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THE HOMOTOPY LIE ALGEBRA OF SYMPLECTOMORPHISM GROUPS OF 3-FOLD BLOW-UPS OF $(S^2 \times S^2, \sigma_{std} \oplus \sigma_{std})$

Sinan Eden

Supervisor: Doctor Sílvia Nogueira Rocha Ravasco dos Anjos

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The homotopy Lie algebra of symplectomorphism groups of 3-fold blow-ups of $(S^2 \times S^2, \sigma_{std} \oplus \sigma_{std})$

Abstract

In symplectic geometry, one of the interesting areas of research is the analysis and description of the symplectomorphism groups of symplectic manifolds. It is closely linked to the study of J-holomorphic curves. Particularly in four dimensions, the adjunction inequality and the positivity of intersections facilitate this investigation.

In all cases studied by now, the main idea behind the study of the homotopy type of the symplectomorphism group of a manifold (M, ω) is to investigate the natural action of this group on the contractible space of ω-compatible almost complex structures, $\mathcal{J}_\omega$. In those cases, namely rational ruled manifolds and some of their blow-ups, the space $\mathcal{J}_\omega$ is a stratified space where each stratum contains an integrable almost complex structure such that its isotropy group is a finite dimensional Lie group.

In this work, we will consider the 3-point blow-up of the manifold $S^2 \times S^2$ equipped with the product $\sigma_{std} \oplus \sigma_{std}$ of the standard symplectic forms on each S^2. So far, the monotone case (that is, when the capacities of the blow-ups are all equal to $1/2$) was studied by J. Evans, where he proved that the symplectomorphism group is contractible. Later, J. Li, T. J. Li and W. Wu showed that the group $\mathrm{Symp}_0(M, \omega)$ of symplectomorphisms that act trivially on homology is always connected. We will study, in more detail, the rational homotopy Lie algebra of this group.

As in the previous cases studied, we will show that the isotropy groups of the integrable almost complex structures generate the full topology of the symplectomorphism group. More precisely, in our case, the isotropy groups are either $\mathbb{T}^2$ or S^1 or the identity, and we will show that the Hamiltonian actions contained in them give the generators of the homotopy graded Lie algebra of the symplectomorphism group.

Our study depends also on Karshon's classification of Hamiltonian circle actions and the inflation technique introduced by Lalonde-McDuff.

Keywords: Symplectic geometry, Symplectomorphism group, Homotopy type, Almost complex structures, J-holomorphic curves.

A álgebra de Lie da homotopia de grupos de simplectomorfismos de blow-ups de $(S^2 \times S^2, \sigma_{std} \oplus \sigma_{std})$ em 3 pontos

Sinan Eden

Doutoramento em Matemática

Orientadora: Doutora Sílvia Nogueira Rocha Ravasco dos Anjos

Resumo

Em geometria simpléctica, uma das áreas interessantes de investigação é a análise e descrição dos grupos de simplectomorfismos de variedades simplécticas. Este estudo está relacionado com o estudo das curvas J-holomorfas. Em dimensão 4, a desigualdade de adjunção e a positividade das intersecções facilita esta investigação.

Em todos os casos estudados até agora, a ideia principal por detrás do estudo de homotopia do grupo da simplectomorfismos duma variedade (M, ω) é a análise da ação natural deste grupo no espaço contrátil $\mathcal{J}_\omega$ das estruturas quase complexas e ω-compatíveis. Nesses casos, nomeadamente em variedades racionais e regradas ("*rational ruled manifolds*") e em alguns blow-ups destas variedades, o espaço $\mathcal{J}_\omega$ é estratificado e cada estrato contém uma estrutura quase complexa integrável, tal que o seu grupo de isotropia é um grupo de Lie de dimensão finita.

Nesta tese, consideraremos o blow-up em 3 pontos de $S^2 \times S^2$ com a forma simpléctica $\sigma_{std} \oplus \sigma_{std}$ obtida como produto das formas simplécticas standard em S^2. Até agora, o caso monótono (isto é, em que as capacidades dos blow-ups são todas iguais a 1/2) foi estudado por J. Evans, tendo o autor provado que o grupo de simplectomorfismos é neste caso contrátil. Mais tarde, J. Li, T. J. Li e W. Wu mostraram que no caso geral o grupo de simplectomorfismos desta variedade simpléctica, que agem trivialmente em homologia, é conexo. Analisaremos, em maior detalhe, a álgebra de Lie da homotopia racional deste grupo.

Como nos casos anteriores, iremos mostrar que os grupos de isotropia das estruturas quase complexas integráveis geram a topologia do grupo de simplectomorfismos. Mais precisamente, no nosso caso, os grupos de isotropia são ou $\mathbb{T}^2$, ou S^1, ou a identidade, e iremos mostrar que as ações Hamiltonianas neles contidas produzem os geradores da homotopia da álgebra de Lie graduada ("*Lie graded algebra*") do grupo de simplectomorfismos. O nosso estudo utiliza também a classificação de Karshon das ações Hamiltonianas do círculo e a técnica de dilatação introduzida por Lalonde-McDuff.

Palavras-chave: Geometria simpléctica, Grupo de simplectomorfismos, Tipo de homotopia, Estruturas quase complexas, Curvas J-holomorfas.

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... 'and what is the use of a book,' thought Alice, 'without pictures or conversations?'

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Contents

Chapter 1

Introduction

1.1 History

The study of how the symplectomorphism groups of 4-manifolds look like was introduced by M. Gromov in [10], while the first systematic developments in the area were initiated by M. Abreu in his doctoral thesis [1], which gave the subject a fresh kick-start.

The long-known example is the symplectomorphism group of $S^2 \times S^2$ equipped with the product of standard symplectic forms on S^2, namely $\omega = \sigma \oplus \sigma$. This was calculated by M. Gromov in [10], among many other groundbreaking results. Then M. Abreu and D. McDuff ([3]) described the group of symplectomorphisms that act as identity on homology, on the same manifold equipped with the symplectic form $\omega_\mu = \mu\sigma \oplus \sigma$ where $\mu \geq 1$.

The group of symplectomorphisms of this manifold with only one blow-up, acting trivially on homology, is calculated by M. Pinsonnault in [28]. Then the 2-fold blow-up is calculated by S. Anjos and M. Pinsonnault in [5].

The next simplest thing to do would be to analyze the 3-fold blow-up. So far, the

monotone case (that is, when $\mu = 1$ and the capacities of the blow-ups are equal to $1/2$) was studied by J. Evans in [8], where he proved that the symplectomorphism group in this case is contractible. Furthermore, J. Li, T. J. Li and W. Wu showed in [18] that its symplectic mapping class group is connected. This work aimed at exploring the case $\mu = 1$ in more detail.

1.2 The story

In all cases studied by now, the main idea behind the study of the homotopy type of the symplectomorphism group of a manifold (M, ω) is to investigate the natural action of this group on the contractible space of ω-compatible almost complex structures, $\mathcal{J}_\omega$. In those cases, namely rational ruled manifolds and some of their blow-ups, the space $\mathcal{J}_\omega$ is a stratified space where each stratum contains an integrable almost complex structure such that its isotropy group is a finite dimensional Lie group.

As in the previous cases, we will show that the isotropy groups of the integrable almost complex structures generate the full topology of the symplectomorphism group. More precisely, in our case, the isotropy groups are either $\mathbb{T}^2$ or S^1 or the identity, and we will show that the Hamiltonian actions contained in them give the generators of the homotopy graded Lie algebra of the symplectomorphism group.

In the next section, we introduce the basic language and notation. In Chapter 2 we will look at the stratification of the space of ω-compatible almost complex structures. In Chapter 3, we study the homotopy groups of the space of symplectomorphisms, state in full detail and prove the main result of this work. Some of the arguments work equally for the case $\mu > 1$, while others are essentially different; we will point these out in Chapter 4 as a brief discussion for further investigations.

1.3 The main result

This work is part of a series of investigations in the area, focusing on the symplectic blow-ups of the space $(S^2 \times S^2, \mu\sigma \oplus \sigma)$, $\mu \geq 1$. In this work, we consider this space with $\mu = 1$, blown up at three balls of capacities of c_1, c_2 and c_3. We denote the end result as $\widetilde{M}_{c_1,c_2,c_3}$, keeping track of the symplectic form $\omega = \omega_{\mu=1,c_1,c_2,c_3} = \omega_{c_1,c_2,c_3}$.

Seeing it as a trivial fibration, the base $B \in H_2(S^2 \times S^2)$ representing $[S^2 \times \{pt\}]$ has area $\mu = 1$, and the fiber $F \in H_2(S^2 \times S^2)$ representing $[\{pt\} \times S^2]$ has area 1.

Remark 1.3.1. *Since we study symplectomorphisms that act trivially on homology, in the calculation of the homotopy type of the symplectomorphism group, we omit the functions that would interchange the roles of B and F. However, since B and F both have area 1, this omission might sometimes cause confusions in the presentation (In particular, see Figures 3.5-3.10 in Section 3.3). In order to avoid this confusion, in some cases we will keep the notation μ to distinguish the area of B from the area of F.*

Equivalently, we can start with $(\mathbb{CP}^2, \omega_\nu)$, where ω_ν is the standard Fubini-Study form rescaled so that $\omega_\nu(\mathbb{CP}^1) = \nu$, and blow up at four balls of capacities $\delta_1, \delta_2, \delta_3$ and δ_4. This gives a space conventionally denoted by $\mathbb{X}_4 = \mathbb{CP}^2 \# 4\overline{\mathbb{CP}^2}$, equipped with the complex structure $\omega_{\nu;\delta_1,\delta_2,\delta_3,\delta_4}$.

One easy way to understand the equivalence between $\mathbb{X}_4$ and $\widetilde{M}_{\mu,c_1,c_2,c_3}$ is as follows: Let $\{L, V_1, V_2, V_3, V_4\}$ be the basis for $H_2(\mathbb{X}_4; \mathbb{Z})$ where L is the class representing a line and the V_i are the exceptional classes. Let $\{B, F, E_1, E_2, E_3\}$ be the basis for $H_2(\widetilde{M}_{\mu,c_1,c_2,c_3}; \mathbb{Z})$ where the E_i represent the exceptional spheres arising from the blow-ups. We identify L with $B+F-E_1$, V_1 with $B-E_1$, V_2 with $F-E_1$, V_3 with E_2, and V_4 with E_3. In order to see this birational equivalence in the symplectic category, we recall "the uniqueness of symplectic blow-ups" due to McDuff (See Corollary 1.3 in [22]): the symplectomorphism type of a symplectic blow-up of a rational ruled manifold along an embedded ball of capacity $c \in (0, 1)$ depends only on the capacity

c and not on the particular embedding used in obtaining the blow-up. Using this result and after rescaling, we obtain the equivalence in the symplectic category:

$$\mu = \frac{\nu - \delta_2}{\nu - \delta_1},\ c_1 = \frac{\nu - \delta_1 - \delta_2}{\nu - \delta_1},\ c_2 = \frac{\delta_3}{\nu - \delta_1},\ \text{and}\ c_3 = \frac{\delta_4}{\nu - \delta_1}.$$

Let $G_{c_1,c_2,c_3} = G_{\mu=1,c_1,c_2,c_3}$ denote the group of symplectomorphisms of $\widetilde{M}_{c_1,c_2,c_3} = \widetilde{M}_{\mu=1,c_1,c_2,c_3}$ that act trivially on homology, and $\mathcal{J}_{c_1,c_2,c_3}$ the space of ω-compatible almost complex structures on $\widetilde{M}_{c_1,c_2,c_3}$.

This work mainly focuses on the case when $\mu = 1$, although some basic results generalize for $\mu \geq 1$. When we point out to results for the latter case in later chapters, we will write $\mu = l + \lambda$ with $l \in \mathbb{N}$ and $0 < \lambda \leq 1$.

In order to describe the algebraic structure of $\pi_*(G_{c_1,c_2,c_3})$, we introduce the Lie product with which we equip it, namely the Samelson product. If G is a connected topological group, the Samelson product $[\cdot,\cdot] : \pi_p(G) \otimes \pi_q(G) \to \pi_{p+q}(G)$ is defined by the commutator map

$$S^{p+q} = S^p \times S^q / S^p \vee S^q \to G \ : \ (s,t) \mapsto a(s)b(t)a^{-1}(s)b^{-1}(t).$$

The Samelson product satisfies the following two properties:

- graded commutativity: $[x,y] = (-1)^{\deg x \deg y + 1}[y,x]$, and
- graded Jacobi identity:

$$(-1)^{\deg x \deg z}[x,[y,z]] + (-1)^{\deg y \deg x}[y,[z,x]] + (-1)^{\deg z \deg y}[z,[x,y]] = 0.$$

The space $\pi_*(G) \otimes \mathbb{Q}$ with the Samelson product is thus a graded Lie algebra (i.e. a graded vector space together with a linear map of degree zero satisfying antisymmetry and the Jacobi identity).

Theorem 1.3.2. *Let $1 > c_1 + c_2 > c_1 + c_3 > c_1 > c_2 > c_3 > 0$. Let $\widetilde{M}_{c_1,c_2,c_3}$ denote the symplectic manifold $(S^2 \times S^2, \sigma \oplus \sigma)$, blown up at three balls of capacities of c_1, c_2 and c_3. Let G_{c_1,c_2,c_3} denote the group of symplectomorphisms of $\widetilde{M}_{c_1,c_2,c_3}$ that act trivially on homology.*

Define $\widetilde{\Lambda}$ as the algebra over $\mathbb{Q}$ generated by nine elements of degree 1 (which we will

define in Section 3.3 and will denote as x_0, x_1, x_2, x_3, x_4, y_0, y_2, y_7, z) subject to a set of relations between their Samelson products, which will be specified in Section 3.4.

Then there is an isomorphism between $\widetilde{\Lambda}$ and the homotopy graded Lie algebra

$$\pi_*(G_{\mu=1,c_1,c_2,c_3}) \otimes \mathbb{Q}.$$

We will see in Section 3.3 that these generators are represented by Hamiltonian S^1-actions on the symplectic manifold, so the rational homotopy type of the symplectomorphism group G_{c_1,c_2,c_3} is generated by these circle actions.

While the general structure of this work is parallel to the study of the homotopy groups of G_{c_1,c_2}, there are a few new phenomena occuring when we blow-up once more. Most importantly, two new relations emerge via the use of auxiliary toric pictures (See Section 3.3.2). These relations appear more naturally when $\mu > 1$.

Chapter 2

The structure of J-holomorphic curves

2.1 Reduced classes

We start by justifying the conditions in Theorem 1.3.2 on the sizes of the capacities, namely the assumption that $1 > c_1 + c_2 > c_1 + c_3 > c_1 > c_2 > c_3 > 0$. We will show that this assumption is sufficient to cover the generic case. The arguments in [5] work equally here to yield:

Lemma 2.1.1. *Every symplectic form on* $\widetilde{M}_{c_1,c_2,c_3}$ *is, after rescaling, diffeomorphic to a form Poincaré dual to* $B + F - c_1E_1 - c_2E_2 - c_3E_3$ *with* $0 < c_3 \leq c_2 \leq c_1 < c_1 + c_3 \leq c_1 + c_2 \leq 1$.

Proof. The key component of this Lemma is the notion of a *reduced class*, because to understand the action of G_{c_1,c_2,c_3} we have the following simplifications:

- Diffeomorphic symplectic forms define symplectomorphism groups that are homeomorphic, and symplectomorphisms are invariant under rescalings of symplectic forms. So we need to describe a fundamental domain for the action of $\mathrm{Diff}^+ \times \mathbb{R}$

on the space Ω_+ of orientation-compatible symplectic forms defined on $\mathbb{X}_4$.

- Taking $\{L, V_1, ..., V_4\}$ as the standard basis of $H_2(\mathbb{X}_4; \mathbb{Z})$ as in Section 1.3, the Poincaré dual of $[\omega_{\nu,\delta_1,\delta_2,\delta_3,\delta_4}]$ is $\nu L - \Sigma_i \delta_i V_i$. Similarly, the first Chern class of any compatible almost-complex structure on $\mathbb{X}_4$ is Poincaré dual to $K := 3L - \Sigma_i V_i$.

  Now if C stands for the Poincaré dual of the symplectic cone of $\mathbb{X}_4$, then by "uniqueness of symplectic blow-ups" in [22], it is enough to describe a fundamental domain on C.

- Moreover, the canonical class K is unique up to orientation preserving diffeomorphisms [19], so it suffices to describe the action of the diffeomorphisms fixing K on

$$C_K = \{A \in H_2(\mathbb{X}_4; \mathbb{R}) \ : \ A = PD[\omega] \text{ for some } \omega \in \Omega_K\}$$

  where Ω_K is the set of orientation-compatible symplectic forms with K as the symplectic canonical class.

