

# HOMOTOPY DECOMPOSITION OF A GROUP OF SYMPLECTOMORPHISMS OF $S^2 \times S^2$

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ABSTRACT. We continue the analysis started by Abreu, McDuff and Anjos [Ab, AM, McD, An] of the topology of the group of symplectomorphisms of  $S^2 \times S^2$  when the ratio of the area of the two spheres lies in the interval  $(1, 2]$ . We express the group, up to homotopy, as the pushout (or amalgam) of certain of its compact Lie subgroups. We use this to compute the homotopy type of the classifying space of the group of symplectomorphisms and the corresponding ring of characteristic classes for symplectic fibrations.

## 1. INTRODUCTION

Let  $M_\lambda$  denote the symplectic manifold  $(S^2 \times S^2, \omega_\lambda = \lambda\sigma_0 \oplus \sigma_0)$  where  $1 \leq \lambda \in \mathbb{R}$  and  $\sigma_0$  is the standard area form on  $S^2$  with total area equal to 1. It is known that any symplectic form on the manifold  $S^2 \times S^2$  is, up to scaling by a constant, diffeomorphic to  $\omega_\lambda$  (see Lalonde-McDuff [LM]). Let  $G_\lambda$  denote the group of symplectomorphisms of  $M_\lambda$ .

By now, much is known about the topology of the group  $G_\lambda$  and, more generally, about symplectomorphism groups of ruled surfaces. Gromov [Gr] first showed that, when  $\lambda = 1$ ,  $G_\lambda$  deformation retracts onto the subgroup of standard isometries  $\mathbb{Z}/2 \times (SO(3) \times SO(3))$ . He also showed that this would no longer be the case if  $\lambda > 1$ , and, confirming this, McDuff [McD1] constructed an element of infinite order in  $H_1(G_\lambda)$ . Later, Abreu [Ab] computed the rational cohomology of  $G_\lambda$  when  $1 < \lambda \leq 2$  and his methods were extended by Abreu and McDuff [AM, McD], who completely described the rational homotopy type of  $G_\lambda$  and  $BG_\lambda$  for all values of  $\lambda$  (see also [McD2] for more information on the integral homotopy type of  $G_\lambda$ , including the fact that the topology of the group changes precisely when  $\lambda$  crosses an integer). In [An], the first author computed the homology Pontryagin ring of  $G_\lambda$  with field coefficients when  $1 < \lambda \leq 2$  and used this to determine the homotopy type of the space  $G_\lambda$ . All these results were derived from a study of the action of  $G_\lambda$  on the contractible space  $\mathcal{J}_\lambda$  of compatible almost complex structures on  $M_\lambda$ .

Our aim in this paper is to continue the analysis of the case  $1 < \lambda \leq 2$  by describing the homotopy type of  $G_\lambda$  as a topological group.

To explain our result, recall from [Ab] that, for  $1 < \lambda \leq 2$ , the space  $\mathcal{J}_\lambda$  is a stratified space with two strata  $U_0$  and  $U_1$ . Each of these contains an integrable almost complex structure  $J_i$ , with isotropy group  $K_i \subset G_\lambda$  where  $K_0 = SO(3) \times SO(3)$  and  $K_1 = S^1 \times SO(3)$ . Following Gromov, Abreu showed that the inclusions

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of the orbits  $G_\lambda/K_i \subset U_i$  are weak equivalences.  $K_0$  and  $K_1$  intersect in  $K_{01} = SO(3)$  which sits inside  $K_0$  as the diagonal and inside  $K_1$  as the second factor. Thus, there is a commutative diagram of group monomorphisms

$$(1) \quad \begin{array}{ccc} SO(3) & \xrightarrow{\Delta} & SO(3) \times SO(3) \\ j \downarrow & & \downarrow i_0 \\ S^1 \times SO(3) & \xrightarrow{i_1} & G_\lambda \end{array}$$

where  $\Delta$  denotes the inclusion of the diagonal,  $j$  the inclusion of the second factor, and  $i_0, i_1$  the inclusions in  $G_\lambda$  of the subgroups  $K_0, K_1$ .

Let  $P$  denote the pushout of the diagram of topological groups

$$(2) \quad \begin{array}{ccc} SO(3) & \xrightarrow{\Delta} & SO(3) \times SO(3) \\ j \downarrow & & \\ S^1 \times SO(3) & & \end{array}$$

This is also known as the amalgam (or amalgamated product, or free product with amalgamation) of the groups  $SO(3) \times SO(3)$  and  $S^1 \times SO(3)$  over the common subgroup  $SO(3)$  and it is characterized as the initial topological group admitting compatible homomorphisms from the diagram (2). We will review this construction in section 2. The universal property of the pushout gives us a canonical continuous homomorphism

$$(3) \quad P \longrightarrow G_\lambda.$$

We need to describe another map derived from (1). We will write

$$\text{hocolim}(X \xleftarrow{f} Y \xrightarrow{g} Z)$$

for the homotopy pushout (or double mapping cylinder) of the maps  $f, g$ . This is the quotient space of

$$X \amalg (Y \times [0, 1]) \amalg Z$$

by the equivalence relation generated by  $(y, 0) \sim f(y); (y, 1) \sim g(y)$ . Applying the classifying space functor to (1) we obtain a canonical map

$$(4) \quad \text{hocolim}\left(BSO(3) \times BSO(3) \xleftarrow{B\Delta} BSO(3) \xrightarrow{Bj} BS^1 \times BSO(3)\right) \rightarrow BG_\lambda.$$

Finally, let  $\text{Met}(S^2 \times S^2)$  denote the space of metrics on  $S^2 \times S^2$ . The usual retraction [MS, Proposition 2.50 (ii)]

$$\text{Met}(S^2 \times S^2) \xrightarrow{r} \mathcal{J}_\lambda$$

is equivariant with respect to the action of the group of symplectomorphisms. Therefore, letting  $K < G_\lambda$  denote a compact subgroup, the fact that the fixed point space  $\text{Met}(S^2 \times S^2)^K$  is convex implies that  $\mathcal{J}_\lambda^K$  is contractible. It follows that we can pick a path

$$(5) \quad [0, 1] \xrightarrow{\gamma} \mathcal{J}_\lambda^{K_{01}}$$

with  $\gamma(0) = J_0$  and  $\gamma(1) = J_1$  and this is unique up to homotopy. A choice of path determines a  $G_\lambda$ -equivariant map (unique up to  $G_\lambda$ -equivariant homotopy)

$$(6) \quad \text{hocolim} \left( G_\lambda / (SO(3) \times SO(3)) \xleftarrow{\pi_0} G_\lambda / SO(3) \xrightarrow{\pi_1} G_\lambda / (S^1 \times SO(3)) \right) \rightarrow \mathcal{J}_\lambda$$

where  $\pi_0, \pi_1$  denote the canonical projections.

We can now state our main result:

**Theorem 1.1.** *Let  $1 < \lambda \leq 2$ . Then the following equivalent statements hold:*

- (i) *The homomorphism (3) is a weak equivalence.*
- (ii) *The map (4) is a weak equivalence.*
- (iii) *The  $G_\lambda$ -equivariant map (6) is a weak equivalence.*

Statement (i) is an immediate consequence of the observation that the homology computations in [An] say that, for  $k$  a field, the Pontryagin ring  $H_*(G_\lambda; k)$  is the pushout in the category of  $k$ -algebras of the diagram obtained by applying  $H_*(-; k)$  to (2), together with Theorem 3.8 below which says that the functor  $H_*(-; k)$  preserves certain pushouts. The equivalence of statements (i)-(iii) is an easy consequence of Theorem 3.8 together with Theorem 3.10 which says that under the same hypotheses, the classifying functor also preserves pushouts. From (ii), it is easy to compute the ring  $H^*(BG_\lambda; \mathbb{Z})$  of integral characteristic classes of symplectic fibrations with fibre  $M_\lambda$  (see Corollary 4.5).

