Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Differential Geometry

Exam 1 - January 11, 2019

Duration: 3 hours
Justify your answers carefully.

(3 val.) 1. Let M be an oriented, smooth n-manifold, ω a smooth n-form on M and let X be a smooth vector field on M with compact support. Prove that

$$\int_M \mathcal{L}_X \omega = 0.$$

(3 val.) 2. Show that any Lie group G is parallelizable, i.e. the tangent bundle TG is a trivial vector bundle. Conclude that an even dimension sphere $\mathbb{S}^{2n}$ does not admit the structure of a Lie group.

(4 val.) 3. Consider the distribution $\mathcal{D}$ in $\mathbb{R}^4$, with coordinates (x, y, z, w), generated by the vector fields $X = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$ and $Y = \frac{\partial}{\partial x} + y\frac{\partial}{\partial w}$. Determine the integral curve of X through the point $(2, 1, -1, 3)$. Show that $\mathcal{D}$ has no integral manifolds of dimension 2.

(5 val.) 4. Let M and N be disjoint connected compact manifolds of dimension d. Let $M\#N$ be the **connected sum** of M and N, i.e., the manifold obtained by deleting the interiors of closed d-balls $B_1 \subset M$ and $B_2 \subset N$ and gluing the resulting boundary spheres ∂B_1 and ∂B_2 via some diffeomorphism between them (note that M and N are glued together through a set which is diffeomorphic to $\mathbb{S}^{d-1}$). Show that the Euler characteristics of M, N and $M\#N$ are related by

$$\mathcal{X}(M\#N) = \mathcal{X}(M) + \mathcal{X}(N) - \mathcal{X}(\mathbb{S}^d).$$

5. Consider the **complex** line bundle (hence a real vector bundle of rank 2) $\xi = (\pi, E, \mathbb{CP}^1)$, where $E = \mathbb{CP}^2\backslash\{[0, 0, 1]\}$, $\mathbb{CP}^1 \simeq \mathbb{S}^2$ and $\pi([z, w, t]) = [z, w]$, defined by the trivializing charts (U_1, ϕ_1) and (U_2, ϕ_2), where

$$U_1 = \{[z, 1] \in \mathbb{CP}^1 \,:\, z \in \mathbb{C}\}, \quad \phi_1([z, 1, t]) = ([z, 1], t);$$
$$U_2 = \{[1, w] \in \mathbb{CP}^1 \,:\, w \in \mathbb{C}\}, \quad \phi_2([1, w, t]) = ([1, w], t).$$

(2 val.) (a) Show that the first Chern class of ξ is $c_1(\xi) = -\mu$ where μ is the canonical generator of $H^2(\mathbb{S}^2) \simeq \mathbb{R}$ (i.e. the class of the volume form on $\mathbb{S}^2$ which gives the same orientation as $dx \wedge dy$, where $z = x + iy$, and has integral 1 over $\mathbb{S}^2$).
Hint: Recall that the curvature form is given in U_1 by $\Omega = \dfrac{dz \wedge d\bar{z}}{(1 + |z|^2)^2}$.

(3 val.) (b) Show that the first Chern class of ξ^* is $c_1(\xi^*) = \mu$.
Hint: Recall that a connection ∇ in a vector bundle ξ induces a connection on ξ^* by $(\nabla_X \sigma)(s) = X(\sigma(s)) - \sigma(\nabla_X s)$.