- A class $A = a_0 L - \sum_{1 \le i \le 4} a_i V_i$ is *reduced* with respect to the basis $\{L, V_1, ..., V_4\}$ if $a_1 \ge a_2 \ge a_3 \ge a_4$ and $a_0 \ge a_1 + a_2 + a_3$. By a result due to Y. Karshon and L. Kessler ([13], Theorem 1.4. Also see [14] Theorem 1.2.), the set of reduced classes is a fundamental domain of $C_K(\mathbb{X}_4)$ under the action of Diff_K; so the discussion on Section 1.3 gives the Lemma.

$\blacksquare$

Remark 2.1.2. *Note that there are several possibilities within this condition. We don't know the relation between $c_1 + c_2 + c_3$ and 1, and between $c_2 + c_3$ and c_1. We will show below that this does not make a difference in the homotopy type of G_{c_1,c_2,c_3}. (See Remark 3.4.5.)*

2.2 Configurations of J-holomorphic curves

Throughout this section we will assume that $1 > c_1 + c_2 > c_1 + c_3 > c_1 > c_2 > c_3 > 0$ (that is, without allowing for the equalities) and study the possible configurations of J-holomorphic curves. In this work, we use two basic properties of J-holomorphic curves on symplectic manifold of dimension 4, namely the positivity of intersections and the adjunction formula. (Theorem 2.6.3 and Theorem 2.6.4 in [24], or Section 1 in [1].)

Positivity of Intersections: Two distinct closed J-curves S and S' in an almost complex 4-manifold (M, J) have only a finite number of intersection points. Each such point x contributes a number $k_x \geq 1$ to the algebraic intersection number $[S] \cdot [S']$. Moreover, $k_x = 1$ if and only if the curves S and S' intersect transversally at x. Thus, $[S] \cdot [S'] = 0$ if and only if $[S] \cdot [S']$ are disjoint, and $[S] \cdot [S'] = 1$ if and only if the curves meet exactly once transversally and at a point which is non-singular on both curves.

Adjunction Formula: For S the image of a J-holomorphic map $u : \Sigma \to M$, we define the virtual genus of S as the number

$$g(S) = 1 + ([S] \cdot [S] - c_1([S])).$$

Then, if u is somewhere injective, $g(S)$ is an integer. Moreover $g(S) \geq \text{genus}(\Sigma)$ with equality if and only if S is embedded.

Lemma 2.2.1. *Let $J \in \mathcal{J}_{c_1,c_2,c_3}$. Suppose $A = pB + qF - r_1E_1 - r_2E_2 - r_3E_3 \in H_2(\widetilde{M}_{c_1,c_2,c_3}, \mathbb{Z})$ has a simple J-holomorphic representative. Then $p \geq 0$. Moreover*

- *if $p = 0$, then A is one of the followings: F, $F - E_i$, $F - E_i - E_j$, E_i, $E_i - E_j$, $E_1 - E_2 - E_3$, $i, j \in \{1, 2, 3\}$ and $i < j$;*
- *if $p = 1$, then $r_1, r_2, r_3 \in \{0, 1\}$.*

Proof. By adjunction inequality, we have

$$2g_v(A) = 2(p-1)(q-1) - r_1(r_1-1) - r_2(r_2-1) - r_3(r_3-1) \geq 0.$$

Also, since $\omega(B) = \omega(F) = 1 \geq \omega(E_1) = c_1 \geq \omega(E_2) = c_2 \geq \omega(E_3) = c_3$, we have $\omega(A) = p + q - c_1r_1 - c_2r_2 - c_3r_3 > 0$.

Now we show $p \geq 0$: Suppose $p < 0$. Then $p < \frac{1}{2}$ and so

$$\begin{aligned}
-2g_v(A) &> (q-1) + r_1(r_1-1) + r_2(r_2-1) + r_3(r_3-1) \\
&\geq (c_1r_1 + c_2r_2 + c_3r_3 - p - 1) + r_1(r_1-1) + r_2(r_2-1) + r_3(r_3-1) \\
&\geq c_1r_1 + c_2r_2 + c_3r_3 + r_1(r_1-1) + r_2(r_2-1) + r_3(r_3-1) \\
&= r_1(r_1 - 1 + c_1) + r_2(r_2 - 1 + c_2) + r_3(r_3 - 1 + c_3) \geq 0
\end{aligned}$$

which yields $g_v(A) < 0$, contradicting the adjunction inequality.

Now suppose $p = 0$. Then the adjunction formula reads

$$2(q-1) + r_1(r_1-1) + r_2(r_2-1) + r_3(r_3-1) \leq 0,$$

and the positivity of intersections yields $q - c_1r_1 - c_2r_2 - c_3r_3 \geq 0$. The values for q, r_1, r_2 and r_3 that satisfy these inequalities are given in Tables 2.1 and 2.2.

q	r_1	r_2	r_3	A
1	0	0	0	F
1	0	0	1	$F - E_3$
1	0	1	0	$F - E_2$
1	0	1	1	$F - E_2 - E_3$

Table 2.1: The values for q, r_1, r_2 and r_3, and the resulting curves.

q	r_1	r_2	r_3	A
1	1	0	0	$F - E_1$
1	1	0	1	$F - E_1 - E_3$
1	1	1	0	$F - E_1 - E_2$
1	1	1	1	$F - E_1 - E_2 - E_3$
0	-1	0	0	E_1
0	-1	0	1	$E_1 - E_3$
0	-1	1	0	$E_1 - E_2$
0	-1	1	1	$E_1 - E_2 - E_3$
0	0	0	-1	E_3
0	0	-1	0	E_2
0	0	-1	1	$E_2 - E_3$

Table 2.2: The values for q, r_1, r_2 and r_3, and the resulting curves, continued.

Note that the curves $F - E_1 - E_2 - E_3$ and $E_1 - E_2 - E_3$ are represented only if $c_1 + c_2 + c_3 < 1$ and $c_2 + c_3 < c_1$, respectively.

Finally if $p = 1$, we get $r_1(r_1-1)+r_2(r_2-1)+r_3(r_3-1) \leq 0$ so that $r_1, r_2, r_3 \in \{0, 1\}$.

■

Lemma 2.2.2. *Let $J \in \mathcal{J}_{c_1,c_2,c_3}$. Then E_3 is represented by a unique embedded J-curve. Hence, if $A = pB + qF - r_1E_1 - r_2E_2 - r_3E_3$ has a simple J-representative, then $r_3 \geq 0$.*

Proof. Suppose, by contradiction, that E_3 is represented by a cusp-curve

$$C = \bigcup_{i=1}^{N} m_i C_i,\ N \geq 2 \text{ and } C_i \text{ simple.}$$

Each C_i is one of the options given in Lemma 2.2.1. By area considerations ($\omega(E_3) > \omega(C_i)$ for all i), C_i cannot be representing $F, F - E_1, F - E_2, E_1, E_1 - E_2, F - E_1 - E_2$

or E_3. But then the options we are left with all include $-E_3$, which is also impossible as $m_i > 0$ and $N \geq 2$. ■

Remark 2.2.3. *The fact that the class E_3 is always represented by a unique embedded J-curve will be extremely useful hereinafter. It allows us to analyze almost-complex structures as well as the symplectomorphisms on the manifold $\widetilde{M}_{c_1,c_2,c_3}$ via the ones on $\widetilde{M}_{c_1,c_2}$, for we can blow-down E_3 to obtain the structures for the 2-point blow-up.*

Remark 2.2.4. *If we start with E_2 (instead of E_3) in Lemma 2.2.2, a similar argument shows that we must have $C_i = E_2 - E_3$ unless we have a simple representative of E_2.*

Lemma 2.2.5. *The set of tamed almost complex structures on $\widetilde{M}_{c_1,c_2,c_3}$ for which the classes $B - E_i$, $i = 1, 2, 3$ are represented by an embedded J-holomorphic sphere is open and dense in $\mathcal{J}_{c_1,c_2,c_3}$.*

If for a given J there are no such spheres, then either one of the classes $B - E_i - E_j$ $i < j$ or $B - E_1 - E_2 - E_3$ is represented by a unique embedded J-holomorphic sphere.

Proof. The proof of this Lemma is similar to the proof of Lemma 2.5 in [28] and of Lemma 2.10 in [5]. We only need to modify their proofs to include the various curves available in our case. We will confine ourselves with giving a sketch of the proof.

Since the curves $B - E_i$, $i = 1, 2, 3$, are exceptional, the set of almost complex structures J for which they have an embedded J-holomorphic representative is open and dense. Since they are primitive, they cannot have multiply-covered representatives. Suppose, then, that the representative of $B - E_i$ is a cusp curve, which due to Lemma 2.2.1 and Lemma 2.2.2 would decompose as $B - r_1E_1 - r_2E_2 - r_3E_3$ and a linear combination of the following classes: F, $F - E_j$, $F - E_j - E_k$, $F - E_1 - E_2 - E_3$, E_j, $E_j - E_k$, $E_1 - E_2 - E_3$. Here all the coefficients in the linear combination would be non-negative and $r_i \in \{0, 1\}$. This implies that at least two of the r_i should be non-zero, hence one of the classes $B - E_i - E_j$ $i < j$ or $B - E_1 - E_2 - E_3$ is represented by a unique embedded J-holomorphic sphere. ■

Remark 2.2.6. *By positivity of intersections, we cannot have $F - E_i$ and $F - E_i - E_j$ simultaneously represented by embedded J-holomorphic spheres. Similarly, it is not possible for the classes $F - E_i$ and $F - E_1 - E_2 - E_3$ to simultaneously have embedded J-holomorphic representatives.*

Using the positivity of intersections and the results of this section, we thus get a set of possible configurations of J-holomorphic curves. Note that all the configurations can be obtained from the configurations in Figure 2 of [5], by blowing up either inside a curve, or at an intersection between curves, or outside. (See Remark 2.2.3.) We will recall the configurations in [5] and draw the configurations resulting from the next blow-up.

For instance, corresponding to the configuration (1) in Figure 2 of [5]

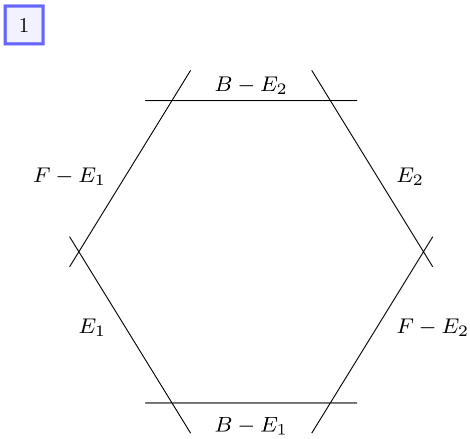


Figure 2.1: Configuration (1) of [5] for $\mathbb{X}_3$

we obtain the following 13 possible configurations of J-holomorphic spheres.