A better proof of Theorem 1.1 (which would, in particular, provide an alternative proof of the main results of [An]) would be to deduce (iii) directly from an analysis of the stratification of  $\mathcal{J}_\lambda$ . Unfortunately, we were unable to do this. The description of the stratification of  $\mathcal{J}_\lambda$  by McDuff in [McD] yields a homotopy pushout decomposition of  $\mathcal{J}_\lambda$  as

$$\text{hocolim}(U_0 \leftarrow NU_1 \setminus U_1 \rightarrow NU_1)$$

where  $NU_1$  denotes a tubular neighborhood of  $U_1$  in  $\mathcal{J}_\lambda$  which fibers over  $U_1$  as a disk bundle. These three spaces have the required homotopy types but we do not know whether  $NU_1$  can be chosen so as to be invariant under the action of  $G_\lambda$ . More precisely, (iii) would follow if we could choose a tubular neighborhood  $NU_1$ , a path  $\gamma$  as in (5), and  $t_0 \in ]0, 1[$  satisfying:

- $G_\lambda \cdot \gamma(t) \subset NU_1$  for  $t > t_0$ ,
- $\gamma(t) \notin U_1$  for  $t < t_0$ .

(ii) and (iii) are the statements one should try to generalize to the cases when  $\lambda > 2$ , although in those cases it might be necessary to use a topological category to index the homotopy colimit decomposition. See [MW, Appendix D] for a relationship between stratifications and homotopy colimit decompositions.

It is an easy consequence of [Se, Theorem 8, p. 36] that any compact subgroup of the amalgam  $P$  is subconjugate in  $P$  to either  $K_0$  or  $K_1$ . In view of Theorem 1.1 (ii), it is natural to ask whether this is also the case in  $G_\lambda$ . Yael Karshon has proved in [Ka] that the answer is yes if one assumes that the subgroup in question is a torus (her result is not circumscribed to the case  $1 < \lambda \leq 2$ ).

Finally, Theorem 1.1 suggests it might be a good idea to explore analogies with infinite dimensional algebraic groups (cf. [Mi, Ki]). In particular, compare Theorem 1.1(ii) with the homotopy decomposition of the classifying space of a rank 2 Kac-Moody group of indefinite type as the homotopy pushout of a diagram of compact subgroups [Ki, Theorem 4.2.3] (see also [ABKS, BrK]).

**Organization of the paper.** In section 2 we begin by recalling the constructions of pushouts in various categories and introducing necessary notation. In section 3, under the assumption that the homomorphisms involved induce injections on homology, we compute the Pontryagin ring and the homotopy type of the classifying space of an amalgam of topological groups whose connected components are compact Lie groups. In section 4, we deduce Theorem 1.1 from the results of section 3 and [An]. We then use it to produce an interesting fiber sequence involving  $BG_\lambda$  and compute the ring  $H^*(BG_\lambda; \mathbb{Z})$  of characteristic classes for symplectic fibrations with fiber  $M_\lambda$ .

## 2. PUSHOUTS AND AMALGAMS.

In this section we begin by reviewing the categorical notions of pushout and coequalizer. We then recall the construction of the pushout in the categories of groups, topological groups and  $k$ -algebras, and introduce some notation that will be necessary in the next section.

**Pushouts and coequalizers.** Everything in this subsection is standard basic category theory. See [Ma], for instance.

Let  $B_0, B_1, B_2$  be objects in a category  $\mathcal{C}$  and  $f_1, f_2$  morphisms in  $\mathcal{C}$ . Given a diagram in  $\mathcal{C}$ ,

$$(7) \quad \begin{array}{ccc} B_0 & \xrightarrow{f_1} & B_1 \\ \downarrow f_2 & & \\ B_2 & & \end{array}$$

a *cone* on this diagram consists of an object  $C$  together with morphisms  $g_1$  and  $g_2$  such that

$$\begin{array}{ccc} B_0 & \xrightarrow{f_1} & B_1 \\ \downarrow f_2 & & \downarrow g_1 \\ B_2 & \xrightarrow{g_2} & C \end{array}$$

commutes. A *pushout* of (7) is a cone  $(C, g_1, g_2)$  with the property that if  $(D, h_1, h_2)$  is any other cone, then there is a unique morphism  $\varphi : C \rightarrow D$  making the following diagram commute:

$$\begin{array}{ccccc} B_0 & \xrightarrow{f_1} & B_1 & & \\ \downarrow f_2 & & \downarrow g_1 & \searrow h_1 & \\ B_2 & \xrightarrow{g_2} & C & \xrightarrow{h_1} & D \\ & \searrow h_2 & \swarrow \varphi & & \\ & & & & \end{array}$$

This universal property characterizes the pushout up to unique isomorphism in  $\mathcal{C}$  (when the pushout exists). The pushout is usually denoted by

$$B_1 \amalg_{B_0} B_2.$$

Given a pair of arrows

$$(8) \quad A \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} B,$$

a cone on (8) is an arrow  $\epsilon : B \rightarrow C$  such that  $\epsilon d_0 = \epsilon d_1$ . A *coequalizer* of (8) is a cone  $\epsilon : B \rightarrow C$  with the property that given any other cone  $\epsilon' : B \rightarrow C'$ , there is a unique map  $\varphi : C \rightarrow C'$  making the following diagram commute

$$\begin{array}{ccccc} A & \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} & B & \xrightarrow{\epsilon} & C \\ & & & \searrow \epsilon' & \downarrow \varphi \\ & & & & C' \end{array}$$

Again this universal property characterizes the coequalizer up to unique isomorphism. Pushouts and coequalizers are instances of a more general construction, that of colimit of a diagram, which is defined by a similar universal property.

**Example 2.1.** The main examples we have in mind are the following:

- (a)  $\mathcal{C}$  is the category of topological spaces. If  $A$  is a topological group acting on the right on the space  $Y$  and on the left on the space  $X$  then the coequalizer of the two maps  $d_0, d_1 : X \times A \times Y \rightarrow X \times Y$  defined by  $d_0(x, a, y) = (xa, y)$  and  $d_1(x, a, y) = (x, ay)$  is the quotient of  $X \times Y$  by the action  $(a, x, y) \rightarrow (xa, a^{-1}y)$ . We write  $X \times_A Y$  for this space.
- (b)  $\mathcal{C}$  is the category of vector spaces over a field  $k$ . If  $A$  a  $k$ -algebra,  $V$  a left  $A$ -module and  $W$  a right  $A$ -module, the coequalizer of the action maps  $d_0, d_1 : V \otimes_k A \otimes_k W \rightarrow V \otimes_k W$  defined as above is the tensor product  $V \otimes_A W$ .

If  $k$  is a field, the universal property of the coequalizer together with the Künneth theorem give us a canonical map

$$(9) \quad H_*(X; k) \otimes_{H_*(A; k)} H_*(Y; k) \rightarrow H_*(X \times_A Y; k).$$

The main theorem of this paper follows immediately from the fact that a similar canonical map is an isomorphism and the proof will consistently exploit such maps.

**Pushouts of groups.** Suppose  $\mathcal{C}$  is the category of groups. A good reference for everything in this subsection is [Se].

