Instituto Superior Técnico
Departamento de Matemática
Secção de Álgebra e Análise

Cálculo Diferencial e Integral 2
Respostas à Ficha de Trabalho 5

1. (a) $\frac{2}{3}$.
   (b) $2 - \operatorname{sen}(2)$.

2. (a) $\frac{1}{6}\left(e^9 - 1\right)$.
   (b) $\frac{1}{6}$.

3. A área é $\frac{7}{6}$. O centro de massa é o ponto $\left(\frac{5}{14}, \frac{38}{35}\right)$. Os momentos de inércia são $I_x = \frac{673}{420}$, $I_y = \frac{13}{60}$ e $I_O = I_x + I_y = \frac{191}{105}$.

4. (a) $\int_{-1}^{0}\left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)dx\right)dy + \int_0^1\left(\int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x,y)dx\right)dy.$
   (b) $\int_0^1\left(\int_{2-y}^{1+\sqrt{1-y^2}} f(x,y)dx\right)dy.$
   (c) $\int_{-1}^{0}\left(\int_{-2\arcsin y}^{\pi} f(x,y)dx\right)dy + \int_0^1\left(\int_{\arcsin y}^{\pi-\arcsin y} f(x,y)dx\right)dy.$
   (d) $\int_0^{\frac{1}{2}}\left(\int_{\frac{5\pi}{2}-\frac{2\pi}{3}y}^{\frac{5\pi}{2}+\frac{2\pi}{3}y} f(x,y)dx\right)dy + \int_{\frac{1}{2}}^{1}\left(\int_{2\pi+\arcsin y}^{3\pi-\arcsin y} f(x,y)dx\right)dy.$
   (e) $\int_{-1}^{0}\left(\int_0^{\pi-\arcsin x} f(x,y)dy + \int_{2\pi+\arcsin x}^{2\pi} f(x,y)dy\right)dx + \int_0^1\left(\int_{\arcsin x}^{\pi-\arcsin x} f(x,y)dy\right)dx.$
   (f) $\int_0^1\left(\int_{y-2}^{-\sqrt{1-y^2}} f(x,y)dx + \int_{\sqrt{1-y^2}}^{1} dx\right)dy + \int_1^2\left(\int_{y-2}^{2-y} f(x,y)dx\right)dy.$

5. (a) $\ln\sqrt{2} - \frac{5}{16}$.
   (b) $\frac{7}{48}$.

6. $\frac{1}{6}$.

7. O volume é $\frac{80}{3}$. O centróide é o ponto $\left(\frac{121}{50}, 0, \frac{182}{75}\right)$.

8. (a) $\int_0^1\left(\int_0^x\left(\int_0^{1-x} dy\right)dz + \int_x^1\left(\int_{z-x}^{1-x} dy\right)dz\right)dx$, e
   $\int_0^1\left(\int_0^z\left(\int_{z-y}^{1-y} dx\right)dy + \int_z^1\left(\int_0^{1-y} dx\right)dy\right)dz.$
   (b) $\int_0^1\left(\int_0^{y^2}\left(\int_0^1 dx\right)dz + \int_{y^2}^{1+y^2}\left(\int_{\sqrt{z-y^2}}^1 dx\right)dz\right)dy$, e
   $\int_0^1\left(\int_0^{\sqrt{z}}\left(\int_{\sqrt{z-y^2}}^1 dx\right)dy + \int_{\sqrt{z}}^1\left(\int_0^1 dx\right)dy\right)dz + \int_1^2\left(\int_{\sqrt{z-1}}^1\left(\int_{\sqrt{z-y^2}}^1 dx\right)dy\right)dz.$
   (c) $\int_{-1}^1\left(\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\right)dx\right)dz$ e $\int_{-1}^1\left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dz\right)dy\right)dx.$