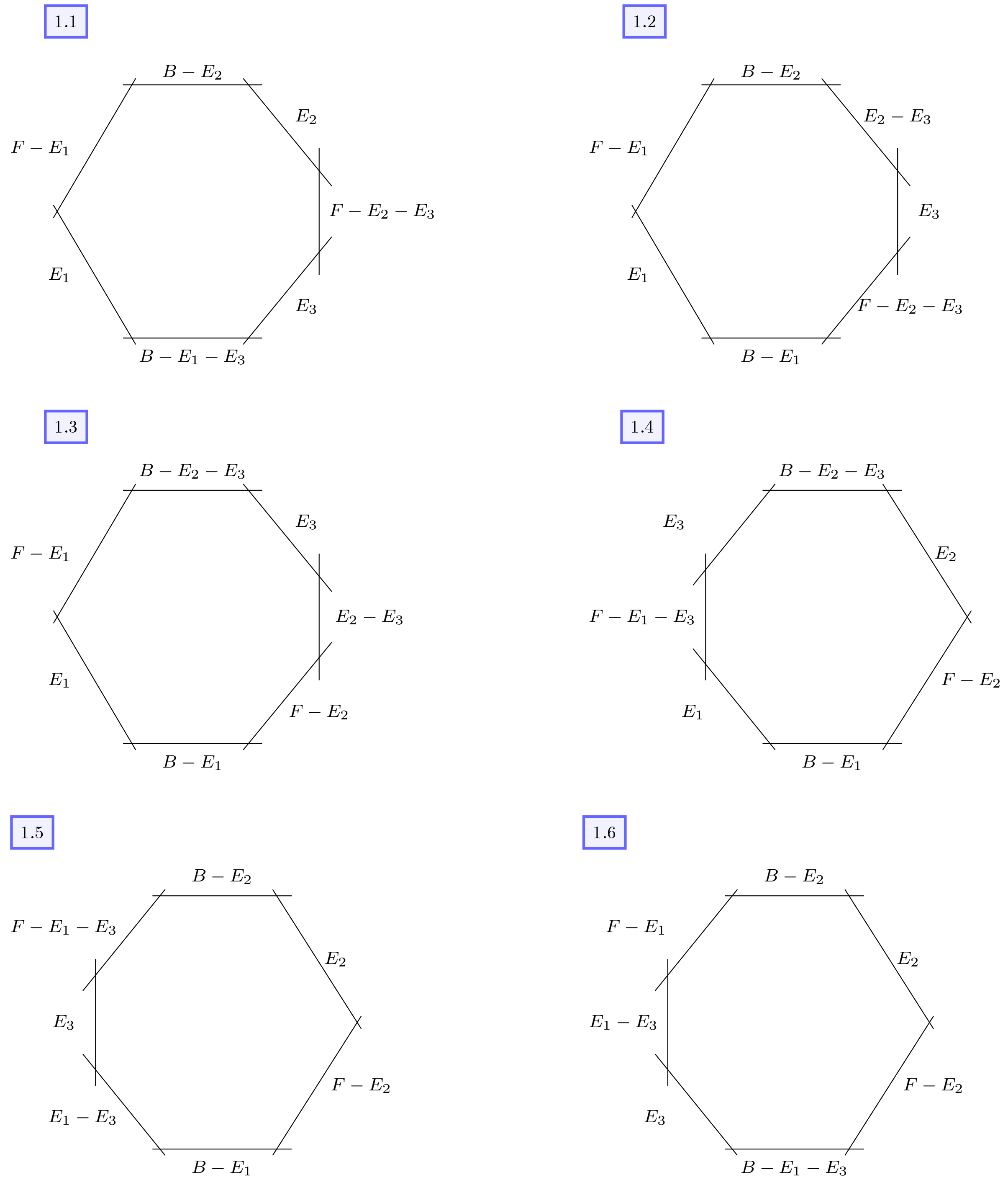


Figure 2.2: Configurations 1.1-6

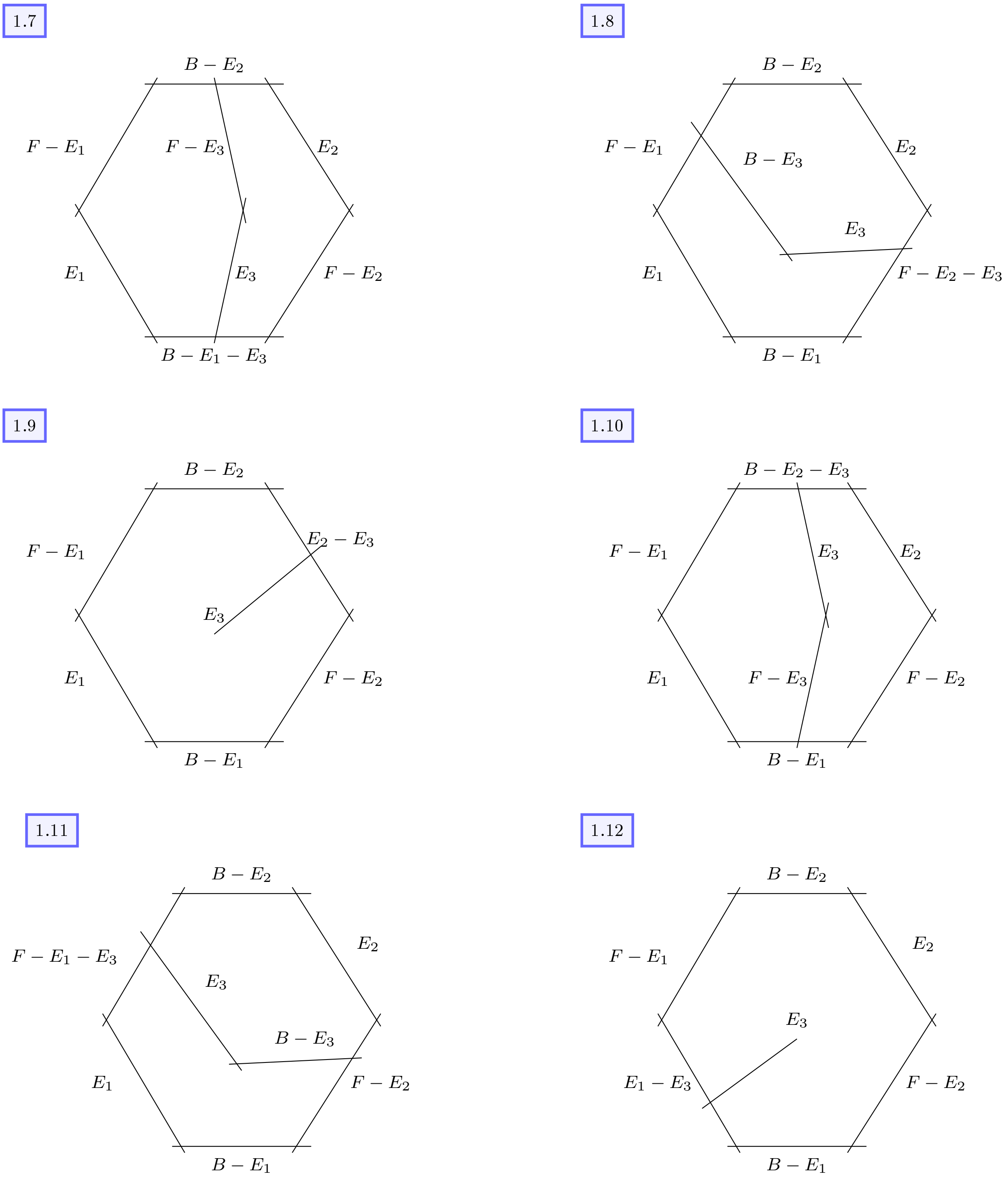


Figure 2.3: Configurations 1.7-12

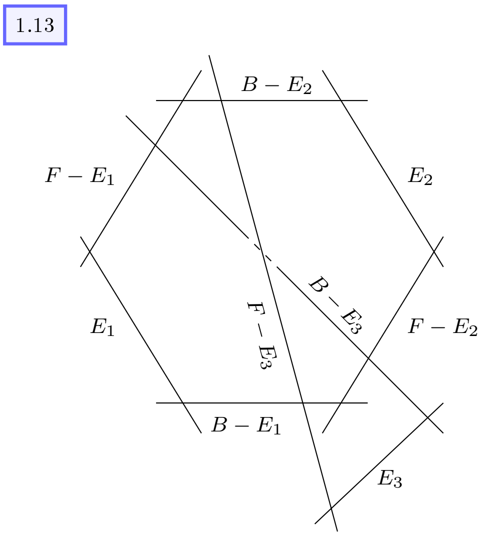


Figure 2.4: Configuration 1.13

Similarly, corresponding to the configuration (3) in Figure 2 of [5] [1]

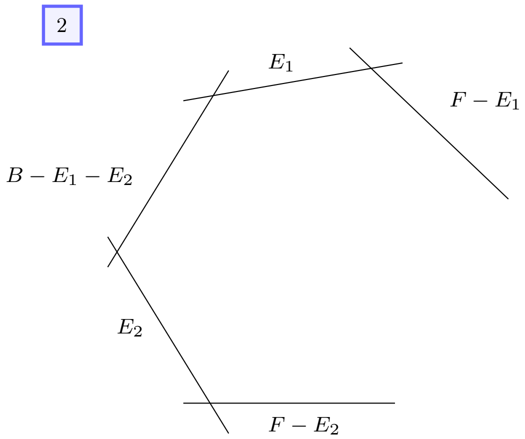


Figure 2.5: Configuration (3) of [5] for $\mathbb{X}_3$

we obtain the following possible configurations of J-holomorphic spheres.

[1] In order to simplify the presentation in further chapters, we will change the labeling of the configurations.

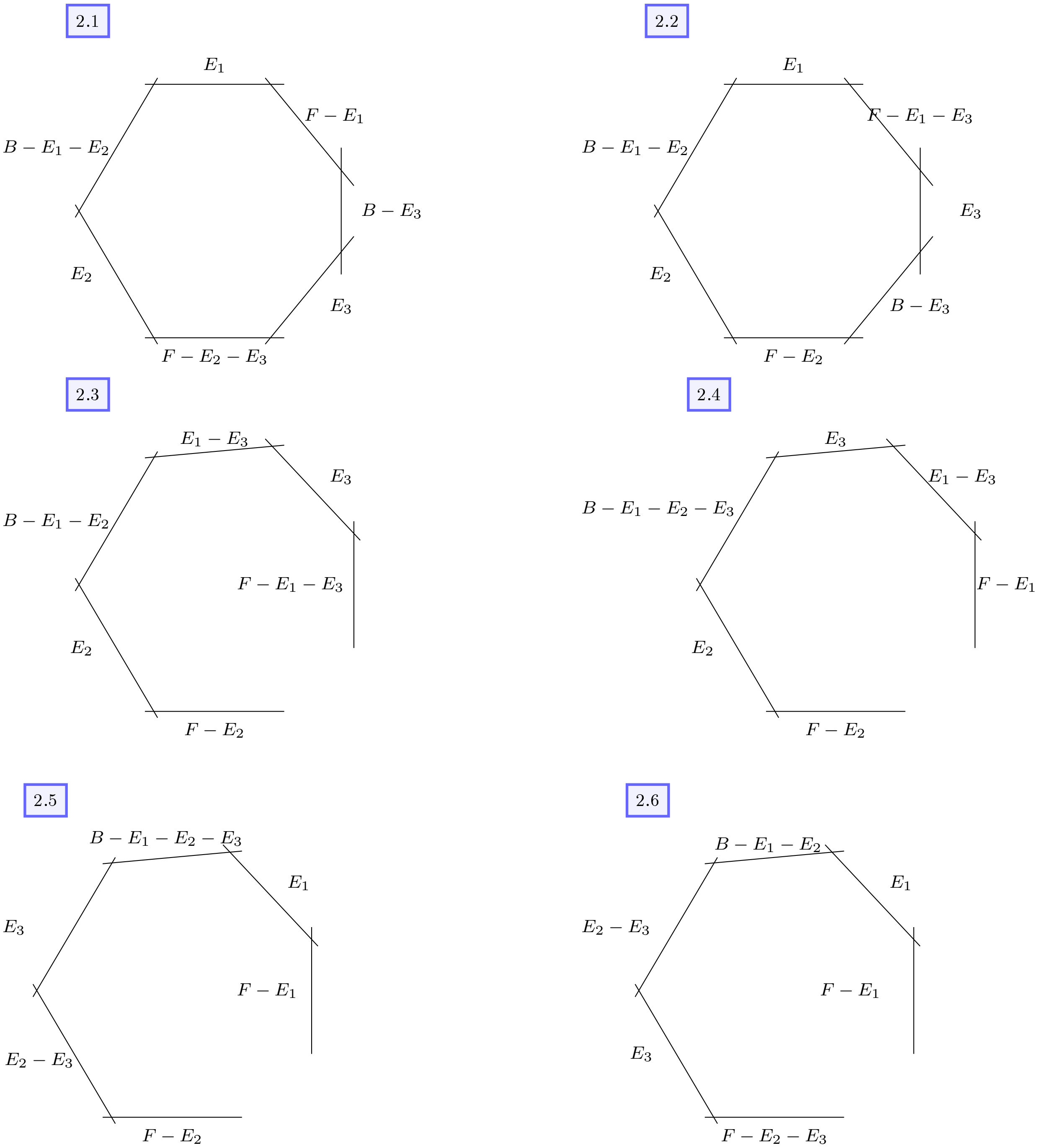


Figure 2.6: Configurations 2.1-6

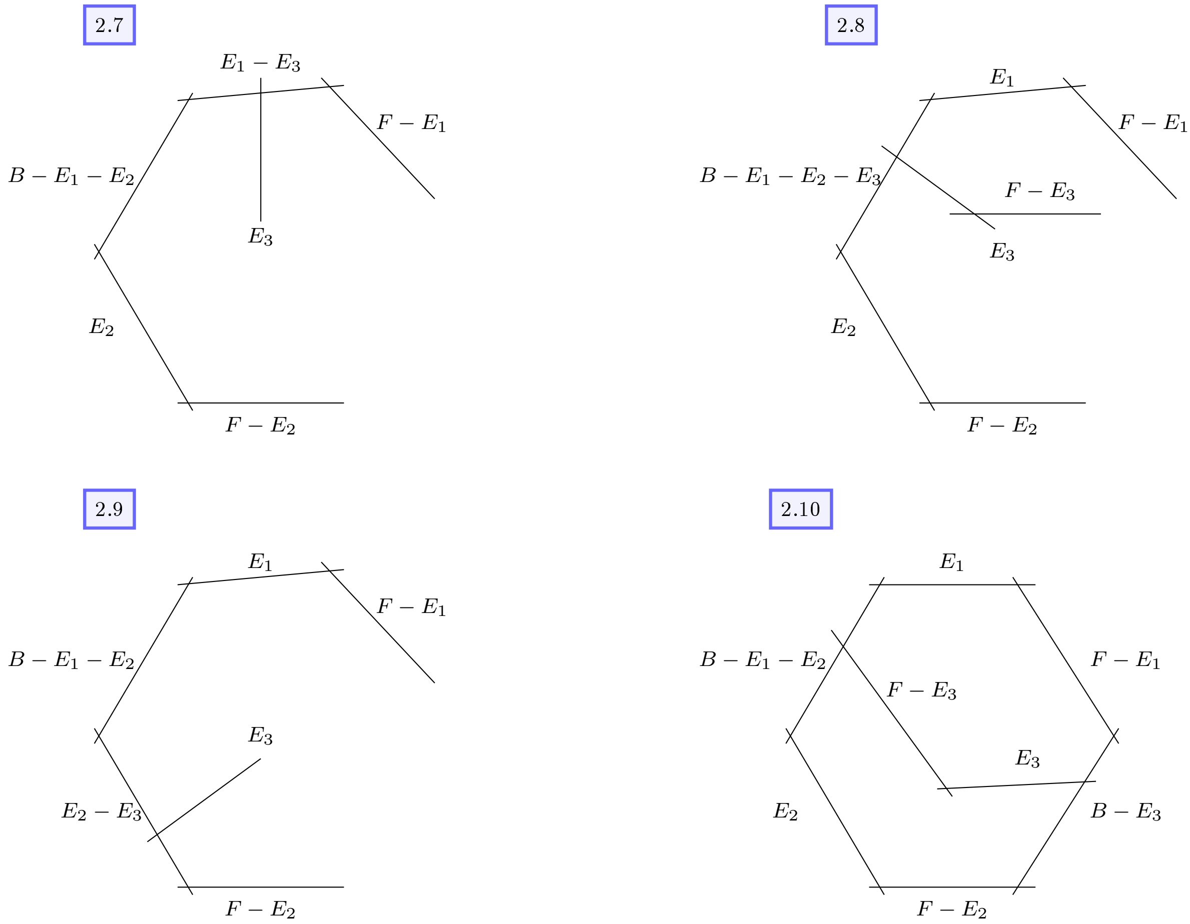


Figure 2.7: Configurations 2.7-10

Corresponding to the configuration (5) in Figure 2 of [5]

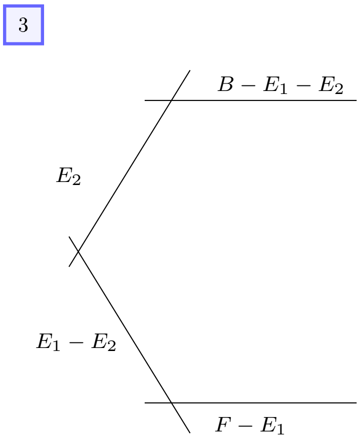


Figure 2.8: Configuration (5) of [5] for $\mathbb{X}_3$

we obtain the following configurations:

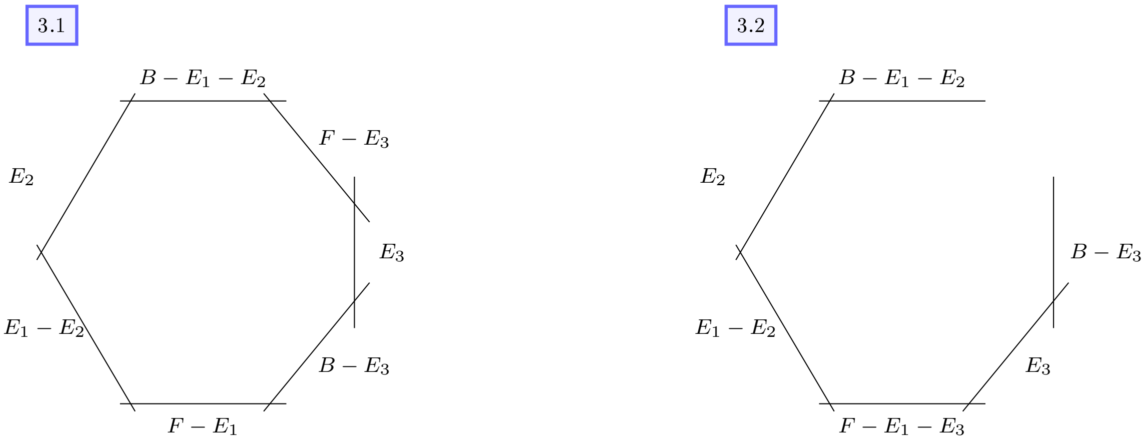


Figure 2.9: Configurations 3.1-2

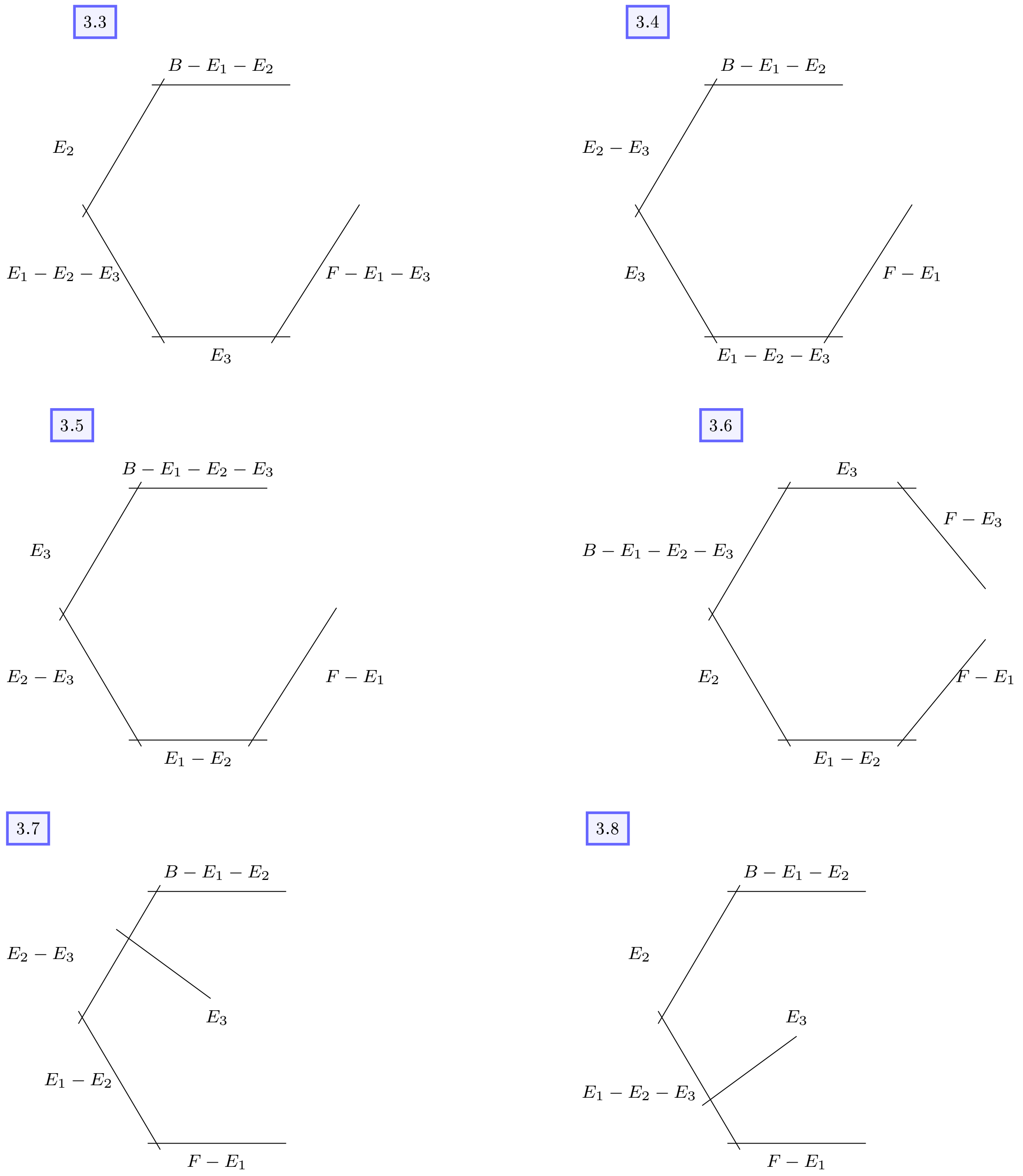


Figure 2.10: Configurations 3.3-8

Corresponding to the configuration (4) in Figure 2 of [5]