Let  $S$  denote the set of finite sequences

$$x_1 x_2 \dots x_n$$

where  $x_i \in B_1$  or  $x_i \in B_2$  and consider the equivalence relation  $\sim$  on  $S$  generated by

- (i)  $x_1 \dots x_n \sim x_1 \dots \hat{x}_i \dots x_n$  if  $x_i = 1$ ,
- (ii)  $x_1 \dots f_1(a) \dots x_n \sim x_1 \dots f_2(a) \dots x_n$  for  $a \in B_0$ ,
- (iii)  $x_1 \dots x_i x_{i+1} \dots x_n \sim x_1 \dots (x_i x_{i+1}) \dots x_n$  when  $x_i, x_{i+1}$  both belong to  $B_1$  or  $B_2$ .

$S$  has an associative unital product defined by concatenation and it is easy to check that this descends to the set  $P = S / \sim$  of equivalence classes and that  $P$  together with the canonical maps  $B_1 \rightarrow P$  and  $B_2 \rightarrow P$  is the pushout of (7).

If the maps  $f_1$  and  $f_2$  are monomorphisms then the pushout of the diagram (7) is also called the *amalgam*<sup>1</sup> of  $B_1$  and  $B_2$  over  $B_0$ . In this case, there is a useful description of the elements of the pushout, called the Normal Form Theorem. To explain this, we will start by introducing some notation. Let  $\mathbf{i} = (i_1, \dots, i_n) \in \{0, 1, 2\}^n$ . Then we define

$$(10) \quad \overline{B}_{\mathbf{i}} = B_{i_1} \times \cdots \times B_{i_n}$$

Note that there is a canonical map

$$(11) \quad \overline{B}_{\mathbf{i}} \xrightarrow{\pi} P$$

determined by multiplication.  $B_0^{n-1}$  acts on the right on  $\overline{B}_{\mathbf{i}}$  by

$$(a_1, \dots, a_{n-1}) \cdot (b_1, \dots, b_n) = (b_1 a_1, a_1^{-1} b_1 a_2, a_2^{-1} b_2 a_3, \dots, a_{n-1}^{-1} b_n).$$

We will write

$$(12) \quad B_{\mathbf{i}} = B_{i_1} \times_{B_0} \cdots \times_{B_0} B_{i_n}$$

for the quotient of  $\overline{B}_{\mathbf{i}}$  by this action. Note that we can express the quotient  $B_{\mathbf{i}}$  as the coequalizer of

$$\overline{B}_{\mathbf{i}} \times B_0^{n-1} \begin{array}{c} \xrightarrow{d_0} \\ \xrightarrow{d_1} \end{array} \overline{B}_{\mathbf{i}}$$

where  $d_0$  is given by the action, and  $d_1$  is the projection on the first factor. The canonical map  $\pi$  of (11) factors through this quotient and we will denote the resulting map also by  $\pi$ .

Let  $B'_i = B_i \setminus B_0$  for  $i = 1, 2$ , and for  $\mathbf{i} = (i_1, \dots, i_n)$  a sequence of alternating 1's and 2's, let

$$B'_{\mathbf{i}} = B'_{i_1} \times_{B_0} B'_{i_2} \times_{B_0} \cdots \times_{B_0} B'_{i_n}$$

denote the quotient of  $B'_{i_1} \times \cdots \times B'_{i_n}$  by the action of  $B_0^{n-1}$ .

**Theorem 2.2.** [Se, Theorem 2, page 4] *If  $f_1, f_2$  are monomorphisms, the canonical maps  $\pi$  determine a bijection*

$$B_0 \amalg \amalg_{\mathbf{i}} B'_{\mathbf{i}} \xrightarrow{\sim} P$$

where  $\mathbf{i}$  runs over all sequences of alternating 1's and 2's.

**Pushouts of topological groups.** We now consider the case when the  $B_i$  are topological groups and the homomorphisms between them continuous. We need to give a topology to the space  $P$  defined above so that  $P$  together with the maps  $B_i \rightarrow P$  is the pushout in the category of topological groups.

In order to do this, it is necessary to work in a category of compactly generated spaces. Any will do for our purposes, so we will work with the simplest, namely that of Vogt [Vo, Example 5.1]. A space  $X$  is said to be compactly generated if a subset  $U \subset X$  is closed iff for every compact Hausdorff space  $K$  and continuous map  $g : K \rightarrow X$ ,  $g^{-1}(U)$  is closed. Given an arbitrary topological space  $X$  we can refine the topology in the obvious way so that it becomes compactly generated. Denoting this space by  $kX$ , the natural transformation

$$kX \rightarrow X$$

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<sup>1</sup>It is also called amalgamated product of  $B_1$  and  $B_2$  over  $B_0$ , and free product of  $B_1$  and  $B_2$  with amalgamation.

determined by the identity maps is a weak equivalence [Vo, Proposition 1.2 (h)]. The (categorical) product in the category of compactly generated spaces does not in general agree with the product in spaces; it is necessary to apply the functor  $k$  to the usual product. In all that follows we will write  $\times$  for the product in the category of compactly generated spaces. By a topological group we mean a group object in the category of compactly generated spaces. Any topological group in the usual sense determines such a group object by applying  $k$  to the multiplication<sup>2</sup>.

We will write

$$P_n \subset P$$

for the image in  $P$  of the  $B_{\mathbf{i}}$  with  $\mathbf{i} \in \{0, 1, 2\}^n$  and  $P_0$  for the image of  $B_0$ . We will also write  $\alpha_n$  and  $\beta_n$  for the two sequences in  $\{0, 1, 2\}^n$  of alternating 1's and 2's and

$$(13) \quad Q_n = B_{\alpha_n} \amalg B_{\beta_n}.$$

Similarly we define

$$(14) \quad \overline{Q}_n = \overline{B}_{\alpha_n} \amalg \overline{B}_{\beta_n}.$$

We will set  $\overline{Q}_0 = Q_0 = B_0$ . With the notation above, we first give  $P_n$  the quotient topology determined by the canonical map

$$(15) \quad Q_n \xrightarrow{\pi} P_n.$$

Note that this is the same as to give  $P_n$  the topology induced by  $\overline{Q}_n$ , since  $Q_n$  has the quotient topology determined by the projection  $\overline{Q}_n \rightarrow Q_n$ .

The inclusion  $\overline{Q}_n \rightarrow \overline{Q}_{n+1}$  defined by

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 1)$$

induces a continuous map

$$P_n \xrightarrow{i_n} P_{n+1}.$$

We give  $P$  the topology of the union of the  $P_n$ 's. The product  $\mu : P \times P \rightarrow P$  clearly restricts to a product  $\mu_{n,m} : P_n \times P_m \rightarrow P_{n+m}$ . Consider the commutative diagram

$$\begin{array}{ccc} \overline{Q}_n \times \overline{Q}_m & \xrightarrow{\overline{\mu}_{n,m}} & \overline{Q}_{n+m} \\ \pi \times \pi \downarrow & & \downarrow \pi \\ P_n \times P_m & \xrightarrow{\mu_{n,m}} & P_{n+m} \end{array}$$

where the map  $\overline{\mu}_{n,m}$  is defined by:

$$\overline{\mu}_{n,m}(\mathbf{x}, \mathbf{y}) = \begin{cases} (x_1, \dots, x_n, y_1, \dots, y_m) & \text{if } x_n \in B_i, y_1 \in B_j \text{ with } i \neq j, \\ (x_1, \dots, x_n y_1, \dots, y_m, 1) & \text{otherwise.} \end{cases}$$

Both vertical maps are quotient maps [Vo, Corollary 3.8] so we conclude that the maps  $\mu_{n,m}$  are continuous. Since in the category of compactly generated spaces, colimits commute with products [Vo, Proposition 3.7 (b)], it follows that the multiplication

$$P \times P \xrightarrow{\mu} P$$

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<sup>2</sup>This is not necessary for groups of diffeomorphisms of compact manifolds which are metrizable and hence compactly generated.

is continuous. Similarly one checks that the inverse map on  $P$  is continuous. It is now easy to check that the topological group  $P$  together with the maps  $B_i \rightarrow P$  has the required universal property and hence is the pushout in the category of topological groups.

