Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Differential Geometry

Exam 2 - January 25, 2019

Duration: 3 hours
Justify your answers carefully.

(3 val.) 1. A symplectic form on a $2n$-manifold M is a closed, non-degenerate 2-form ω. Show that S^{2n} has no symplectic structure if $n > 1$.
Hint: Recall that ω^n is a volume form in M.

2. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Consider the adjoint representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$ defined by $\mathrm{Ad}(g) := d_e i_g$ where $i_g : G \to G$ is given by $i_g(h) = ghg^{-1}$, and let $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ be the induced Lie algebra representation. Show that:

(2 val.) (a) If $g \in G$ and $Y \in \mathfrak{g}$ then

$$\exp(t\mathrm{Ad}_g Y) = g(\exp(tY))g^{-1} \quad \forall t \in \mathbb{R}.$$

(2 val.) (b) If $X, Y \in \mathfrak{g}$ then

$$\exp(X)\exp(Y)\exp(-X) = \exp\left(\left(\sum_{n=0}^{+\infty} \frac{1}{n!}(\mathrm{ad}(X))^n\right) Y\right).$$

(3 val.) 3. Show that a vector field on a compact, oriented, surface of genus $g \neq 1$, must have at least one zero.
Hint: Recall that if M and N are connected compact manifolds of dimension d then $\mathcal{X}(M\#N) = \mathcal{X}(M) + \mathcal{X}(N) - \mathcal{X}(\mathbb{S}^d)$.

(2 val.) 4. Find the index of the zeros of the vector field $X = (x^2y + y^3)\frac{\partial}{\partial x} - (x^3 + xy^2)\frac{\partial}{\partial y}$ in $\mathbb{R}^2$.

(3 val.) 5. Let $d \geq 2$. Using the Poincaré Lemma, show that $H^1(\mathbb{S}^d) = 0$.

6. The sphere $\mathbb{S}^4$ can be given by two parametrizations $\phi_1^{-1}, \phi_2^{-1} : \mathbb{H} \to \mathbb{S}^4$ defined in the space $\mathbb{H}$ of the quaternions such that the transition function is given by

$$y = \phi_2 \circ \phi_1^{-1}(x) = x^{-1}.$$

The corresponding local charts ϕ_1, ϕ_2 defined in U_1 and U_2, respectively, are the stereographic projections from the poles.

(2 val.) (a) Show that the local forms

$$\omega = \frac{xd\bar{x} - dx\bar{x}}{2(1 + |x|^2)} \quad \text{and} \quad \theta = \frac{yd\bar{y} - dy\bar{y}}{2(1 + |y|^2)}$$

satisfy

$$\theta = \left(\frac{x}{|x|}\right)^{-1} \omega \ \frac{x}{|x|} + \left(\frac{x}{|x|}\right)^{-1} d\left(\frac{x}{|x|}\right).$$

Therefore they determine one connection on a (rank 2) complex vector bundle ξ over $\mathbb{S}^4$, such that the transition functions take values on $SU(2)$.
Hint: Note that $dy = -x^{-1}dx\, x^{-1}$ and $d|x| = \frac{dx\bar{x}+xd\bar{x}}{2|x|}$.

(0.5 val.) (b) Justify that $c_1(\xi) = 0$.

(2.5 val.) (c) Knowing that the curvature is given in U_1 by

$$\Omega = \frac{dx \wedge d\bar{x}}{(1+|x|^2)^2}$$

show that $c_2(\xi) = -\mu$ where μ is the canonical generator of $H^4(\mathbb{S}^4) \simeq \mathbb{R}$ (i.e. μ is the class of the volume form on $\mathbb{S}^4$ which has integral 1 over $\mathbb{S}^4$ and gives the same orientation as $da \wedge dv_1 \wedge dv_2 \wedge dv_3$, where $x = a + v_1 i + v_2 j + v_3 k$).
Hint: Recall that the quaternions i, j, k can be identified with

$$i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

and satisfy $ij = k, jk = i, ki = j, i^2 = j^2 = k^2 = ijk = -1$. Recall also that the spherical coordinates in $\mathbb{R}^4$ are given by $(r, \varphi_1, \varphi_2, \varphi_3)$ where $0 \le r < \infty$, $0 < \varphi_1, \varphi_2 < \pi$ and $0 < \varphi_3 < 2\pi$, and the Jacobian of the change of variables is given by $dV_4 = r^3 \sin^2 \varphi_1 \sin \varphi_2$.