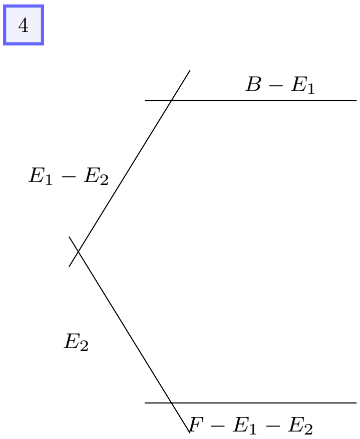


Figure 2.11: Configuration (4) of [5] for $\mathbb{X}_3$

we get 4.1-4.8 similar to the previous case. The configurations can be obtained simply from 3.1-3.8 by interchanging the roles F and B. Also, from the configuration (2) in Figure 2 of [5]

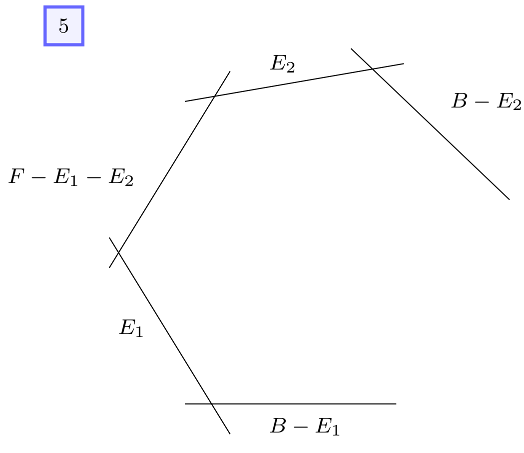


Figure 2.12: Configuration (2) of [5] for $\mathbb{X}_3$

we obtain 5.1-5.10 and the corresponding configurations can be obtained by inter-

changing the roles of F and B in 2.1-2.10 respectively. And finally, using the configuration (6)

6

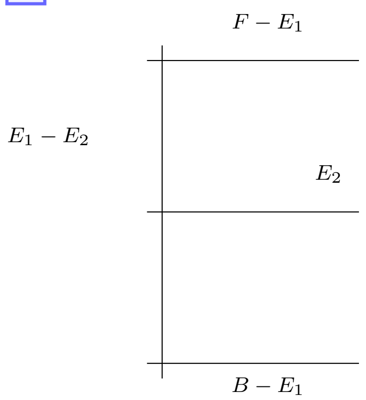


Figure 2.13: Configuration (6) of [5] for $\mathbb{X}_3$

we obtain the following possible configurations:

6.1 6.2

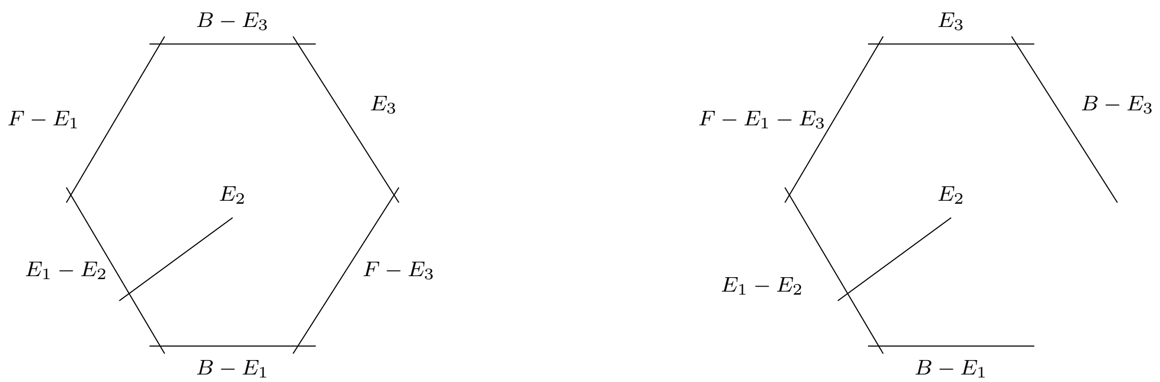


Figure 2.14: Configurations 6.1-2

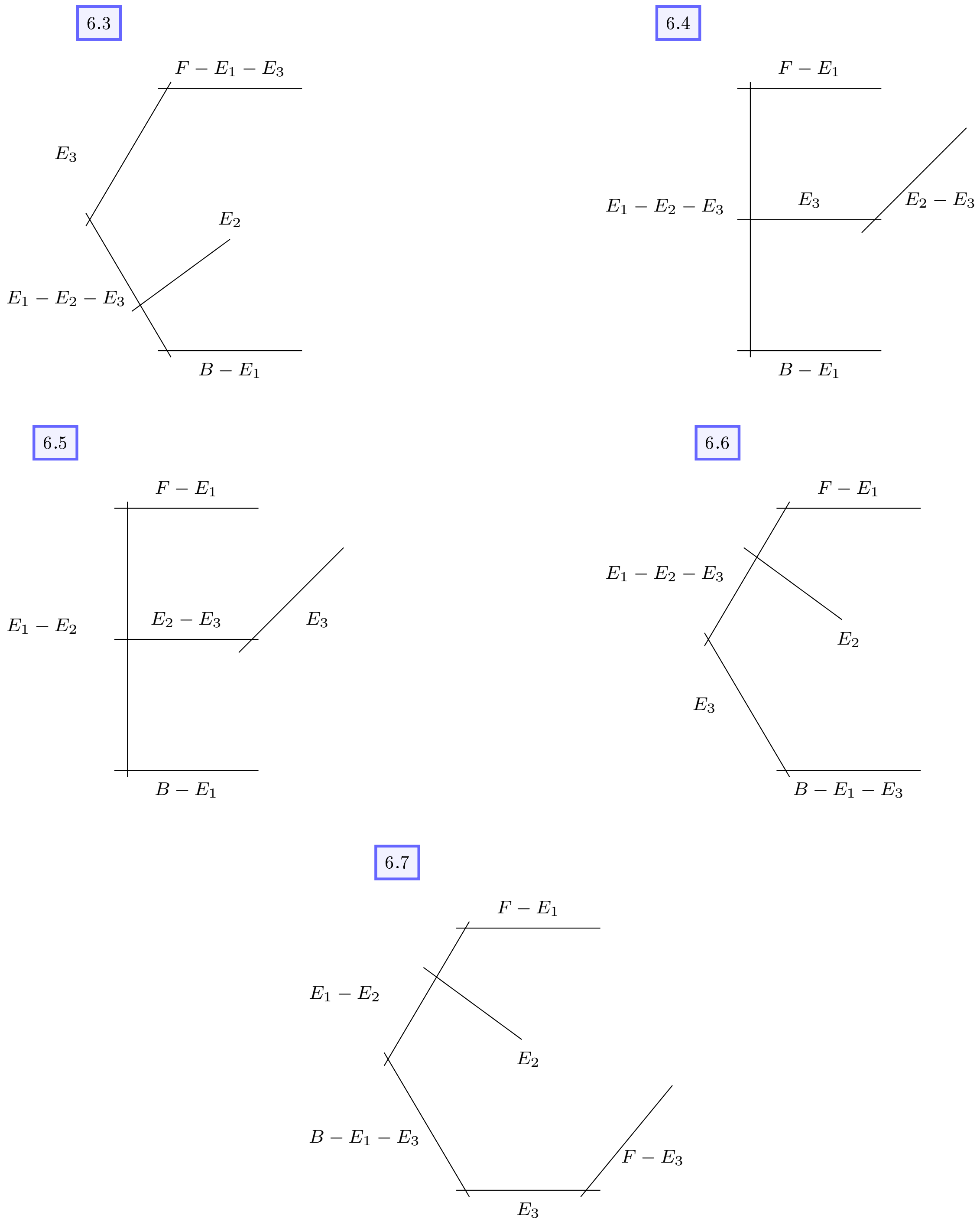


Figure 2.15: Configurations 6.3-7

Chapter 3

Homotopy type of symplectomorphism groups

We start by citing a result by J.Li, T.Li and W.Wu:

Proposition 3.0.1. *([18], Theorem 1.1) The space* G_{μ,c_1,c_2,c_3} *is connected for any* $\mu \geq 1$.

Proof. (Outline) We will give the main ingredients of the proof. We start by establishing the basic concepts, definitions and notation.

A *stable (spherical symplectic) configuration* in a symplectic manifold (M, ω) is an ordered configuration of symplectic spheres $C = \bigcup_{i=1}^{N} C_i$ such that

- $[C_i] \cdot [C_i] \geq 1$ for all i,
- for $i \neq j$, $[C_i] \neq [C_j]$ and $[C_i] \cdot [C_j]$ is zero or one.
- there is a compatible almost complex structure J such that all the C_i's are J-holomorphic.

A stable configuration is *standard* if the C_i's intersect ω-orthogonally at every intersection point of the configuration. We denote by $\mathcal{C}_0$ the space of all standard stable configurations. A standard stable configuration $C \in \mathcal{C}_0$ is *full* if $H^2(M, C; \mathbb{R}) = 0$.

Note that the configuration 1.13 is full standard stable. From now on, we will restrict our attention to this choice of C as our working configuration for the case $\mu = 1$.

The first step in the proof is to show that the group $\mathrm{Symp}_c(U)$ of compactly supported symplectomorphisms of $(U = M \setminus C, \omega|_U)$ is contractible. By Moser's theorem, $\mathrm{Symp}_c(U)$ is homotopy equivalent to the space $\mathrm{Stab}^1(C)$ of symplectomorphisms on M that fix C pointwise and that act trivially on the normal bundles of the C_i's.

Next, we consider the fibration $\mathrm{Stab}^1(C) \to \mathrm{Stab}^0(C) \to \mathcal{G}(C)$, where $\mathrm{Stab}^0(C)$ is the set of symplectomorphisms on M that fix C pointwise, and $\mathcal{G}(C)$ is defined as follows:

Let $\mathcal{G}_{k_i}(C_i)$ denote the group of gauge transformations of the normal bundle to C_i which equal the identity at the k_i intersection points. In our case, each of these are homotopy equivalent to $\mathcal{G}_{k_i}(S^2)$. Then, for $C = \bigcup\limits_{i=1}^{N} C_i$, we define $\mathcal{G}(C) = \prod_{i=1}^{N} \mathcal{G}_{k_i}(C_i)$. So, by the first step, we obtain $\mathrm{Stab}^0(C) \simeq \mathcal{G}(C)$.

The third step is to consider the set $\mathrm{Stab}(C)$ of symplectomorphisms on M that fix C as a set. For our choice of C, it can be shown that $\mathrm{Stab}^0(C) \to \mathrm{Stab}(C) \to \mathrm{Symp}(C)$ is a homotopy fibration. Considering the long exact sequence

$$... \to \pi_1(\mathrm{Stab}(C)) \to \pi_1(\mathrm{Symp}(C)) \xrightarrow{\psi} \pi_0(\mathrm{Stab}^0(C)) \to \pi_0(\mathrm{Stab}(C)) \to \pi_0(\mathrm{Symp}(C)),$$

it is possible to show that ψ is surjective and $\mathrm{Stab}(C)$ is connected.

Finally, we consider $\mathrm{Stab}(C) \to \mathrm{Symp}_h(M, \omega) \to \mathcal{C}_0$. Here, the action of $\mathrm{Symp}_h(M, \omega)$ on $\mathcal{C}_0$ is transitive and $\mathcal{C}_0$ is connected. Since by the third step $\mathrm{Stab}(C)$ is connected, the homotopy fibration yields that $\mathrm{Symp}_h(M, \omega)$ is also connected. This finishes the proof the proposition.

To summarize, the proof relies on the analysis of the following diagram, for a clever choice of C:

$$\begin{tikzcd}
\mathrm{Symp}_c(U) \arrow[r] & \mathrm{Stab}^1(C) \arrow[r] & \mathrm{Stab}^0(C) \arrow[r] \arrow[d] & \mathrm{Stab}(C) \arrow[r] \arrow[d] & \mathrm{Symp}_h(M,\omega) \arrow[d] \\
 & & \mathcal{G}(C) & \mathrm{Symp}(C) & \mathcal{C}_0.
\end{tikzcd}$$

In fact, this diagram gives a uniform approach to show the connectedness of the space $\mathrm{Symp}_h(M, \omega)$, for M a n-fold blow-up of $\mathbb{CP}^2$ for $n \leq 4$. For each case, the authors give the configuration that can be used to run the proof. ([18], Remark 3.4)

■

3.1 Stability of symplectomorphism groups

Theorem 3.1.1. *Consider $c_1, c_2, c_3, c_1', c_2', c_3'$ such that*

$$0 < c_3 < c_2 < c_1 < c_1 + c_3 < c_1 + c_2 < 1,$$

$$0 < c_3' < c_2' < c_1' < c_1' + c_3' < c_1' + c_2' < 1,$$

Then the symplectomorphism groups G_{c_1,c_2,c_3} and $G_{c_1',c_2',c_3'}$ are homotopy equivalent.

Proof. The proof of this theorem follows the same line of thought as the proof of Proposition 3.1 in [28] and uses the inflation technique introduced by Lalonde [15] and McDuff [23].

We start by considering the natural action of the identity component of the diffeomorphism group of $\widetilde{M}_{c_1,c_2,c_3}$ on the space $\mathcal{S}_{c_1,c_2,c_3}$ of all symplectic forms on $\widetilde{M}_{c_1,c_2,c_3}$ in the cohomology class $[\omega_{c_1,c_2,c_3}]$ that are isotopic to ω_{c_1,c_2,c_3}. By Moser's theorem, this action is transitive, and we obtain the following fibration:

$$G_{c_1,c_2,c_3} = \mathrm{Symp}(\widetilde{M}_{c_1,c_2,c_3}) \cap \mathrm{Diff}_0(\widetilde{M}_{c_1,c_2,c_3}) \longrightarrow \mathrm{Diff}_0(\widetilde{M}_{c_1,c_2,c_3}) \longrightarrow \mathcal{S}_{c_1,c_2,c_3}.$$

Doing the same with c_1', c_2', c_3', we see that to show the required homotopy equivalence,

it is sufficient to find a homotopy equivalence between the spaces $\mathcal{S}_{c_1,c_2,c_3}$ and $\mathcal{S}_{c_1',c_2',c_3'}$ that makes the below diagram commutative up to homotopy.

$$\begin{tikzcd}
G_{c_1,c_2,c_3} \arrow[r] & \mathrm{Diff}_0 \arrow[r] \arrow[d, equal] & \mathcal{S}_{c_1,c_2,c_3} \arrow[d, leftrightarrow] \\
G_{c_1',c_2',c_3'} \arrow[r] & \mathrm{Diff}_0 \arrow[r] & \mathcal{S}_{c_1',c_2',c_3'}
\end{tikzcd}$$

To this end, we follow an idea introduced by McDuff [23] and consider the larger space $\mathcal{X}_{c_1,c_2,c_3}$ defined as

$$\mathcal{X}_{c_1,c_2,c_3} = \{(\omega, J) \in \mathcal{S}_{c_1,c_2,c_3} \times \mathcal{A}_{c_1,c_2,c_3} \ : \ \omega \text{ tames } J\},$$

where $\mathcal{A}_{c_1,c_2,c_3}$ is the space of almost complex structures that are tamed by some form in $\mathcal{S}_{c_1,c_2,c_3}$. Then both projection maps $\mathcal{X}_{c_1,c_2,c_3} \to \mathcal{A}_{c_1,c_2,c_3}$ and $\mathcal{X}_{c_1,c_2,c_3} \to \mathcal{S}_{c_1,c_2,c_3}$ are fibrations with contractible fibers, and therefore they are homotopy equivalences. Below, we will prove that $\mathcal{A}_{c_1,c_2,c_3}$ and $\mathcal{A}_{c_1',c_2',c_3'}$ are in fact equal.