**Pushouts of  $k$ -algebras.** Let  $k$  be a field. There is an entirely analogous description of the pushout of (7) in the category of associative  $k$ -algebras. We will soon want to compare the construction for algebras and the homology of the analogous construction for topological groups and to make the notation more clear we will decorate the algebraic constructions with a superscript *alg*.

Let  $S^{alg}$  denote the tensor algebra generated by the  $k$ -vector spaces  $B_1^{alg}$  and  $B_2^{alg}$ . Thus, as a vector space,  $S^{alg}$  is the direct sum

$$k \oplus B_1^{alg} \oplus B_2^{alg} \oplus (B_1^{alg} \otimes B_1^{alg}) \oplus (B_1^{alg} \otimes B_2^{alg}) \oplus \dots$$

where  $\otimes$  denotes the tensor product of  $k$ -vector spaces. Then the pushout of (7) in  $k$ -algebras is

$$P^{alg} = S^{alg} / I$$

together with the canonical maps  $B_i^{alg} \rightarrow P^{alg}$ , where  $I$  is the ideal generated by

- (i)  $x_1 \otimes \dots \otimes x_n - x_1 \otimes \dots \otimes \hat{x}_i \otimes \dots \otimes x_n$  if  $x_i = 1$ ,
- (ii)  $x_1 \otimes \dots \otimes f_1(a) \otimes \dots \otimes x_n - x_1 \otimes f_2(a) \otimes \dots \otimes x_n$  for  $a \in B_0^{alg}$ ,
- (iii)  $x_1 \otimes \dots \otimes x_i \otimes x_{i+1} \otimes \dots \otimes x_n - x_1 \otimes \dots \otimes (x_i x_{i+1}) \otimes \dots \otimes x_n$  when  $x_i, x_{i+1}$  both belong to  $B_1^{alg}$  or  $B_2^{alg}$ .

In some circumstances it is possible to give a more precise description of the pushout. To do this, we need some notation. Let  $\mathbf{i} = (i_1, \dots, i_n) \in \{0, 1, 2\}^n$ . Then we define

$$(16) \quad \overline{B}_{\mathbf{i}}^{alg} = B_{i_1}^{alg} \otimes \dots \otimes B_{i_n}^{alg}$$

Note that there is a canonical map

$$(17) \quad \overline{B}_{\mathbf{i}}^{alg} \xrightarrow{\pi} P^{alg}$$

determined by the multiplication on  $P^{alg}$ .

Given a sequence  $\mathbf{i} \in \{0, 1, 2\}^n$ , let  $\bar{\mathbf{i}} \in \{0, 1, 2\}^{2n-1}$  denote the sequence obtained from  $\mathbf{i}$  by inserting 0's between each two entries of  $\mathbf{i}$ . For example, if  $\mathbf{i} = (i_1, i_2, i_3)$  then  $\bar{\mathbf{i}} = (i_1, 0, i_2, 0, i_3)$ . Then we define maps

$$\overline{B}_{\bar{\mathbf{i}}}^{alg} \xrightleftharpoons[d_1]{d_0} \overline{B}_{\mathbf{i}}^{alg}$$

by (setting  $f_0 = \text{id}$ )

$$\begin{aligned} d_0(x_1 \otimes a_1 \otimes x_2 \otimes \dots \otimes x_n) &= x_1 f_{i_1}(a_1) \otimes x_2 f_{i_2}(a_2) \otimes \dots \otimes x_n \\ d_1(x_1 \otimes a_1 \otimes x_2 \otimes \dots \otimes x_n) &= x_1 \otimes f_{i_2}(a_1) x_2 \otimes \dots \otimes f_{i_n}(a_{n-1}) x_n. \end{aligned}$$

and we define  $B_{\mathbf{i}}^{alg}$  to be the coequalizer (in  $k$ -vector spaces) of these maps. That is,

$$(18) \quad B_{\mathbf{i}}^{alg} = B_{i_1}^{alg} \otimes_{B_0^{alg}} B_{i_1}^{alg} \otimes_{B_0^{alg}} \dots \otimes_{B_0^{alg}} B_{i_n}^{alg}$$

The canonical map  $\pi$  of (17) factors through  $B_{\mathbf{i}}^{alg}$  and we will denote the resulting map also by  $\pi$ . Moreover we will write as in (13)

$$(19) \quad Q_n^{alg} = B_{\alpha_n}^{alg} \oplus B_{\beta_n}^{alg}.$$



and

$$(20) \quad P_n^{alg} \subset P^{alg}$$

for the image of  $Q_n^{alg}$  in  $P^{alg}$ .

**Remark 2.3.** *If the  $k$ -algebras  $B_i^{alg}$  are the homology Pontryagin rings of topological groups  $B_i$ , then they have added structure: they are Hopf algebras (see [MM]). In particular there are antiautomorphisms*

$$c : B_i^{alg} \rightarrow B_i^{alg}$$

induced by the inverse map on the group, as well as augmentations (i.e. maps of  $k$ -algebras)

$$B_i^{alg} \xrightarrow{\epsilon} k$$

induced by the map to the trivial group. Let  $B_0^{n \text{ alg}}$  denote the  $k$ -algebra

$$B_0^{n \text{ alg}} = B_0^{alg} \otimes \dots \otimes B_0^{alg}$$

augmented in the obvious way. For  $\mathbf{i} = (i_1, \dots, i_n) \in \{0, 1, 2\}^n$ ,  $\overline{B}_{\mathbf{i}}^{alg}$  is a right  $B_0^{n-1 \text{ alg}}$ -module under the action

$$(b_1 \otimes \dots \otimes b_n) \cdot (a_1 \otimes \dots \otimes a_{n-1}) = (b_1 a_1 \otimes c(a_1) b_1 a_2 \otimes \dots \otimes c(a_{n-1}) b_n)$$

where we have omitted the maps  $f_i$  from the notation. Moreover, it is easy to see that

$$\overline{B}_{\mathbf{i}}^{alg} \otimes_{B_0^{n-1 \text{ alg}}} k = B_{\mathbf{i}}^{alg}.$$

As in (9) it is then an immediate consequence of the Künneth theorem and the universal property of a coequalizer that there is a canonical map

$$(21) \quad B_{\mathbf{i}}^{alg} \xrightarrow{\phi} H_*(B_{\mathbf{i}}; k)$$

Moreover, writing  $P^{alg}$  for the pushout of the Pontryagin rings, the diagram

$$\begin{array}{ccc} B_{\mathbf{i}}^{alg} & \xrightarrow{\phi} & H_*(B_{\mathbf{i}}; k) \\ \downarrow & & \downarrow \\ P^{alg} & \longrightarrow & H_*(P; k) \end{array}$$

commutes.

We will need the following analog of Theorem 2.2 for pushouts of  $k$ -algebras, due to P.M. Cohn.