(d) $\int_{-1}^{0}\left(\int_{-1}^{-\sqrt{\frac{1-z}{2}}}\left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\right) dy + \int_{-\sqrt{\frac{1-z}{2}}}^{-\sqrt{-z}}\left(\int_{-\sqrt{y^2+z}}^{\sqrt{y^2+z}} dx\right) dy +\right.$
$+\int_{\sqrt{-z}}^{\sqrt{\frac{1-z}{2}}}\left(\int_{-\sqrt{y^2+z}}^{\sqrt{y^2+z}} dx\right) dy + \left.\int_{\sqrt{\frac{1-z}{2}}}^{1}\left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\right) dy\right) dz +$
$+\int_{0}^{1}\left(\int_{-1}^{-\sqrt{\frac{1-z}{2}}}\left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\right) dy + \int_{-\sqrt{\frac{1-z}{2}}}^{\sqrt{\frac{1-z}{2}}}\left(\int_{-\sqrt{y^2+z}}^{\sqrt{y^2+z}} dx\right) dy +\right.$
$+\left.\int_{\sqrt{\frac{1-z}{2}}}^{1}\left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\right) dy\right) dz$, e
$\int_{-1}^{1}\left(\int_{2x^2-1}^{x^2}\left(\int_{-\sqrt{1-x^2}}^{-\sqrt{x^2-z}} dy + \int_{\sqrt{x^2-z}}^{\sqrt{1-x^2}} dy\right) dz + \int_{x^2}^{1}\left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\right) dz\right) dx.$

(e) $\int_0^1\left(\int_{\frac{x}{2}}^{x}\left(\int_0^x dz\right) dy\right) dx$, $\int_0^{\frac{1}{2}}\left(\int_y^{2y}\left(\int_0^x dz\right) dx\right) dy+\int_{\frac{1}{2}}^{1}\left(\int_y^{1}\left(\int_0^x dz\right) dy\right) dx$,
e
$\int_0^{\frac{1}{2}}\left(\int_{\frac{z}{2}}^{z}\left(\int_z^{2y} dx\right) dy + \int_z^{\frac{1}{2}}\left(\int_y^{2y} dx\right) dy + \int_{\frac{1}{2}}^{1}\left(\int_y^{1} dx\right) dy\right) dz +$
$\int_{\frac{1}{2}}^{1}\left(\int_{\frac{z}{2}}^{\frac{1}{2}}\left(\int_z^{2y} dx\right) dy + \int_{\frac{1}{2}}^{z}\left(\int_z^{1} dx\right) dy + \int_z^{1}\left(\int_y^{1} dx\right) dy\right) dz.$

9. (a) $\int_0^{\sqrt{2}}\int_{\frac{3\pi}{4}}^{\frac{9\pi}{4}} f(r\cos\theta, r\operatorname{sen}\theta) r d\theta dr.$

   (b) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_1^2 f(r\cos\theta, r\operatorname{sen}\theta) r dr d\theta.$

   (c) $\int_0^{\frac{\pi}{2}}\int_0^{\frac{1}{\cos\theta+\operatorname{sen}\theta}} f(r\cos\theta, r\operatorname{sen}\theta) r dr d\theta.$

   (d) $\int_0^{\frac{\pi}{4}}\int_0^{\frac{\operatorname{sen}\theta}{\cos^2\theta}} f(r\cos\theta, r\operatorname{sen}\theta) r dr d\theta+\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_0^{\frac{1}{\operatorname{sen}\theta}} f(r\cos\theta, r\operatorname{sen}\theta) r dr d\theta+$
   $\int_{\frac{3\pi}{4}}^{\pi}\int_0^{\frac{\operatorname{sen}\theta}{\cos^2\theta}} f(r\cos\theta, r\operatorname{sen}\theta) r dr d\theta.$

   (e) $\int_{-\frac{\pi}{4}}^{0}\int_0^{\frac{1}{\cos\theta}} f(r\cos\theta, r\operatorname{sen}\theta) r dr d\theta + \int_0^{\frac{\pi}{2}}\int_0^1 f(r\cos\theta, r\operatorname{sen}\theta) r dr d\theta.$

   (f) $\int_0^{\frac{\pi}{4}}\int_{\frac{\operatorname{sen}\theta}{\cos^2\theta}}^{\frac{1}{\cos\theta}} f(r\cos\theta, r\operatorname{sen}\theta) r dr d\theta.$

10. (a) $\pi\left(1-\frac{1}{e^4}\right).$

    (b) $\frac{\pi\log 2}{2}.$

    (c) $\frac{3\pi}{4}.$

    (d) $\pi\operatorname{sen}(2).$

    (e) $\pi - 2\arctan\frac{1}{2}.$

11. (a) A imagem de T é $\left\{(x,y)\in\mathbb{R}^2\colon 0\le x\le 2, -x^2\le y\le x\right\}.$