We start by recalling the following result due to Lalonde and McDuff.

Lemma 3.1.2 (Inflation Lemma, [23]). *Let J be an τ_0-tame almost complex structure on a symplectic 4-manifold (M, τ_0) that admits a J-holomorphic curve Z with non-negative self-intersection. Then there is a family τ_t, $t \geq 0$, of symplectic forms that all tame J and have cohomology class $[\tau_t] = [\tau_0] + tPD(Z)$, where $PD(Z)$ is the Poincaré dual to the homology class Z.*

This result was generalized by Buse to curves with negative self-intersection.

Lemma 3.1.3 (Buse, [6]). *Let J be an τ_0-tame almost complex structure on a symplectic 4-manifold (M, τ_0) that admits a J-holomorphic curve Z with $Z \cdot Z = -m$, $m \in \mathbb{N}$. Then for all $\epsilon > 0$ there is a family τ_t of symplectic forms that all tame J and have cohomology class $[\tau_t] = [\tau_0] + tPD(Z)$ for all $0 \leq t \leq \frac{\tau_0(Z)}{m} - \epsilon$, where $PD(Z)$ is the Poincaré dual to the homology class Z.*

We prove $\mathcal{A}_{c_1,c_2,c_3} = \mathcal{A}_{c'_1,c'_2,c'_3}$ in three steps.

Step 1: $\mathcal{A}_{c'_1,c_2,c_3} = \mathcal{A}_{c_1,c_2,c_3}$ for $c_1 \leq c'_1$.

Step 2: $\mathcal{A}_{c_1,c'_2,c_3} = \mathcal{A}_{c_1,c_2,c_3}$ for $c_2 \leq c'_2$.

Step 3: $\mathcal{A}_{c_1,c_2,c'_3} = \mathcal{A}_{c_1,c_2,c_3}$ for $c_3 \leq c'_3$.

Here, we will only prove $\mathcal{A}_{c'_1,c_2,c_3} \subset \mathcal{A}_{c_1,c_2,c_3}$ for $c_1 \leq c'_1$ in full detail. The remaining steps and inclusions are similar and are left to Appendix A.

Take $J \in \mathcal{A}_{c'_1,c_2,c_3}$ such that the embedded J-holomorphic spheres satisfy one of the following configurations from Chapter 2: #1: 1, 2, 3, 4, 7, 8, 9, 10, 11, 13; #2:1, 2, 5, 6, 8, 9, 10. Note that the curve E_1 is represented by a J-holomorphic curve in all these configurations. By definition, there is a symplectic form $\omega_{c'_1,c_2,c_3}$ taming J and satisfying $\omega_{c'_1,c_2,c_3}(B) = \omega_{c'_1,c_2,c_3}(F) = 1$, $\omega_{c'_1,c_2,c_3}(E_1) = c'_1$ and $\omega_{c'_1,c_2,c_3}(E_i) = c_i$ for $i = 2, 3$. To show $J \in \mathcal{A}_{c_1,c_2,c_3}$ we need to find a symplectic form ω such that $\omega(B) = \omega(F) = 1$ and $\omega(E_i) = c_i$ for $i = 1, 2, 3$. We can use negative inflation along the curve E_1 to define one-parameter family of symplectic forms all taming J, $\omega_t = \omega_{c'_1,c_2,c_3} + t\mathrm{PD}(E_1)$ for $0 \leq t \leq c'_1 - \epsilon$, where ϵ can be chosen small enough so that we have $t = t_0 = c'_1 - c_1$. For this value of t, we obtain $\omega = \omega_{c'_1,c_2,c_3} + (c'_1 - c_1)\mathrm{PD}(E_1)$, which satisfies $\omega(B) = \omega(F) = 1$ and $\omega(E_i) = c_i$ for $i = 1, 2, 3$, as desired.

Next, we consider $J \in \mathcal{A}_{c'_1,c_2,c_3}$ with the following configurations: #1: 5, 6, 12; #2: 3, 4, 7. In this case, we first inflate along $B + F - E_2$ and $B + F$, and then apply negative inflation to the resulting symplectic form along $E_1 - E_3$:

$$\omega_{b,e} = \frac{\omega_{c'_1,c_2,c_3} + b\mathrm{PD}(B + F - E_2) + e\mathrm{PD}(B + F)}{1 + b + e}, \ b, e \geq 0,$$

$$\text{and } \omega_a = \omega_{b,e} + a\mathrm{PD}(E_1 - E_3), \text{ with } a < \frac{\omega_{b,e}(E_1 - E_3)}{2}.$$

We get the desired symplectic form by setting $e = \dfrac{(1 - c_2)(c'_1 - c_1)}{(c_1 + c_3)}$, $b = \dfrac{c_2 e}{1 - c_2}$ and $a = \dfrac{c_3(b + e)}{1 + b + e}$.

These choices cover all the possible configurations of type #1 and #2. Note that,

as explained at the end of Chapter 2, the configurations #5 can be obtained by interchanging the roles of B and F in #2. Hence, we also covered the configurations of type #5.

For the configurations #3: 1, 2 and #6: 1, 2; we utilize the curves $B+F-E_3$, $B+F$ and then E_1-E_2. This case is similar to the last one, and we only need to interchange the roles of E_2 and E_3 (and therefore of c_2 and c_3).

For the configurations #3: 3, 4, 8 and #6: 3, 4, 6; we inflate along the curves $2B+2F-E_1-E_2$, $B+F$, and $E_1-E_2-E_3$. In this case, we set

$$\omega_{b,e} = \frac{\omega_{c_1',c_2,c_3} + b\mathrm{PD}(2B+2F-E_1-E_2) + e\mathrm{PD}(B+F)}{1+2b+e},\ b,e \geq 0,$$

$$\text{and } \omega_a = \omega_{b,e} + a\mathrm{PD}(E_1-E_2-E_3), \text{ with } a < \frac{\omega_{b,e}(E_1-E_2-E_3)}{3}.$$

We get the desired symplectic form by setting

- $e = \dfrac{(1-2c_2+2c_3)(c_1'-c_1)}{(c_1-c_2+2c_3)}$,
- $b = \dfrac{(c_2-c_3)e}{1-2c_2-2c_3}$, and
- $a = \dfrac{c_3(2b+e)}{1+2b+e}$.

Note that here, to prove $a < \dfrac{\omega_{b,e}(E_1-E_2-E_3)}{3}$, we use the fact that $c_1 > c_2+c_3$ as otherwise $E_1-E_2-E_3$ could not have a J-holomorphic representative in the first place.

Then, for the configurations #3: 5, 6, 7 and #6: 5, 7; we inflate along the curves $3B+3F-E_1-E_2-E_3$, $B+F$, and E_1-E_2. This case is similar to the previous case, and we use the fact that $c_1+c_2+c_3 < 1$, which makes it possible to have $B-E_1-E_2-E_3$ in the configuration.

Since the configurations of type #4 can be obtained from #3 by interchanging the roles of B and F, this finishes the proof of $\mathcal{A}_{1,c_1',c_2,c_3} \subset \mathcal{A}_{c_1,c_2,c_3}$ for $c_1 \leq c_1'$. The inverse inclusion, as well as Step 2 and 3, are similar. In Appendix A, we give a list

of curves along which the inflation procedure can be used to produce the symplectic forms required in each case. ∎

We can get information on G_{c_1,c_2,c_3} via the group $\mathrm{Symp}_p(\widetilde{M}_{c_1,c_2})$ of symplectomorphisms that fix a point p in the manifold $\widetilde{M}_{c_1,c_2}$, which is the manifold studied by Anjos and Pinsonnault (Proposition 1.2 in [5]):

Theorem 3.1.4. *Consider $c_1, c_2, c_3 \in (0,1)$ such that $c_3 < c_2 \leq c_1 < c_1 + c_3 < c_1 + c_2 \leq 1$. Then G_{c_1,c_2,c_3} is homotopy equivalent to $Symp_p(\widetilde{M}_{c_1,c_2})$.*

Proof. Let $\mathrm{Symp}^{U(2)}(\widetilde{M}_{c_1,c_2,c_3}, E_3)$ denote the group of symplectomorphisms of the manifold $\widetilde{M}_{c_1,c_2,c_3}$ that act linearly in a small neighborhood of the exceptional fiber whose cohomology class is E_3. The first step in the proof is that $\mathrm{Symp}^{U(2)}(\widetilde{M}_{c_1,c_2,c_3}, E_3)$ is homotopy equivalent to $\mathrm{Symp}_p(\widetilde{M}_{c_1,c_2})$: Every element of $\mathrm{Symp}^{U(2)}(\widetilde{M}_{c_1,c_2,c_3}, E_3)$ gives rise to a symplectomorphism of $\widetilde{M}_{c_1,c_2}$ acting linearly near the embedded ball B_{c_3} and fixing its center p. Conversely, every homotopy class of the stabilizer $\mathrm{Symp}_p(\widetilde{M}_{c_1,c_2})$ can be realized by a family of symplectomorphisms that act linearly on a ball $B_{c_3'}$ of sufficiently small capacity c_3' centered at p. So we can lift such a representative to the group $\mathrm{Symp}^{U(2)}(\widetilde{M}_{c_1,c_2,c_3}, E_3)$. If r is the restriction map, then the directed system of homotopy maps

$$r_{c_3',c_3} \;:\; \mathrm{Symp}^{U(2)}(\widetilde{M}_{c_1,c_2,c_3'}, E_3) \to \mathrm{Symp}^{U(2)}(\widetilde{M}_{c_1,c_2,c_3}, E_3)$$

gives homotopy equivalences when $c_3 \leq c_3'$ are sufficiently small, by Theorem 3.1.1. This system together with maps

$$g_{c_3} \;:\; \mathrm{Symp}^{U(2)}(\widetilde{M}_{c_1,c_2,c_3}, E_3) \to \mathrm{Symp}_p(\widetilde{M}_{c_1,c_2})$$

yield a commutative diagram for each pair $c_3 \leq c_3'$. It is then obvious that each g_{c_3} is a weak homotopy equivalence.

The second step in the proof is that $\mathrm{Symp}^{U(2)}(\widetilde{M}_{c_1,c_2,c_3'}, E_3)$ is homotopy equivalent to $\mathrm{Symp}(\widetilde{M}_{c_1,c_2,c_3'}, E_3)$. This is Lemma 2.3 in [17].

Finally, by Lemma 2.2.2, we know that that the curve E_3 cannot degenerate, that is, for every J in $\mathcal{J}_{c_1,c_2,c_3}$ there is a unique embedded J-holomorphic representative of E_3 and no cusp-curve in class E_3. Hence by Lemma 2.4 in [17] we conclude that G_{c_1,c_2,c_3} retracts onto its subgroup $\mathrm{Symp}(\widetilde{M}_{c_1,c_2,c_3'}, E_3)$. ■

3.2 Homotopy groups of G_{c_1,c_2,c_3}

We use the same tools (Poincaré series, Poincaré-Birkoff-Witt Theorem, spectral sequence) as in Section 4.1 of [5] to calculate

$$\widetilde{r}_n = \dim \pi_n(\Omega\widetilde{M}_{c_1,c_2}) \otimes \mathbb{Q} = \dim \pi_{n+1}(\widetilde{M}_{c_1,c_2}) \otimes \mathbb{Q}.$$

It is known ([9]) that a simply-connected space with rational cohomology of finite type and finite category is either *rationally elliptic* (i.e. its rational homotopy is finite dimensional) or *rationally hyperbolic* (i.e. the dimensions of the rational homotopy groups grow exponentially). Also, since $b^2(\widetilde{M}_{c_1,c_2}) = \dim H^2(\widetilde{M}_{c_1,c_2};\mathbb{Q}) > 3$ and since a 4-dimensional simply-connected finite CW-complex has $cat(X) \leq 2$, the following theorem shows that $\widetilde{M}_{c_1,c_2}$ is rationally hyperbolic.

Theorem 3.2.1 (Part VI in [9]). *If an n-dimensional simply-connected space X is rationally elliptic, then*

1. $\dim \pi_{even}(X) \otimes \mathbb{Q} \leq \dim \pi_{odd}(X) \otimes \mathbb{Q} \leq cat(X)$,
2. $\chi(X) \geq 0$, *where χ denotes the Euler characteristic.*

For a rationally hyperbolic space X, the Poincaré series is the formal series

$$P_X = \textstyle\sum_0^\infty \dim H_n(X;\mathbb{Q}) z^n.$$

The Poincaré-Birkoff-Witt Theorem applied to topolog,cal spaces (see [9] Section 33) yields

$$P_{\Omega X} = \frac{\prod_{n=0}^\infty (1+z^{2n+1})^{\widetilde{r}_{2n+1}}}{\prod_{n=1}^\infty (1-z^{2n})^{\widetilde{r}_{2n}}}$$

so that we can recover the $\widetilde{r}_n$ from $\widetilde{h}_n = \dim H_n(\Omega\widetilde{M}_{c_1,c_2};\mathbb{Q})$. We compute the latter

using the Serre spectral sequence of the path fibration

$$\Omega\widetilde{M}_{c_1,c_2} \to P\widetilde{M}_{c_1,c_2} \to \widetilde{M}_{c_1,c_2},$$

where $P\widetilde{M}_{c_1,c_2}$ is the space of paths starting at a point x_0 and the projection map $p : P\widetilde{M}_{c_1,c_2} \to \widetilde{M}_{c_1,c_2}$ sends each path to its endpoint. The spectral sequence yields $\widetilde{h}_0 = 1$, $\widetilde{h}_1 = 4$ and $\widetilde{h}_n = 4\widetilde{h}_{n-1} - \widetilde{h}_{n-2}$, which in turn gives, for instance, $\widetilde{r}_1 = 4$, $\widetilde{r}_2 = 9$, $\widetilde{r}_3 = 16$ and $\widetilde{r}_4 = 27$.

Proposition 3.2.2. *Let $0 < c_3 < c_2 < c_1 < c_1 + c_3 < c_1 + c_2 \leq 1$. Then*

$$\dim \pi_1(G_{c_1,c_2,c_3}) \leq 9,$$

and the homotopy groups satisfy

$$\dim \pi_n(G_{c_1,c_2,c_3}) \leq \dim \pi_{n+1}(\widetilde{M}_{c_1,c_2}) + \dim \pi_n(G_{c_1,c_2}).$$

Proof. We recall part of Proposition 4.2 in [5] that would be relevant in our case:

Proposition 3.2.3. *Let $0 < c_2 < c_1 < c_2 + c_1 < 1$. Then*

$$\pi_1(Symp(\widetilde{M}_{c_1,c_2})) = \mathbb{Q}^5, \text{ and}$$

$$\pi_n(Symp(\widetilde{M}_{c_1,c_2})) = \mathbb{Q}^{r_n} \text{ for } n \geq 2,$$

where $r_n = \dim \pi_n(\Omega\widetilde{M}_{c_1}) \otimes \mathbb{Q} = \dim \pi_{n+1}(\widetilde{M}_{c_1}) \otimes \mathbb{Q}$ is calculated as $r_2 = r_3 = 5$, $r_4 = 10$, and $r_5 = 24$.

From the evaluation fibration

$$Symp_p(\widetilde{M}_{c_1,c_2}) \to Symp(\widetilde{M}_{c_1,c_2}) \to \widetilde{M}_{c_1,c_2},$$

using Lemma 3.1.4 and Theorem 3.1.1, we get a long exact sequence that ends with

$$\pi_5(\widetilde{M}_{c_1,c_2}) \to \pi_4(G_{c_1,c_2,c_3}) \to \pi_4(G_{c_1,c_2}) \to$$
$$\to \pi_4(\widetilde{M}_{c_1,c_2}) \to \pi_3(G_{c_1,c_2,c_3}) \to \pi_3(G_{c_1,c_2}) \to$$
$$\to \pi_3(\widetilde{M}_{c_1,c_2}) \to \pi_2(G_{c_1,c_2,c_3}) \to \pi_2(G_{c_1,c_2}) \to$$
$$\to \pi_2(\widetilde{M}_{c_1,c_2}) \to \pi_1(G_{c_1,c_2,c_3}) \to \pi_1(G_{c_1,c_2}) \to \pi_1(\widetilde{M}_{c_1,c_2}).$$

Using Theorem 1.1 in [5], we can write it as

$$\mathbb{Q}^{27} \to \pi_4(G_{c_1,c_2,c_3}) \to \mathbb{Q}^{10} \to$$
$$\to \mathbb{Q}^{16} \to \pi_3(G_{c_1,c_2,c_3}) \to \mathbb{Q}^{5} \to$$
$$\to \mathbb{Q}^{9} \to \pi_2(G_{c_1,c_2,c_3}) \to \mathbb{Q}^{5} \to$$
$$\to \mathbb{Q}^{4} \to \pi_1(G_{c_1,c_2,c_3}) \to \mathbb{Q}^{5} \to \{0\}.$$

■

So we expect to find at most 9 generators for π_1. In the next section, we will look at toric pictures and see that the methods hitherto used in literature do not reduce the rank to less than 11 generators. We address this in Section 3.3.2 .