**Theorem 2.4.** [Co3, Proof of Theorem 3.1] *If  $f_1, f_2$  are monomorphisms, and there exist right  $B_0^{alg}$ -modules  $B_i'^{alg}$  such that  $B_i^{alg} = f_i(B_0^{alg}) \oplus B_i'^{alg}$  as right  $B_0^{alg}$ -modules. Then regarding  $B_i'^{alg}$  as a  $B_0^{alg}$ -bimodule via the isomorphism  $B_i'^{alg} \simeq B_i^{alg} / B_0^{alg}$ , the canonical maps  $\pi$  determine an isomorphism of  $B_0^{alg}$ -bimodules*

$$B_0^{alg} \oplus \bigoplus_{\mathbf{i}} B_{\mathbf{i}}'^{alg} \xrightarrow{\sim} \bigoplus_{n \geq 0} P_n^{alg} / P_{n-1}^{alg}$$

where  $\mathbf{i}$  runs over all sequences of alternating 1's and 2's.

For future reference, we also note that there are exhaustive filtrations

$$B_i^{alg} \subset \dots \subset P_n^{alg} B_i^{alg} \subset \dots \subset P^{alg}$$

of  $P^{alg}$  by right  $B_i^{alg}$ -modules and that (cf. [Co3, (32), p. 61]) we have canonical isomorphisms

$$(22) \quad \bigoplus_{\mathbf{i}} B_{\mathbf{i}}^{alg} \otimes_{B_0^{alg}} B_j^{alg} \xrightarrow{\sim} \bigoplus_{n \geq 0} P_n^{alg} B_j^{alg} / P_{n-1}^{alg} B_j^{alg}$$

of right  $B_j^{alg}$ -modules, where  $\mathbf{i}$  runs over all sequences of alternating 1's and 2's not ending in  $j$ .

**Remark 2.5.** *Theorem 2.4 also holds under the assumption that the  $f_i$  are monomorphisms and  $B_i^{alg} = B_i^{alg}/f_i(B_0^{alg})$  are flat right  $B_0^{alg}$ -modules (cf. [Co2, (1.6)-(1.8) p. 436]) but the above formulation is sufficient for our purposes.*

### 3. AMALGAMS OF TOPOLOGICAL GROUPS.

In this section, under the assumption that the homomorphisms involved induce injections on homology, we compute the Pontryagin ring of an amalgam of topological groups whose connected components are compact Lie groups. Using this and Puppe's theorem, we obtain a homotopy decomposition of the classifying space.

Although the results are stated for the case when the free products involved have only two factors, it is clear that all the statements hold (with the same proof) for an arbitrary number of factors.

**Homology of the pushout.** In order to compute the homology of the pushout of a diagram of topological groups we require some assumptions.

**Definition 3.1.** *We will say that a pushout diagram (7) of topological groups is homologically free if it satisfies the following conditions:*

- (i) *The connected components of the  $B_i$  are compact Lie groups,*
- (ii) *The continuous homomorphisms  $f_1, f_2$  are monomorphisms.*
- (iii) *The homomorphisms  $H_*(f_i; k)$  are injective for every field  $k$ .*

The first condition of the previous definition can certainly be weakened. We make this assumption because it suffices for the application we have in mind and it considerably simplifies the point set topology involved.

**Lemma 3.2.** *Under the assumptions (i)-(ii) of Definition 3.1, and with the notation of (15) we have*

- (a)  *$P_n$  is a Hausdorff space.*
- (b)  *$(Q_n, \pi^{-1}(P_{n-1})) \xrightarrow{\pi_n} (P_n, P_{n-1})$  is a relative homeomorphism.*

*Proof.* (a) is easy and (b) follows easily from compactness and the Normal Form Theorem 2.2.  $\square$

The following lemma will be used often in the rest of this section.

**Lemma 3.3.** *Let  $X \rightarrow B$  be a principal fibration with fiber  $A$  and  $k$  be a field. If the inclusion of the fiber induces an injection  $H_*(A; k) \rightarrow H_*(X; k)$  then*

- (a)  *$H_*(X; k)$  is a free (and hence flat) right  $H_*(A; k)$ -module,*
- (b) *The canonical map*

$$H_*(X; k) \otimes_{H_*(A; k)} k \rightarrow H_*(B; k)$$

*is an isomorphism.*

*Proof.* Since the inclusion of the fiber  $A \rightarrow X$  induces an injection on connected components, it suffices to consider the case when  $A$  is connected. It follows that the action of  $\pi_1(B)$  on  $H_*(A; k)$  is trivial. The homology Leray-Serre spectral sequence is a spectral sequence of right  $H_*(A; k)$ -modules. Since  $H_*(A; k) \rightarrow H_*(X; k)$  is an injection, spectral sequence collapses. The  $E_2$  and hence the  $E_\infty$  terms are free  $H_*(A; k)$ -modules on  $H_*(B; k)$ . It follows that  $H_*(X; k)$  is a free  $H_*(A; k)$ -module and the projection induces an isomorphism

$$H_*(X; k) \otimes_{H_*(A; k)} k \rightarrow H_*(B; k)$$

as required.  $\square$

**Remark 3.4.** *The first part of Lemma 3.3 is also a consequence of [MM, Theorem 4.4].*

In order to compute the homology of the pushout, we need one more lemma. Let  $\Lambda$  denote the partially ordered set  $\{0, 1, 2\}$  where the order is defined by  $0 \leq 1$  and  $0 \leq 2$  ( $1, 2$  are incomparable). Given  $n > 0$ ,  $\Lambda^n$  is again a poset. The order relation is

$$(i_1, \dots, i_n) \leq (j_1, \dots, j_n) \iff i_k \leq j_k \text{ for each } i.$$

For  $\mathbf{j} \leq \mathbf{i} \in \Lambda^n$  there are obvious inclusions

$$B_{\mathbf{j}} \longrightarrow B_{\mathbf{i}}.$$

**Lemma 3.5.** *Suppose the diagram (7) satisfies conditions (i) and (ii) of Definition 3.1. Let  $\mathbf{i} \in \Lambda^n$  and  $\Pi \subset \Lambda^n$  be such that*<sup>3</sup>

- (a)  $\mathbf{j} \leq \mathbf{i}$  for every  $\mathbf{j} \in \Pi$ ,
- (b)  $\mathbf{j} \in \Pi$  and  $\mathbf{k} \leq \mathbf{j} \Rightarrow \mathbf{k} \in \Pi$ .

*Then the canonical map*

$$\operatorname{colim}_{\mathbf{j} \in \Pi} B_{\mathbf{j}} \rightarrow B_{\mathbf{i}}$$

*is a closed cofibration.*

*Proof.* If  $\Pi = \{\mathbf{j}\}$  consists of a single element, this follows from the fact that  $\overline{B_{\mathbf{j}}} \subset \overline{B_{\mathbf{i}}}$  has a  $B_0^{n-1}$ -equivariant tubular neighborhood, from which we can obtain a neighborhood deformation retraction of  $B_{\mathbf{j}} \subset B_{\mathbf{i}}$ .

The result now follows by induction using the union theorem for cofibrations [Li, Corollary 2] (which states that if  $A \subset X$ ,  $B \subset X$  and  $A \cap B \subset X$  are closed cofibrations then  $A \cup B \subset X$  is again a closed cofibration).  $\square$

For the rest of this section we set

$$B_i^{alg} := H_*(B_i; k).$$

**Corollary 3.6.** *If the pushout diagram (7) is homologically free, then for any  $\mathbf{i} \in \Lambda^n$  the canonical map*

$$B_{\mathbf{i}}^{alg} \xrightarrow{\phi} H_*(B_{\mathbf{i}}; k)$$

*is an isomorphism.*

---

<sup>3</sup>These conditions ensure that the colimit is just the union of the images of the spaces  $B_{\mathbf{j}}$  in  $B_{\mathbf{i}}$ .

