3 val.) 1. Let ω be the 1–form on $S^1 = \mathbb{R}\backslash\mathbb{Z}$ whose pull-back by the projection $\pi : \mathbb{R} \to S^1$ is dx (where x is the standard coordinate on $\mathbb{R}$). Let M be a smooth compact manifold without boundary and let $\mu : M \to S^1$ be a smooth map whose derivative is never 0. Show that $\mu^*\omega$ is not an exact 1–form. Conclude that the de Rham group $H^1(M) \neq 0$.

(2 val.) 2. Let G be a compact Lie group. Show that $\chi(G) = 0$.

3. Let D_1 and D_2 denote the one and two dimensional distributions on $\mathbb{R}^3$ defined by

$$D_1 = \langle X \rangle \quad \text{e} \quad D_2 = \langle X, Y \rangle,$$

where X, Y are the smooth vector fields on $\mathbb{R}^3$ defined by

$$X = \sin y \frac{\partial}{\partial x} + 8y^3 \frac{\partial}{\partial y} + \frac{\partial}{\partial z};$$

$$Y = \cos y \frac{\partial}{\partial x} + \sin(x+z) \frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

(2 val.) (a) Is there a one dimensional foliation of $\mathbb{R}^3$ tangent to D_1?

(2 val.) (b) Is there a two dimensional foliation of $\mathbb{R}^3$ tangent to D_2?

(3 val.) (c) Are the vector fields X and Y complete?

(4 val.) 4. Let $M = \mathbb{R}^d\backslash\{x_1, \ldots, x_n\}$, where $x_1, \ldots, x_n \in \mathbb{R}^d$ are n distinct points and $d \geq 2$. Show that

$$H^k(M) = \begin{cases} \mathbb{R} & \text{if } k = 0, \\ \mathbb{R}^n & \text{if } k = d-1, \\ 0 & \text{otherwise.} \end{cases}$$

(4 val.) 5. The matrix of the curvature 2-forms for the Fubini-Study metric in $\mathbb{CP}^2$ is given by

$$\Omega = \begin{pmatrix} 2\alpha \wedge \bar{\alpha} + \beta \wedge \bar{\beta} & \alpha \wedge \bar{\beta} \\ \beta \wedge \bar{\alpha} & \alpha \wedge \bar{\alpha} + 2\beta \wedge \bar{\beta} \end{pmatrix}$$

in a certain orthonormal frame $\{E_1, E_2, E_3, E_4\}$, where we identify $T_p\mathbb{CP}^2 \simeq \mathbb{C}^2$ through $X^1E_1 + X^2E_2 + X^3E_3 + X^4E_4 \mapsto (X^1 + iX^2, X^3 + iX^4)$ and we write $\alpha = \omega^1 + i\omega^2$ and $\beta = \omega^3 + i\omega^4$, where $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ is the dual co-frame. Knowing that the volume of $\mathbb{CP}^2$ for this metric is $V_4(\mathbb{CP}^2) = \frac{\pi^2}{2}$, show that

$$p_1(T\mathbb{CP}^2) = \frac{1}{4\pi^2}[\sigma_2(\Omega)] = 3[\mu],$$

where $[\mu]$ is the canonical generator of $H^4(\mathbb{CP}^2)$.

Hint: Note that if A is a $n \times n$ complex matrix then it corresponds to a $2n \times 2n$ real matrix A' such that $\text{tr}A' = 2Re(\text{tr}A)$.