3.3 The generators

This section is dedicated to the description of the generators of the homotopy Lie algebra of G_{c_1,c_2,c_3}. As these generators will appear as Hamiltonian circle actions, which then can be seen as elements of $\pi_1(G_{c_1,c_2,c_3})$, we will make extensive use of Delzant's classification of toric actions and Karshon's classification of Hamiltonian circle actions on symplectic manifolds. We first give a quick overview how these classifications work.

A *Delzant polytope* in $\mathbb{R}^2$ is a convex polytope satisfying

- simplicity, i.e., there are two edges meeting each vertex,
- rationality, i.e., the edges meeting at the vertex p are rational in the sense that each edge is of the form $p + tu_i$, $t \geq 0$, where $u_i \in \mathbb{Z}^2$;
- smoothness, i.e., for each vertex, the corresponding u_1, u_2 can be chosen to be a $\mathbb{Z}$-basis of $\mathbb{Z}^2$.

Delzant's classification tells us that toric manifolds up to equivariant symplectomorphisms are classified by Delzant polytopes up to transformations by $GL(2,\mathbb{Z})$. The Delzant polytope corresponding to a toric manifold $(M,\omega,\mathbb{T}^2,\phi)$ is given by the image $\phi(M)$ of the moment map. (See [7], Chapter 29, for the reverse direction in the correspondence.) Note that in our case since we work with symplectomorphisms that act trivially on homology, we do not allow symplectomorphisms that interchange the roles of B and F. Therefore, we shall consider Delzant polytopes up to transformations by $SL(2,\mathbb{Z})$.

On the other hand, Karshon's classification [12] yields the correspondence between Hamiltonian circle actions and "decorated graphs". A Hamiltonian circle action comes with a real-valued momentum map $\Phi : M \to \mathbb{R}$, which is a Morse-Bott function with critical set corresponding to the fixed points. When the manifold is four-dimensional, the critical set consists of isolated points and two-dimensional submanifolds, and the latter can only occur at the extrema of Φ.

To (M,ω,Φ), Karshon associates the following graph: For each isolated fixed point p, there is a vertex $\langle p\rangle$, labeled by the real number $\Phi(p)$. For each two-dimensional invariant surface S, there is a fat vertex $\langle S\rangle$, labeled by two real numbers and one integer: the momentum map label $\Phi(S)$, the area label $\frac{1}{2\pi}\int_S \omega$, and the genus g of the surface S. A $\mathbb{Z}_k$*-sphere* is a gradient sphere in M on which S^1 acts with isotropy $\mathbb{Z}_k$. For each $\mathbb{Z}_k$-sphere containing two fixed points p and q, the graph has an edge connecting the vertices $\langle p\rangle$ and $\langle q\rangle$ labeled by the integer k. As vertical translations of the graph correspond to equivariant symplectomorphisms, and flips correspond to automorphisms of the circle. (See [11], Section 3, for more details.)

Karshon's classification tells us that there is a correspondence between these decorated graphs up to translation and flipping, and Hamiltonian circle actions up to equivariant symplectomorphisms that respect the moment maps. Namely, for two compact four-dimensional Hamiltonian S^1 spaces (M,ω,Φ) and (M',ω',Φ'), an equivariant symplectomorphism $F : M \to M'$ such that $\Phi = F^{-1}\circ\Phi'\circ F$ induces an isomorphism

on the corresponding graphs.

Furthermore, this classification serves to keep track of symplectic blow-ups, as in Figure 3.1 and 3.2.

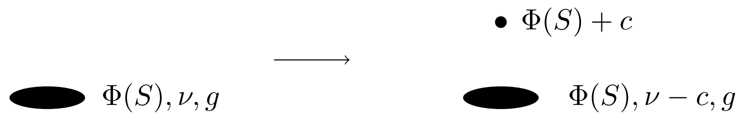


Figure 3.1: Blowing up at a point inside an invariant surface at the minimum value of Φ

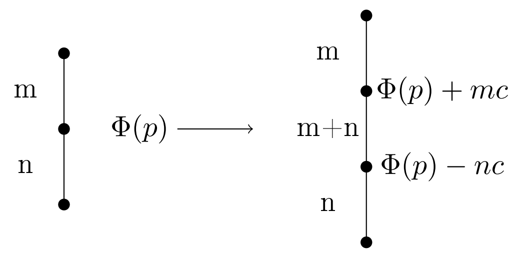


Figure 3.2: Blowing at an interior fixed point

Toric actions are generated by two Hamiltonian circle actions, which in the Delzant polytope can be seen as the projection to the axes. This relationship between polytopes and decorated graphs will be particularly useful in the following subsections.

3.3.1 Toric actions

We start by drawing the Delzant polytopes of the symplectic manifold $\widetilde{M}_{c_1,c_2,c_3}$ equipped with all possible toric actions. In Chapter 4, we will go through the more general construction for $\widetilde{M}_{\mu,c_1,c_2,c_3}$.

Let $T^4 \subset U(4)$ act in the standard way on $\mathbb{C}^4$. Given an integer $n \geq 0$, the action of the subtorus $T_0^2 := (s+t, t, s, s)$ is Hamiltonian with moment map

$$(z_1, ..., z_4) \mapsto (|z_1|^2 + |z_3|^2 + |z_4|^2, |z_1|^2 + |z_2|^2)$$

so that we identify $(S^2 \times S^2, \sigma \oplus \sigma)$ with the toric Hirzebruch surface $\mathbb{F}_0$ defined as the symplectic quotient $\mathbb{C}^4//T_0^2$ at the regular value $(1, 1)$ endowed with the residual

action of the torus $T(0) := (0, u, v, 0) \subset T^4$. The image $\Delta(0)$ of the moment map is the convex hull of

$$\{(0,0),(1,0),(1,1),(0,1)\}.$$

We identify the symplectic blow-up $\widetilde{M}_{c_1}$ at a ball of capacity c_1 with the equivariant blow-up of the Hirzebruch surface $\mathbb{F}_0$.

We define the even torus action $\widetilde{T}(0)$ as the equivariant blow-up of the toric action of $T(0)$ on $\mathbb{F}_0$ at the fixed point $(0,0)$ with capacity c_1. The image of the moment map then is the convex hull of

$$\{(1,1),(0,1),(0,c_1),(c_1,0),(1,0)\}.$$

The Kähler isometry group of $\mathbb{F}_1$ is $N(T_0^2)/T_0^2$ where $N(T_0^2)$ is the normalizer of T_0^2 in $U(4)$. There is a natural isomorphism $N(T_0^2)/T_0^2 \simeq SO(3) \times SO(3) := K(0)$, and its restriction to the maximal torus is given in coordinates by

$$(u,v) \mapsto (-u,v) \in T(0) := S^1 \times S^1 \subset K(0)$$

This identification implies that the moment polygon associated to the maximal tori $T(0) = S^1 \times S^1 \subset K(0)$ and $\widetilde{T}(0)$ are the images of $\Delta(0)$ and $\widetilde{\Delta}(0)$, respectively, under the transformation

$$C_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Under the blow-down map, $\widetilde{T}(0)$ is sent to the maximal torus of $K(0)$. By [17] $\mathrm{Symp}(\widetilde{M}_{c_1})$ is connected, hence the choices involved in the identification give the same maps up to homotopy.

We identify the symplectic blow-up $\widetilde{M}_{c_1,c_2}$ with the equivariant two blow-up of the Hirzebruch surface $\mathbb{F}_0$ and obtain inequivalent toric structures. We define the torus actions $\widetilde{T}_i(0)$, $i = 1, ..., 5$ as the equivariant blow-ups of the toric action of $T(0)$ on $\widetilde{\mathbb{F}}_0$, with capacity c_2, at each one of the five fixed points, which correspond to the vertices of the moment polygon $\widetilde{\Delta}(0)$.

We blow-up each of the resulting toric actions in their six fixed points and obtain $\widetilde{T}_{i,j}(0)$, $j = 1, ..., 6$. These toric pictures arise from the ones described in Section 4.2 of [5] by applying one more symplectic blow-up, of capacity c_3, at each corner. In Figures 3.3 and 3.4, we draw the toric pictures for $\widetilde{T}_i(0)$, $i = 1, ..., 5$, and plot the corners where the next blow-ups take place. Then, in each case we pick two Hamiltonian S^1-actions, $x_{i,j}$ and $y_{i,j}$, that generate the depicted toric action. More precisely, $x_{i,j}$ and $y_{i,j}$ are circle actions whose moment maps are, respectively, the first and second components of the moment map associated to the torus action $\widetilde{T}_{i,j}(0)$. In fact, Y. Karshon explains in [12] how to collect Hamiltonian S^1-actions from these figures via graphs and how these graphs classify the circle actions. For a brief summary on decorated graphs, Delzant polytopes and the relationship between the two, see the beginning of this section.

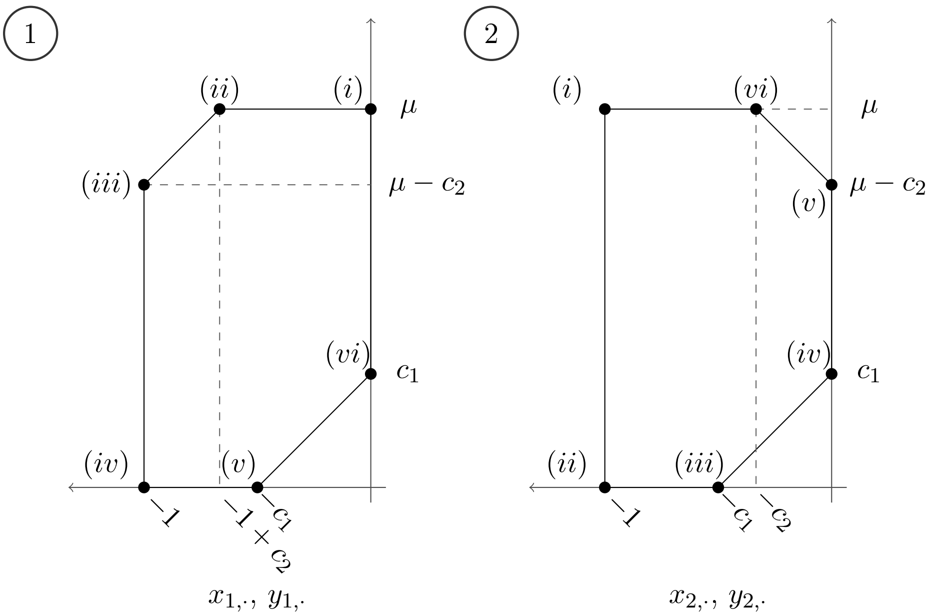


Figure 3.3: Toric pictures for $\widetilde{T}_1(0)$ and $\widetilde{T}_2(0)$ with the next blow-ups plotted

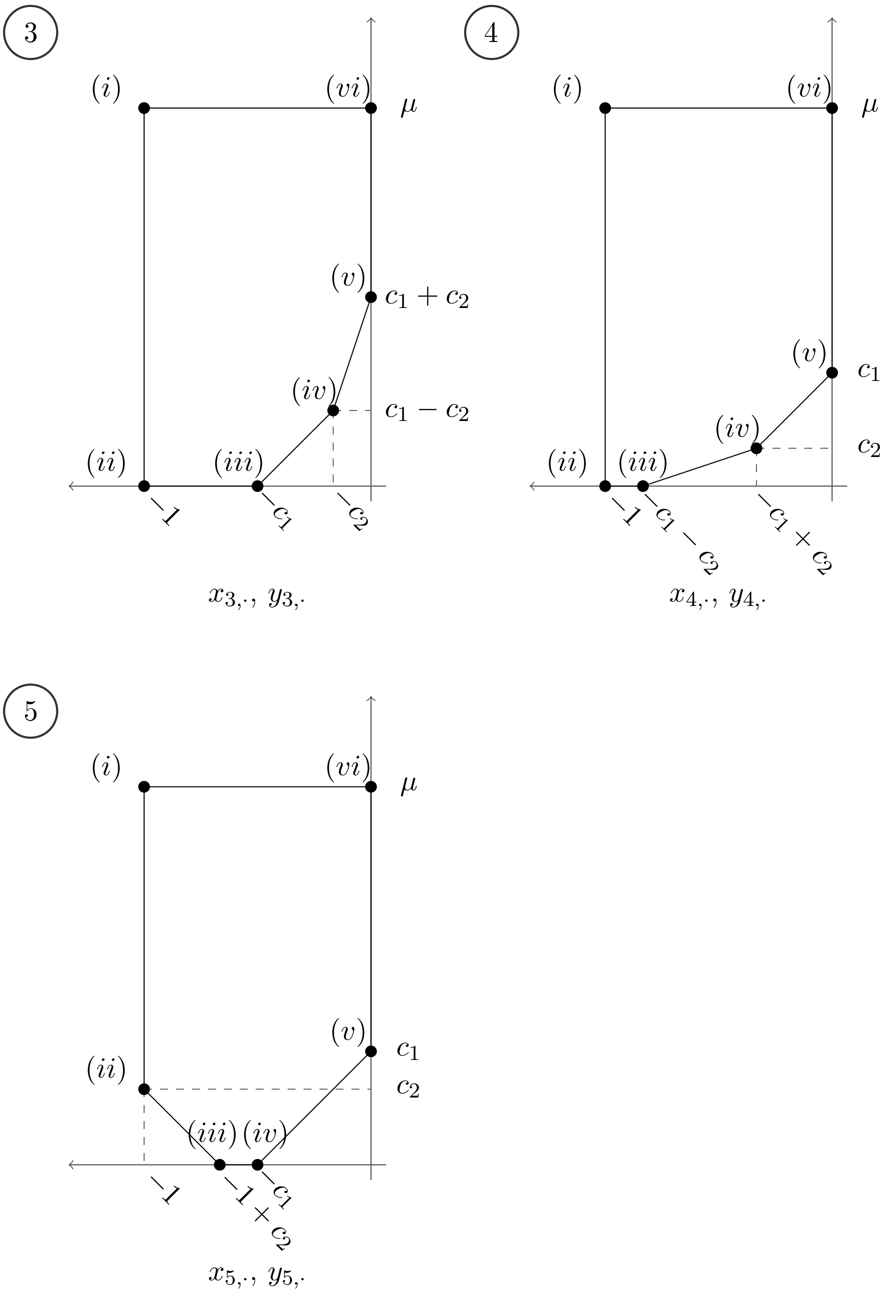


Figure 3.4: Toric pictures for $\widetilde{T}_3(0)$, $\widetilde{T}_4(0)$ and $\widetilde{T}_5(0)$ with the next blow-ups plotted

Some strata whose configurations are depicted in the figures in Chapter 2 correspond to a toric structure on $\widetilde{M}_{c_1,c_2,c_3}$, unique up to equivariant symplectomorphism. These toric structures are given by the tori $\widetilde{T}_{i,j}(0)$. We will demonstrate this correspondence

later in Appendix B. (See Propositon B.0.4, Lemma B.0.6 and Corollary B.0.7.) More accurately, the relationship between the toric pictures and the configurations is, conveniently,

$$\widetilde{T}_{i,j}(0) \longleftrightarrow \text{Configuration (i,j)}$$

for $i = 1, ..., 5$, and $j = 1, ..., 6$.

Beside the circle actions coming from these tori $\widetilde{T}_{i,j}(0)$, there are circle actions obtained by blowing up the toric actions $\widetilde{T}_i(0)$ at an interior point of a curve, as for example in configurations 1.7-1.12. Moreover, there is one configuration for which there is no correspondent S^1-action, namely 1.13. We will now study each of these situations successively, starting with the toric pictures.