*Proof.* It is easy to check that, under our assumptions, the inclusion of the fiber in the total space of the principal  $B_0^{n-1}$ -fibration

$$\overline{B}_i \rightarrow B_i$$

is an injection. The result now follows immediately from Lemma 3.3 and Remark 2.3.  $\square$

Note that if the diagram (7) is homologically free, then the inclusions  $B_j \subset B_i$  induce inclusions on homology with field coefficients, i.e. the canonical maps

$$B_j^{alg} \longrightarrow B_i^{alg}$$

are injective. This follows from the obvious fact that

$$\overline{B}_j^{alg} \longrightarrow \overline{B}_i^{alg}$$

is injective, together with the fact that the  $H_*(B_0^{n-1}; k)$ -modules  $\overline{B}_i^{alg}$  are flat by Lemma 3.3(a). More generally, the same argument implies that the canonical map

$$(23) \quad \operatorname{colim}_{j \in \Pi} B_j^{alg} \longrightarrow B_i^{alg}$$

is injective, when  $\Pi \subset \Lambda^n$  satisfies the two conditions of Lemma 3.5.

**Lemma 3.7.** *Suppose the pushout diagram (7) is homologically free. Let  $\Pi \subset \Lambda^n$  satisfy the two conditions of Lemma 3.5. Then the canonical map*

$$\operatorname{colim}_{j \in \Pi} B_j^{alg} \longrightarrow H_*(\operatorname{colim}_{j \in \Pi} B_j; k)$$

*is an isomorphism.*

*Proof.* Suppose  $\Pi' = \Pi \cup \{i\}$  and let  $\Gamma = \{j \in \Pi : j \leq i\}$ . Then we have a pushout square

$$(24) \quad \begin{array}{ccc} \operatorname{colim}_{j \in \Gamma} B_j & \longrightarrow & \operatorname{colim}_{j \in \Pi} B_j \\ \eta \downarrow & & \downarrow \\ B_i & \longrightarrow & \operatorname{colim}_{j \in \Pi'} B_j \end{array}$$

By Lemma 3.5 the map  $\eta$  is a cofibration. If we assume that the result is true for  $\Gamma$  then  $\eta$  induces an injection in homology by (23) and so the Mayer-Vietoris sequence for (24) splits. This implies that the result holds for  $\Pi'$ , and hence it is true in general, by induction.  $\square$

**Theorem 3.8.** *If the pushout diagram (7) is homologically free then the canonical map*

$$P^{alg} = H_*(B_1; k) \amalg_{H_*(B_0; k)} H_*(B_2; k) \longrightarrow H_*(P; k)$$

*is an isomorphism.*

*Proof.* Since homology commutes with sequential colimits of  $T_1$  spaces along closed inclusions, it suffices to show that the canonical map

$$P_n^{alg} \xrightarrow{\gamma_n} H_*(P_n; k)$$

is an isomorphism for each  $n$ . This is obvious for  $n = 0$ . Assume it is true for  $n \leq m$  and consider the following diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & P_m^{alg} & \longrightarrow & P_{m+1}^{alg} & \longrightarrow & P_{m+1}^{alg}/P_m^{alg} \longrightarrow 0 \\
 & & \downarrow \gamma_m & & \downarrow \gamma_{m+1} & & \downarrow \phi_{m+1} \\
 & & H_*(P_m; k) & \longrightarrow & H_*(P_{m+1}; k) & \longrightarrow & H_*(P_{m+1}, P_m; k)
 \end{array}$$

It is enough to see that the induced map  $\phi_{m+1}$  is an isomorphism since it then follows that  $\psi_n$  is surjective and then, by the 5-lemma, that  $\gamma_{m+1}$  is an isomorphism. To simplify notation, write

$$W_n = \pi^{-1}(P_{n-1}) \subset Q_n \quad ; \quad W_n^{alg} = \pi^{-1}(P_{n-1}^{alg}) \subset Q_n^{alg}.$$

It is clear from the Normal Form Theorem 2.2 that

$$W_n = \operatorname{colim}_{\mathbf{j} < \alpha_n} B_{\mathbf{j}} \amalg \operatorname{colim}_{\mathbf{j} < \beta_n} B_{\mathbf{j}}.$$

Since  $B_1^{alg}$  and  $B_2^{alg}$  are free right  $B_0^{alg}$ -modules by Lemma 3.3, it follows from Theorem 2.4 that we have similarly

$$(25) \quad W_n^{alg} = \operatorname{colim}_{\mathbf{j} < \alpha_n} B_{\mathbf{j}}^{alg} \oplus \operatorname{colim}_{\mathbf{j} < \beta_n} B_{\mathbf{j}}^{alg}.$$

Consider the diagram

$$\begin{array}{ccccc}
 W_{m+1}^{alg} & \xrightarrow{\iota} & Q_{m+1}^{alg} & \longrightarrow & Q_{m+1}^{alg}/W_{m+1}^{alg} \\
 \downarrow & & \downarrow & & \downarrow \psi_{m+1} \\
 H_*(W_{m+1}, k) & \xrightarrow{i_{m+1}} & H_*(Q_{m+1}; k) & \longrightarrow & H_*(Q_{m+1}, W_{m+1}; k)
 \end{array}$$

Lemma 3.7 says that the left vertical map is an isomorphism and Lemma 3.6 says that the middle vertical one is. By (23),  $\iota$  is an inclusion so  $\psi_{m+1}$  is an isomorphism.

The desired result now follows from the commutativity of the diagram

$$\begin{array}{ccc}
 Q_{m+1}^{alg}/W_{m+1}^{alg} & \xrightarrow{\eta} & P_{m+1}^{alg}/P_m^{alg} \\
 \downarrow \psi_{m+1} & & \downarrow \phi_{m+1} \\
 H_*(Q_{m+1}, W_{m+1}; k) & \xrightarrow{\xi} & H_*(P_{m+1}, P_m; k)
 \end{array}$$

where  $\eta$  is an isomorphism by definition and  $\xi$  by Lemma 3.2(b).  $\square$

**The homotopy decomposition of  $BP$ .** We will need the following corollary of Puppe's theorem (see [DF, Proposition, page 180] or [MaWa, Theorem 1]).

**Theorem 3.9.** *Given a diagram of spaces*

$$\begin{array}{ccccc}
 X_1 & \longleftarrow & X_0 & \longrightarrow & X_2 \\
 \downarrow & & \downarrow & & \downarrow \\
 Y & \longleftarrow & Y & \longrightarrow & Y
 \end{array}$$

then denoting by  $F_i$  the homotopy fiber of the map  $X_i \rightarrow Y$ , and by  $F$  the homotopy fiber of the canonical map

$$\operatorname{hocolim}(X_1 \longleftarrow X_0 \longrightarrow X_2) \rightarrow Y,$$

the canonical map

$$\mathrm{hocolim}(F_1 \longleftarrow F_0 \longrightarrow F_2) \rightarrow F$$

is a weak equivalence.

We now compute the homotopy type of the classifying space  $BP$ . Note that this result generalizes the well known theorem of J.H.C. Whitehead for discrete groups [Br, Theorem II.7.3].

**Theorem 3.10.** *If the pushout diagram (7) is homologically free then, the canonical map*

$$\mathrm{hocolim}(BB_1 \longleftarrow BB_0 \longrightarrow BB_2) \xrightarrow{\phi} BP$$

is a weak equivalence.