    (b) $2+\frac{2}{\sqrt{3}}\left(\arctan\left(\frac{1}{\sqrt{3}}\right)-\arctan\left(\frac{5}{\sqrt{3}}\right)\right).$

12. $\frac{1}{16}\left(1-\frac{1}{e}\right).$

13. (a) $\int_0^1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 f(\rho\cos\theta, \rho\,\mathrm{sen}\,\theta, z)\rho dz d\theta d\rho$ e
$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_0^{\frac{\pi}{4}} \int_0^{\frac{1}{\cos\phi}} f(r\cos\theta\,\mathrm{sen}\,\phi, r\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\phi, r\cos\phi) r^2\,\mathrm{sen}\,\phi dr d\phi + \right.$
$\left. \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{1}{\mathrm{sen}\,\phi}} f(r\cos\theta\,\mathrm{sen}\,\phi, r\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\phi, r\cos\phi) r^2\,\mathrm{sen}\,\phi dr d\phi \right) d\theta.$

(b) $\int_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} \int_\rho^{\sqrt{1-\rho^2}} f(\rho\cos\theta, \rho\,\mathrm{sen}\,\theta, z)\rho dz d\rho d\theta$ e
$\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^1 f(r\cos\theta\,\mathrm{sen}\,\phi, r\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\phi, r\cos\phi) r^2\,\mathrm{sen}\,\phi dr d\phi d\theta.$

(c) $\int_0^{\frac{\pi}{4}} \left(\int_0^1 \int_{\sqrt{1-\rho^2}}^{\sqrt{2-\rho^2}} f(\rho\cos\theta, \rho\,\mathrm{sen}\,\theta, z)\rho dz d\rho + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-\rho^2}} f(\rho\cos\theta, \rho\,\mathrm{sen}\,\theta, z)\rho dz d\rho \right) d\theta$
e $\int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_1^{\sqrt{2}} f(r\cos\theta\,\mathrm{sen}\,\phi, r\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\phi, r\cos\phi) r^2\,\mathrm{sen}\,\phi dr d\phi d\theta.$

14. (a) $\int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_1^2 r^4\,\mathrm{sen}^3\,\phi dr d\phi d\theta$ e
$\int_0^{\frac{\pi}{2}} \left(\int_{-\sqrt{2}}^{-\frac{1}{\sqrt{2}}} \int_{-z}^{\sqrt{4-z^2}} \rho^3 d\rho dz + \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{\sqrt{1-z^2}}^{\sqrt{4-z^2}} \rho^3 d\rho dz + \int_{\frac{1}{\sqrt{2}}}^{\sqrt{2}} \int_z^{\sqrt{4-z^2}} \rho^3 d\rho dz \right) d\theta.$

(b) $\frac{127 \cdot 43\pi}{420\sqrt{2}}$.

15. (a) $\frac{16\pi}{3\sqrt{3}}$.

(b) $\frac{63}{32}$.

16. 5π.

17. $\frac{31}{20}$.

18. (a) $\frac{2\pi}{3}$.

(b) $4\pi^2$.

19. (a) $-\frac{26}{3}$.

(b) $\frac{\partial F}{\partial x}(x,y) = \int_0^1 \left(\int_0^u \frac{u}{1+(xu+y^2v)^2} dv \right) du$ e $\frac{\partial F}{\partial y}(x,y) = \int_0^1 \left(\int_0^u \frac{2yv}{1+(xu+y^2v)^2} dv \right) du.$

(c) $G'(x) = -\,\mathrm{sen}\,x \log\left(1+e^{x\cos x}\right) - 2x\log\left(1+e^{x^3}\right) + \int_{x^2}^{\cos x} \frac{te^{tx}}{1+e^{tx}}\, dt.$

20. $F'(x) = 3x^2 f(x^4, x^6+x^3) - f(x^2, x^2+x^3) + \int_x^{x^3} \left(\frac{\partial f}{\partial u}(tx, t^2+x^3)t + \frac{\partial f}{\partial v}(tx, t^2+x^3)3x^2 \right) dt.$

21. $\frac{2\pi}{t}\left(2e^{t^2} - e^t\right)$.