Using Karshon's classification, we use projections onto the x-axis and onto the y-axis to elucidate the two selected Hamiltonian S^1-actions. This procedure gives the following figures:

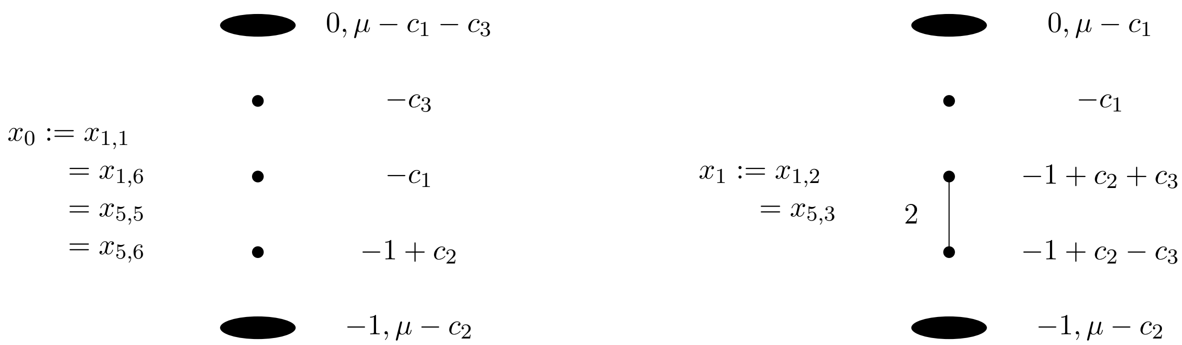


Figure 3.5: The S^1-actions x_0 and x_1

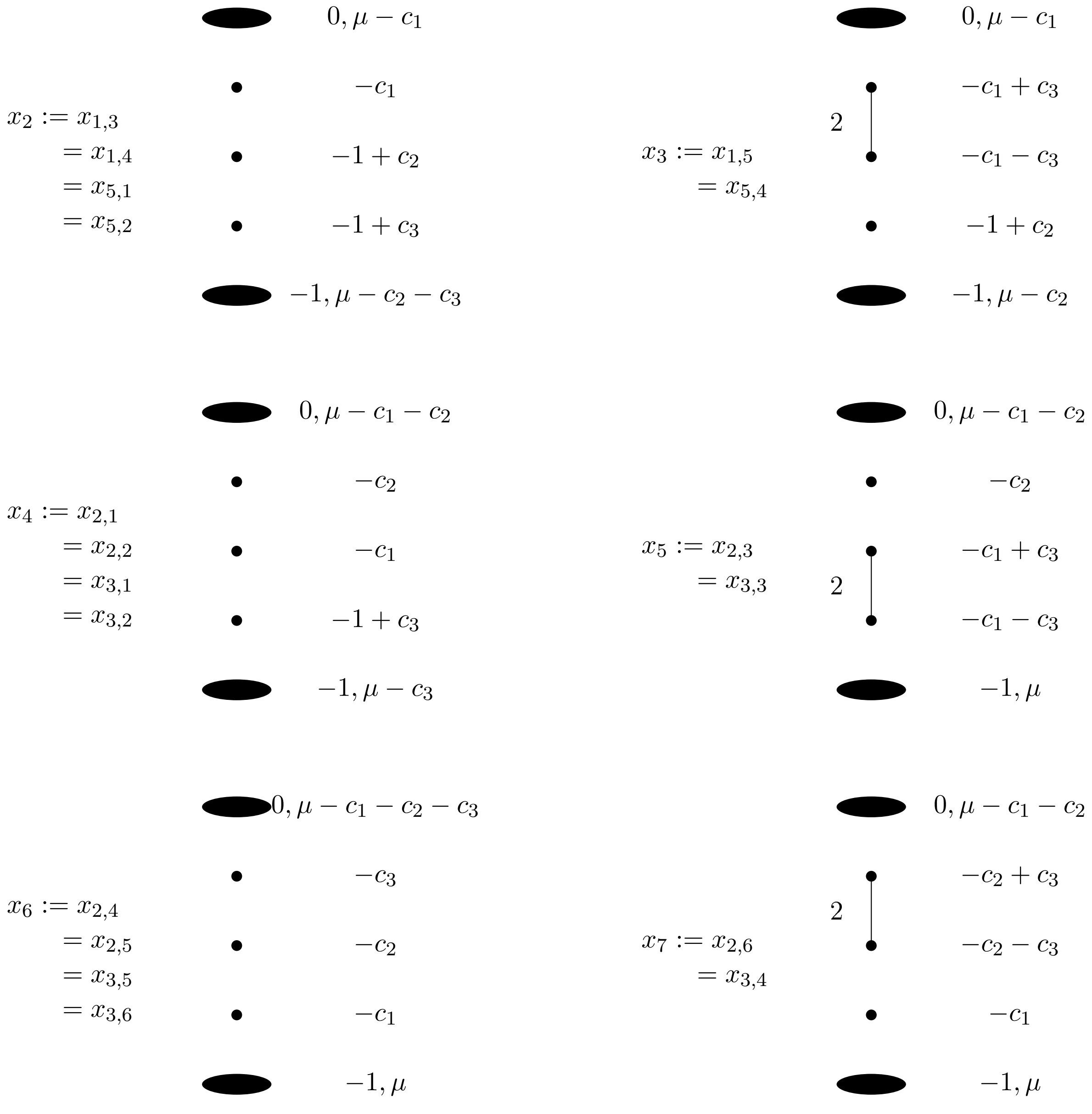


Figure 3.6: The S^1-actions x_2, x_3, x_4, x_5, x_6 and x_7

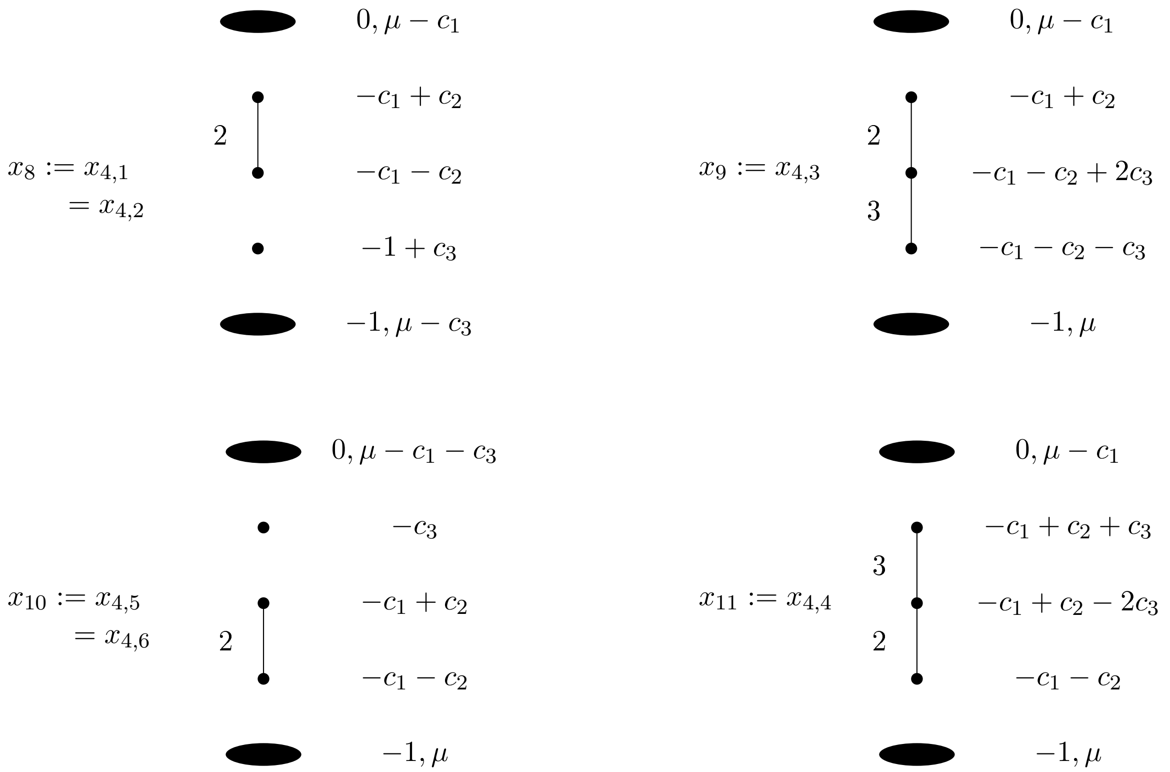


Figure 3.7: The S^1-actions x_8, x_9, x_{10} and x_{11}

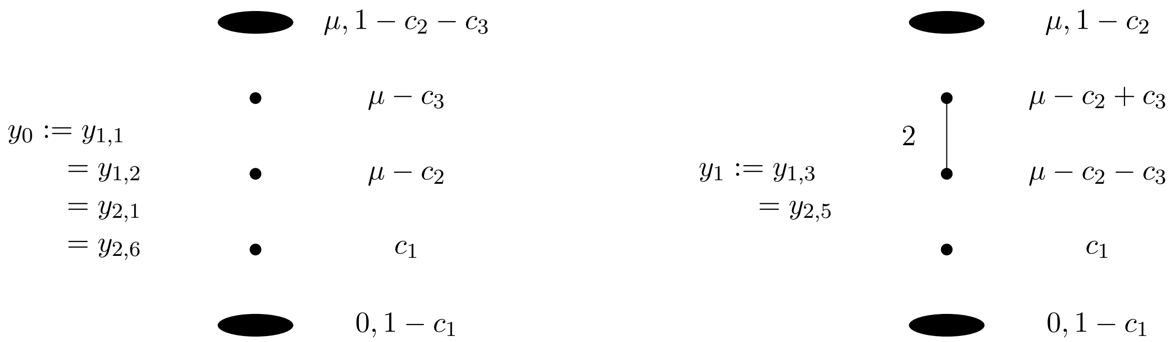


Figure 3.8: The S^1-actions y_0 and y_1

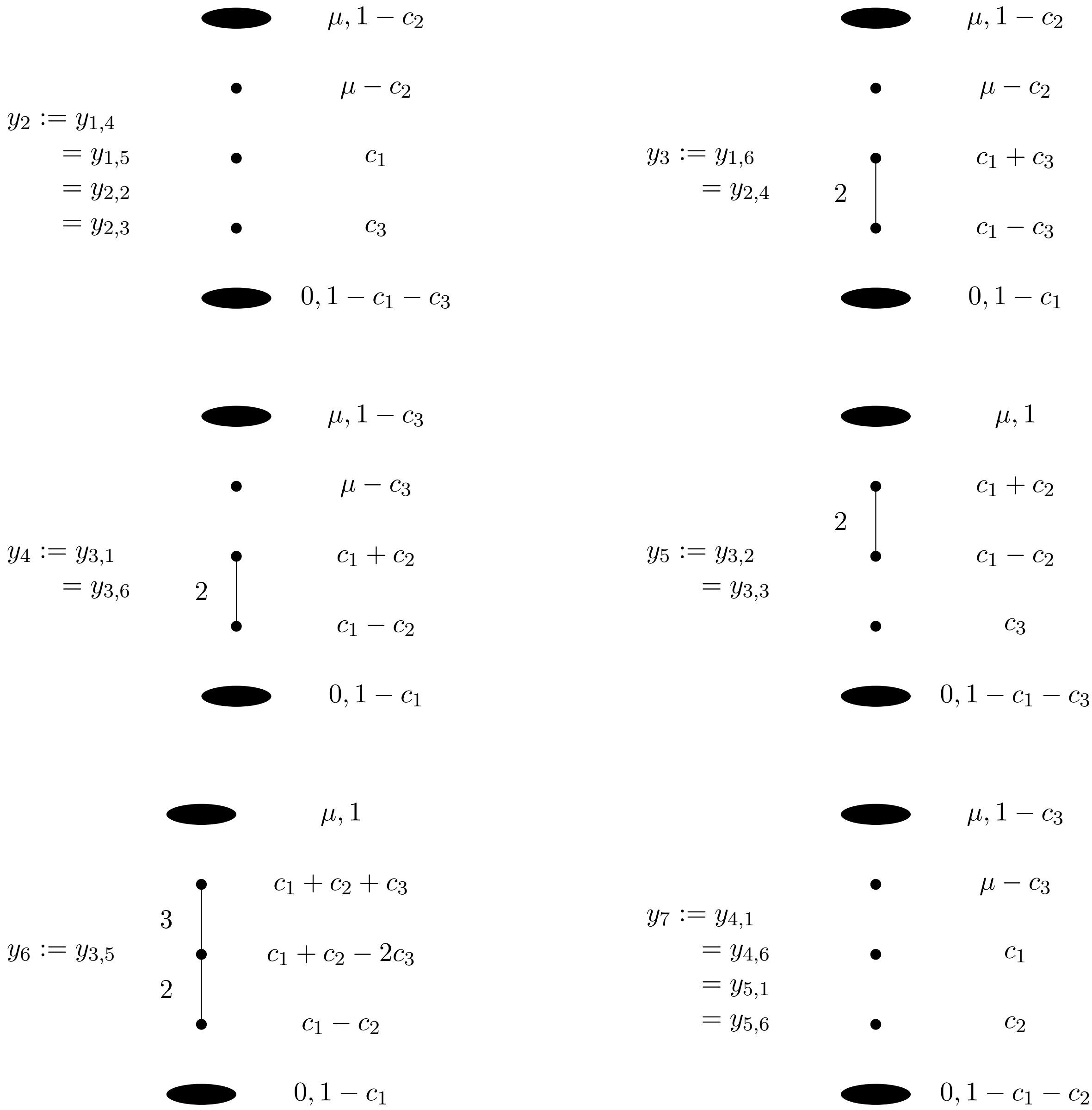


Figure 3.9: The S^1-actions y_2, y_3, y_4, y_5, y_6 and y_7

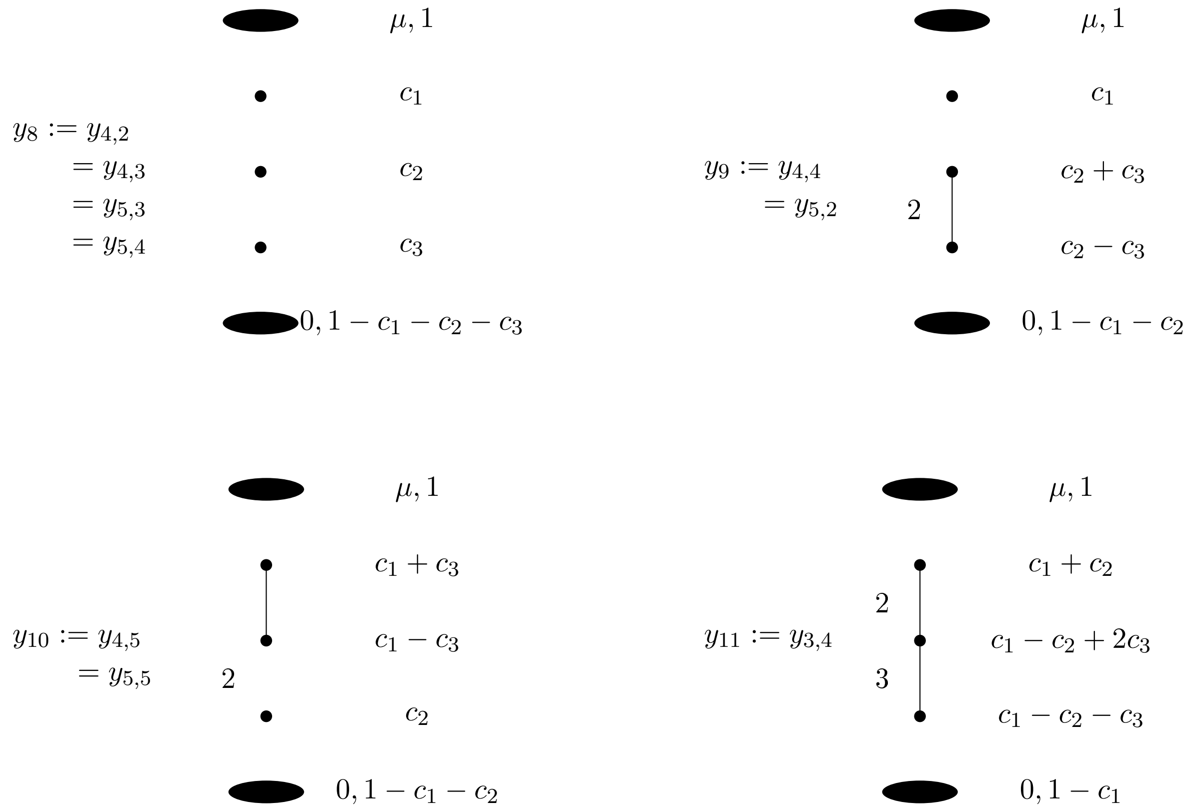


Figure 3.10: The S^1-actions y_8, y_9, y_{10} and y_{11}

Remark 3.3.1. *As explained in [25], the maps* $T_{i,j}(0) \to G_{c_1,c_2,c_3}$ *induce injective maps of fundamental groups. We briefly recall their result here.*

For a simple polytope Δ *and* c_Δ *its center of mass function, an affine function* $H : \mathbb{R}^2 \to \mathbb{R}$ *is called mass linear with respect to* Δ *if the composite* $H \circ c_\Delta$ *is linear.*

[Theorem 1.3, [25]] *A simple 2-dimensional polytope supports a nonconstant mass linear function exactly if it is a triangle, a trapezoid or a parallelogram.*

[Theorem 1.25(i), [25]] *Let* (M, ω, T, ν) *be a symplectic toric manifold with moment polytope* Δ*. The map* $\pi_1(T) \to \pi_1(Symp_0(M, \omega))$ *is an injection if there are no mass linear functions on* Δ*.*

Therefore, combining these results, we can see the actions coming from the toric structures as elements of the fundamental group of G_{c_1,c_2,c_3}*. Using Karshon's classification,*

we can now find relations between the corresponding elements in π_1.