*Proof.* Consider the diagram

$$\begin{array}{ccccc} P/B_1 & \longleftarrow & P/B_0 & \longrightarrow & P/B_2 \\ \downarrow & & \downarrow & & \downarrow \\ BB_1 & \longleftarrow & BB_0 & \longrightarrow & BB_2 \\ \downarrow & & \downarrow & & \downarrow \\ BP & \xleftarrow{=} & BP & \xrightarrow{=} & BP \end{array}$$

where all the vertical maps are fiber sequences. It follows from Van-Kampen's theorem that  $\phi$  induces an isomorphism on  $\pi_1$ . Therefore the homotopy fiber of  $\phi$ , which by Theorem 3.9 is

$$(26) \quad F \simeq \mathrm{hocolim}(P/B_1 \longleftarrow P/B_0 \longrightarrow P/B_2),$$

has abelian fundamental group. It now suffices to show that for any field  $k$ , we have

$$(27) \quad H_*(F; k) \simeq k.$$

By Lemma 3.3 it is sufficient to show that the map

$$(28) \quad P^{alg} \otimes_{B_0^{alg}} k \rightarrow P^{alg} \otimes_{B_1^{alg}} k \oplus P^{alg} \otimes_{B_2^{alg}} k$$

is an isomorphism in positive degrees and injective with cokernel  $k$  in degree 0.

Consider the filtrations  $P_n^{alg}, P_n^{alg} B_1^{alg}, P_n^{alg} B_2^{alg}$  of  $P^{alg}$  respectively by right  $B_0^{alg}, B_1^{alg}$  and  $B_2^{alg}$ -modules. The map (28) is clearly filtration preserving so we get canonical maps of graded  $k$ -vector spaces

$$P_n^{alg} \otimes_{B_0^{alg}} k \rightarrow P_n^{alg} B_1^{alg} \otimes_{B_1^{alg}} k \oplus P_n^{alg} B_2^{alg} \otimes_{B_2^{alg}} k$$

It follows immediately from Theorem 2.4 together with (22) that these maps induce isomorphisms on the associated graded vector space of the filtrations (except in filtration 0 where the cokernel is  $k$ ). This completes the proof.  $\square$

**Remark 3.11.** *It would be interesting to know whether Theorem 3.10 holds under the assumption that  $H_*(B_i; k)$  are  $H_*(B_0; k)$ -flat but the maps  $H_*(f_i; k)$  are not necessarily injective. This is true for discrete groups according to [Fi, Theorem 4.1].*

4. THE HOMOTOPY DECOMPOSITION OF  $G_\lambda$  FOR  $1 < \lambda \leq 2$ .

The following is a reformulation of [An, Theorem 1.2, Theorem 3.1] in a form which is suitable for our purposes.

**Theorem 4.1** (Anjos). *Let  $k$  be a field. The diagram*

$$\begin{array}{ccc} H_*(SO(3); k) & \xrightarrow{\Delta} & H_*(SO(3) \times SO(3); k) \\ \downarrow & & \downarrow \\ H_*(S^1 \times SO(3); k) & \longrightarrow & H_*(G_\lambda; k) \end{array}$$

is a pushout square in the category of  $k$ -algebras.

We are now ready to prove our main result.

*Proof of Theorem 1.1.* The hypotheses of Theorem 3.8 are clearly satisfied. Therefore Theorem 4.1 implies that the canonical map

$$(SO(3) \times SO(3)) \amalg_{SO(3)} (S^1 \times SO(3)) \longrightarrow G_\lambda$$

is a homology equivalence with any field coefficients. Hence it is an equivalence on integral homology. Since both spaces are  $H$ -spaces it follows that it is in fact a weak equivalence. This proves (i).

It remains to show the equivalence of (i)-(iii).

(iii)  $\Rightarrow$  (ii): This follows immediately by applying the Borel construction

$$EG_\lambda \times_{G_\lambda} -$$

to the the weak equivalence of (iii).

(ii)  $\Rightarrow$  (i): By Theorem 3.10, (ii) implies that the map  $BP \rightarrow BG_\lambda$  is a weak equivalence and looping it get that

$$P \longrightarrow G_\lambda$$

is a weak equivalence.

(i)  $\Rightarrow$  (iii): Saying that the map (iii) is a weak equivalence amounts to saying that

$$\text{hocolim}(G_\lambda / (SO(3) \times SO(3)) \longleftarrow G_\lambda / SO(3) \longrightarrow G_\lambda / (SO(3) \times S^1))$$

is weakly contractible. Assuming (i), this follows from (26) and (27).  $\square$

Here is an interesting consequence of Theorem 1.1(ii).

**Proposition 4.2.** *There is a fiber sequence*

$$(29) \quad \Sigma^2 SO(3) \longrightarrow BG_\lambda \longrightarrow BS^1 \times BSO(3) \times BSO(3)$$

*Proof.* Let  $X = BSO(3)$  and  $Y = BS^1$ . Apply Theorem 3.9 to the bottom half of the diagram, where all vertical sequences are homotopy fiber sequences

$$\begin{array}{ccccc} \Omega Y & \longleftarrow & \Omega X \times \Omega Y & \longrightarrow & \Omega X \\ \downarrow & & \downarrow & & \downarrow \\ X \times X & \longleftarrow & X & \longrightarrow & X \times Y \\ \downarrow & & \downarrow & & \downarrow \\ X \times X \times Y & \longleftarrow & X \times X \times Y & \longrightarrow & X \times X \times Y \end{array}$$

we see that the fiber of the map

$$BG_\lambda \rightarrow BS^1 \times BSO(3) \times BSO(3)$$

is

$$\text{hocolim}(\Omega Y \longleftarrow \Omega X \times \Omega Y \longrightarrow \Omega X) = \Omega X * \Omega Y \simeq \Sigma(\Omega X \wedge \Omega Y)$$

(where  $*$  denotes the join). This completes the proof.  $\square$

In particular, we see that  $G_\lambda$  has a canonical homotopy representation

$$(30) \quad BG_\lambda \rightarrow BS^1 \times BSO(3) \times BSO(3)$$

even though, by Banyaga's theorem [MS, Theorem 10.25],  $G_\lambda$  is a simple group and hence admits no nontrivial homomorphisms to compact Lie groups.

**Remark 4.3.** *The identification of the homotopy type of  $G_\lambda$  [An, Theorem 1.1] is an immediate consequence of Proposition 4.2. In fact, looping (29) we get a fiber sequence of loop maps*

$$\Omega \Sigma^2 SO(3) \longrightarrow G_\lambda \longrightarrow S^1 \times SO(3) \times SO(3)$$

*This has a section (given by including the subgroups in  $G_\lambda$  and multiplying) so  $G_\lambda$  is weakly equivalent to the product of the base and the fiber.*

**Remark 4.4.** *Let  $X \subset SO(3) \times SO(3)$  denote the image of any section of the principal fiber bundle*

$$SO(3) \times SO(3) \longrightarrow SO(3)$$

*determined by quotienting  $SO(3) \times SO(3)$  by the diagonal  $SO(3)$  action. Denoting by  $P$  the pushout of (2), it is not hard to see that the kernel of the canonical group homomorphism*

$$P \rightarrow S^1 \times SO(3) \times SO(3)$$

*is the free group generated by the set of commutators  $[S^1, X] \subset P$  (this set is independent of the choice of section). The space  $[S^1, X] \subset P$  is homeomorphic to  $\Sigma SO(3)$ . Writing  $FY$  for the free topological group generated by the space  $Y$ , we have  $FY \simeq \Omega \Sigma Y$ . Thus (29) is the fiber sequence obtained by applying the classifying space functor to the short exact sequence of topological groups*

$$F[S^1, X] \longrightarrow P \longrightarrow S^1 \times SO(3) \times SO(3).$$

Using Theorem 1.1(ii), it is easy to compute the ring of integral characteristic classes of symplectic fibrations with fibre  $M_\lambda$ .

**Corollary 4.5.**

$$H^*(BG_\lambda; \mathbb{Z}) = \mathbb{Z}[T, X_1, X_2, Y_1, Y_2, Z] / \langle T(X_i - Y_i), TZ, 2X_1, 2Y_1, 2Z, Z^2 \rangle$$

where  $|Y_i| = |X_i| = i + 2$ ,  $|T| = 2$ , and  $|Z| = 5$ .

*Proof.* Since

$$H^*(BSO(3) \times BSO(3); \mathbb{Z}) = \mathbb{Z}[X_1, X_2, Y_1, Y_2, Z] / \langle 2X_1, 2Y_1, 2Z, Z^2 \rangle$$

this follows easily from the Mayer-Vietoris sequence

$$\begin{aligned} \cdots \longrightarrow H^*(BG_\lambda; \mathbb{Z}) \longrightarrow H^*(BSO(3) \times BSO(3); \mathbb{Z}) \oplus H^*(BS^1 \times BSO(3); \mathbb{Z}) \\ \xrightarrow{\phi} H^*(BSO(3); \mathbb{Z}) \longrightarrow \cdots \end{aligned}$$

which identifies  $H^*(BG_\lambda; \mathbb{Z})$  with the kernel of the map  $\phi$ .  $\square$



In particular, the map  $H^*(BS^1 \times BSO(3) \times BSO(3); \mathbb{Z}) \rightarrow H^*(BG_\lambda; \mathbb{Z})$  induced by (30) is surjective.

**Remark 4.6.** *Using the computation of  $H^*(BG_\lambda; \mathbb{Q})$  by Abreu and McDuff [AM], Januszkiewicz and Kedra [JK] have shown that all the characteristic classes in  $H^*(BG_\lambda; \mathbb{R})$  come from integrating monomials on the Chern classes of the vertical tangent bundle and the coupling class<sup>4</sup> over the fibers.*

*In more detail, the fibration*

$$(31) \quad M \hookrightarrow M_{hG_\lambda} := M \times_{G_\lambda} EG_\lambda \rightarrow BG_\lambda$$

*yields a vertical tangent bundle*

$$TM_{hG_\lambda} := TM \times_{G_\lambda} EG_\lambda \rightarrow M_{hG_\lambda}.$$

*Since  $\mathcal{J}_\lambda$  is contractible, this bundle has (up to homotopy) a canonical complex structure. Integrating monomials on the Chern classes  $c_k$  and the coupling class  $\Omega$  over the fiber of (31) yields elements in  $H^*(BG_\lambda; \mathbb{R})$ .*

*If  $\alpha : T \rightarrow G_\lambda$  is a torus action on  $M_\lambda$ , the commutativity of the diagram*

$$\begin{array}{ccc} H^*(M_{hT}) & \xleftarrow{\tilde{B}_\alpha^*} & H^*(M_{hG_\lambda}) \\ \downarrow \pi_*^T & & \downarrow \pi_*^{G_\lambda} \\ H^{*-2n}(BT) & \xleftarrow{B_\alpha^*} & H^{*-2n}(BG_\lambda) \end{array}$$

*together with the fact that  $\pi_*^T$  can be computed by localization methods gives information on the corresponding classes in  $H^*(BG_\lambda)$ .*

*For example, Januszkiewicz and Kedra's calculations [JK, Proposition 4.1.1] say that the obvious map  $BG_\lambda \rightarrow BS^1$  derived from (30) classifies the class  $-\pi_*^{G_\lambda}(c_1^3)/8 \in H^2(BG_\lambda; \mathbb{Z})$ .*

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## REFERENCES

- [Ab] M. Abreu, Topology of symplectomorphism groups of  $S^2 \times S^2$ , *Invent. Math.*, **131** (1998), 1-23.
- [AM] M. Abreu and D. McDuff, Topology of symplectomorphism groups of rational ruled surfaces, *J. Amer. Math. Soc.*, **13** (2000), 971-1009.
- [ABKS] J. Aguadé, C. Broto, N. Kitchloo, and L. Saumell, Cohomology of classifying spaces of central quotients of rank 2 Kac-Moody groups, preprint.
- [An] S. Anjos, Homotopy type of symplectomorphism groups of  $S^2 \times S^2$ , *Geometry and Topology*, **6** (2002), 195-218.
- [BrK] C. Broto and N. Kitchloo, Classifying spaces of Kac-Moody groups, *Math. Zeitschr.* **240** (2002), 621-649.
- [Br] K. Brown, *Cohomology of groups*, Grad. Texts in Math. **87**, Springer (1982).
- [Co1] P. M. Cohn, On the free product of associative rings., *Math. Zeitschr.* **71**, 380-398, (1959).
- [Co2] P. M. Cohn, On the free product of associative rings. II., *Math. Zeitschr.* **73**, 433-456, (1960).

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<sup>4</sup>The coupling class  $\Omega \in H^2(M_{hG_\lambda}; \mathbb{R})$  is determined by the requirements that it restricts to  $[\omega_\lambda]$  on the fiber  $M_\lambda$  and satisfies  $\pi_*^{G_\lambda}(\Omega^{n+1}) = 0$  (cf. [MS, Chapter 6]).

- [Co3] P. M. Cohn, Free ideal rings., Journal of Algebra, **1**, (1964), 47-69.
- [DF] Emmanuel Dror-Farjoun, Cellular spaces, null spaces and homotopy localizations, Lect. Notes in Math **1622**, Springer (1996).
- [Fi] Z. Fiedorowicz, Classifying spaces of monoids and topological categories, Amer. Journal of Math. **106** (1984), 301-350.
- [Gr] M. Gromov, Pseudo holomorphic curves in symplectic manifolds, Invent. Math., **82** (1985), 307-347.
- [JK] T. Januszkiewicz and J. Kędra, Characteristic classes of smooth fibrations, math.SG/0209288.
- [Ka] Y. Karshon, Maximal tori in the symplectomorphism groups of Hirzebruch surfaces, math.SG/0204347.
- [Ki] N. Kitchloo, Topology of Kac-Moody groups, Ph.D. Thesis, M.I.T. (1998).
- [LM] F. Lalonde and D. McDuff, The classification of ruled symplectic 4-manifolds, Math. Research Letters **3** (1996), 769-778.
- [Li] J. Lillig, A union theorem for cofibrations, Archiv der Mathematik **24** (1973), 410-415.
- [Ma] S. MacLane, Categories for the working mathematician, Grad. Texts in Math **5**, Springer (1971).
- [MW] I. Madsen and M. Weiss, The stable moduli space of Riemann surfaces: Mumford's conjecture, math.AT/0212321.
- [MaWa] M. Mather and M. Walker, Commuting homotopy limits and colimits, Math. Zeit. **175** (1980), 77-80.
- [MS] D. McDuff and D. Salamon, Introduction to symplectic topology, Oxford University Press, (1996).
- [McD] D. McDuff, Almost complex structures on  $S^2 \times S^2$ , Duke Math. Journal **101** (2000), 135-177.
- [McD1] D. McDuff, Examples of symplectic structures, Invent. Math **89** (1987), 13-36.
- [McD2] D. McDuff, Symplectomorphism groups and almost complex structures, L'Enseignement Mathématique, **38** (2001), 527-556.
- [MM] J. Milnor and J. Moore, On the structure of Hopf algebras, Ann. of Math (2) **81** (1965), 211-264.
- [Mi] S. Mitchell, Quillen's theorem on buildings and the loops on a symmetric space, L'Enseignement Mathématique (2) **34** (1988), 123-166.
- [Se] J-P. Serre, Trees, Springer-Verlag, Berlin (1980).
- [Vo] R. Vogt, Convenient categories of topological spaces for homotopy theory, Archiv der Mathematik **22** (1971), 545-555.

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