Transforming the polytopes by $SL(2,\mathbb{Z})$-actions, we get the following relations:

(0) Type #0 relations

0.1. $y_0 - x_1 = y_1 - x_2$, 1(ii) $\leftrightarrow$ 1(iii)

0.2. $y_2 - x_3 = y_3 - x_0$, 1(v) $\leftrightarrow$ 1(vi)

0.3. $y_2 - x_5 = y_3 - x_6$, 2(iii) $\leftrightarrow$ 2(iv)

0.4. $y_1 + x_6 = y_0 + x_7$, 2(v) $\leftrightarrow$ 2(vi)

0.5. $y_4 - x_4 = y_7 - x_8$, 3(i) $\leftrightarrow$ 4(i)

0.6. $y_5 - x_4 = y_8 - x_8$, 3(ii) $\leftrightarrow$ 4(ii)

0.7. $y_8 - x_9 = y_6 - x_6$, 3(v) $\leftrightarrow$ 4(iii)

0.8. $y_4 - x_6 = y_7 - x_{10}$, 3(vi) $\leftrightarrow$ 4(vi)

0.9. $y_9 + x_2 = y_8 + x_1$, 5(ii) $\leftrightarrow$ 5(iii)

0.10. $y_8 - x_3 = y_{10} - x_0$, 5(iv) $\leftrightarrow$ 5(v)

0.11. $y_5 - x_5 = y_{11} - x_7$, 3(iii) $\leftrightarrow$ 3(iv)

0.12. $y_{11} - 2x_7 = y_6 - 2x_6$, 3(iv) $\leftrightarrow$ 3(v)

0.13. $-2y_8 + x_9 = -2y_9 + x_{11}$, 4(iii) $\leftrightarrow$ 4(iv)

0.14. $y_9 - x_{11} = y_{10} - x_{10}$, 4(iv) $\leftrightarrow$ 4(v)

0.15. $y_{11} - x_7 = y_{10} - x_{10}$, 3(iv) $\leftrightarrow$ 4(v)

0.16. $y_5 - x_5 = y_9 - x_{11}$, 3(iii) $\leftrightarrow$ 4(iv)

3.3.2 New relations

There are two more relations of a somewhat different nature:

Proposition 3.3.2. *Picking elements in π_1 as in Section 3.3.1, we obtain the follow-*

ing relations:

$$x_0 + x_4 = x_2 + x_6. \tag{3.3.1}$$

$$y_2 + y_7 = y_0 + y_8 \tag{3.3.2}$$

Proof. We start by noting that for any $\mu > 1$, we can draw the auxiliary Delzant polytopes in Figure 3.11. We name the S^1-actions corresponding to these polytopes in accordance with the more general case explained later in Appendix C.

Using Karshon's classification, we obtain the following relations, designated in the chart in Figure 3.12.

(S1) $x_{2,2,5} = x_{2,3,5}$, $\quad$ 18(v) $\leftrightarrow$ 17(v)

(S2) $x_{2,2,6} = x_{2,3,6}$, $\quad$ 18(vi) $\leftrightarrow$ 17(vi)

(S3) $y_{2,2,5} = x_{1,5} + y_{1,5}$, $\quad$ 18(v) $\leftrightarrow$ 1(v)

(S4) $y_{2,2,6} = x_{1,4} + y_{1,4}$, $\quad$ 18(vi) $\leftrightarrow$ 1(iv)

(S5) $y_{2,3,5} = x_{3,3} + y_{3,3}$, $\quad$ 17(v) $\leftrightarrow$ 3(iii)

(S6) $y_{2,3,6} = x_{3,2} + y_{3,2}$, $\quad$ 17(vi) $\leftrightarrow$ 3(ii)

(S7) $x_{2,2,5} + y_{2,2,5} = x_{2,2,6} + y_{2,2,6}$, $\quad$ 18(v) $\leftrightarrow$ 18(vi)

(S8) $x_{2,3,5} + y_{2,3,5} = x_{2,3,6} + y_{2,3,6}$, $\quad$ 17(v) $\leftrightarrow$ 17(vi)

Starting with (S8) and using (S1) and (S2), we get $x_{2,2,5} + y_{2,3,5} = x_{2,2,6} + y_{2,3,6}$. Then (S7) yields $y_{2,2,6} + y_{2,3,5} = y_{2,2,5} + y_{2,3,6}$. Combining (S3), (S4), (S5) and (S6) gives

$$x_{3,3} + y_{3,3} + x_{1,4} + y_{1,4} = x_{3,2} + y_{3,2} + x_{1,5} + y_{1,5}.$$

By Figures 3.6 and 3.9, this is equivalent to

$$x_5 + y_5 + x_2 + y_2 = x_4 + y_5 + x_3 + y_2,$$

$$x_5 + x_2 = x_4 + x_3.$$

Finally using the type #0 relations (0.2) and (0.3) to replace x_3 and x_5 respectively

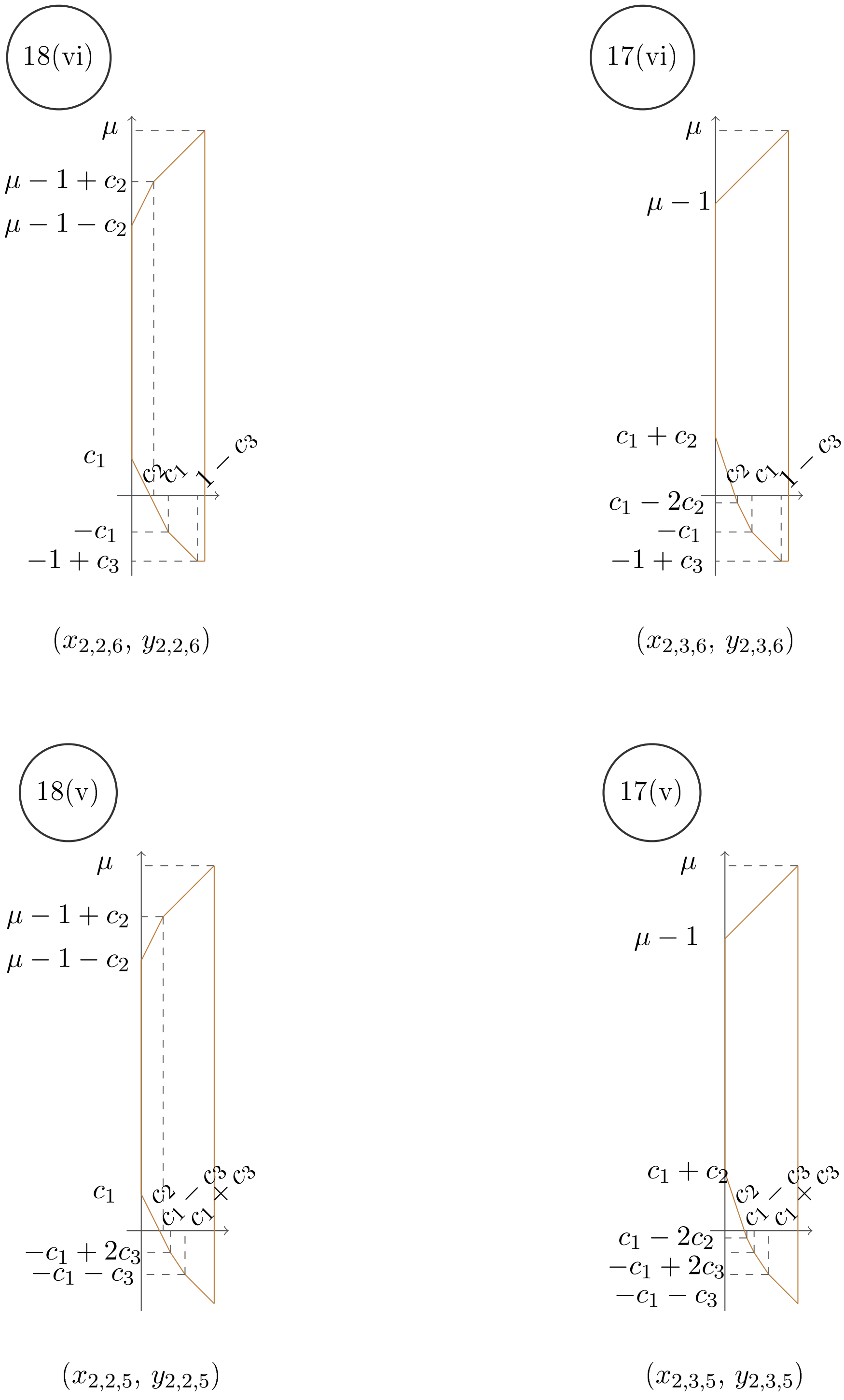


Figure 3.11: The auxiliary Delzant polytopes

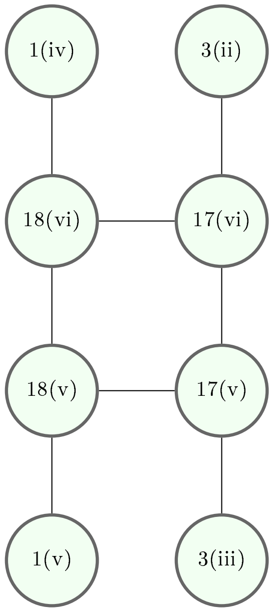


Figure 3.12: The summary of the relations between the Delzant polytopes

we obtain the relation 3.3.1:

$$y_2 - y_3 + x_6 + x_2 = x_4 + y_2 - y_3 + x_0,$$

$$x_2 + x_6 = x_0 + x_4.$$

There is a more geometrical interpretation of this relation (see Figure 3.13). We trace the blow-ups of $\mathbb{CP}^2\#2\overline{\mathbb{CP}^2}$ at the invariant surfaces of the action whose moment map corresponds to the first component of the torus action described in the Hirzebruch surface $\mathbb{F}_0$. Tracking each one of the blow-ups via the arrows, we summarize below how the relation emerges.

As can be seen in Figure 3.13, pairing the elements yields $x_6 - x_4 = x_{10} - x_8 = x_0 - x_2$. Using type $\#0$ relations we note that indeed these are equivalent to (3.3.1). As graphs, the value of μ does not matter for these relations to hold. Since the map $T_{i,j} \to G_{\mu,c_1,c_2,c_3}$ is injective, the relation must hold for $\mu = 1$ as well.

By symmetry, we obtain the relation 3.3.2. That is, we have a similar picture for the y's, where we start with Figure 3.14 and follow the blow-ups at analogous points,

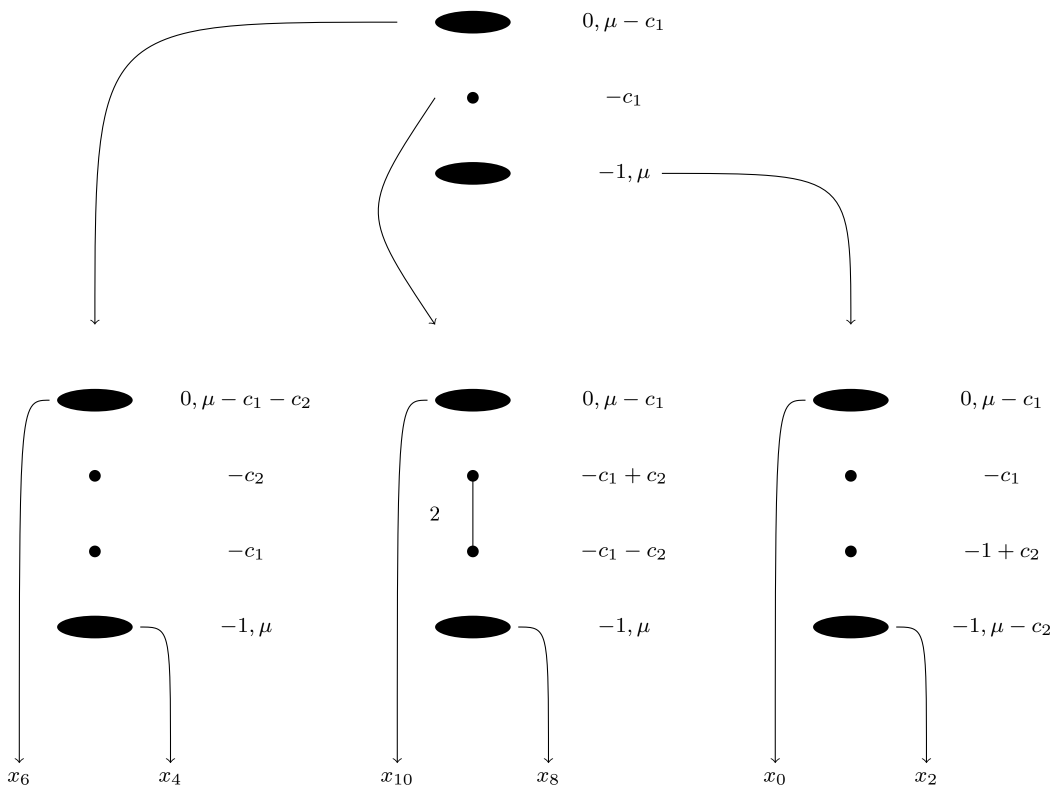


Figure 3.13: Comparison of the S^1-actions in the new relation 3.3.1

which yields $y_8 - y_7 = y_5 - y_4 = y_2 - y_0$.

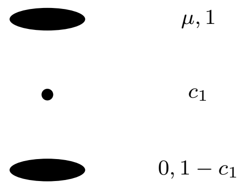


Figure 3.14: The formation of the S^1-actions for relation 3.3.2

∎

If we pick the elements in π_1 corresponding to the actions x_0, x_1, x_2, x_3, x_4, y_0, y_2, y_7, $z = y_4 - x_4$, we can write the rest in terms of them as in Table 3.1.

$x_5 = -x_2 + x_3 + x_4$	$x_6 = x_0 - x_2 + x_4$
$x_7 = x_0 - x_1 + x_4$	$x_8 = y_8 - z$
$x_9 = -x_1 + x_3 + y_7 - z$	$x_{10} = x_0 - x_2 + y_7 - z$
$x_{11} = x_1 - 2x_2 + x_3 + y_7 - z$	$y_1 = -x_1 + x_2 + y_0$
$y_3 = x_0 - x_3 + y_2$	$y_4 = x_4 + z$
$y_5 = x_4 - y_0 + y_2 + z$	$y_6 = x_0 + x_1 - x_2 - x_3 + x_4 - y_0 + y_2 + z$
$y_8 = -y_0 + y_2 + y_7$	$y_9 = x_1 - x_2 - y_0 + y_2 + y_7$
$y_{10} = x_0 - x_3 - y_0 + y_2 + y_7$	$y_{11} = x_0 - x_1 + x_2 - x_3 + x_4 - y_0 + y_2 + z$

Table 3.1: The elements written in terms of the generators

3.3.3 Other circle actions

Beside the circle actions coming from the Delzant polytopes outlined above, there are other circle actions obtained by blowing up the toric actions at interior points of J-holomorphic curves. Again, using Karshon's classification theorem, we aquire an equivalent of Lemma 4.7 of [5] to express them in terms of the existing generators. This is an easy (yet cumbersome) task, we will not write the calculations up. Instead, we will show on one elucidatory example how the calculations go through:

We recall the Configuration 5.8 (See Figure 3.15). [1] Although this configuration does not derive from any toric action, we observe that blowing down the exceptional curve E_3 would yield Configuration 5, which in turn corresponds to the toric action whose moment polytope was given in 5 above, with the corresponding S^1-actions $x_{0,5}$ and $y_{0,5}$. Because the blow-up takes place at an interior point of the curve representing the class $F - E_1 - E_2$, it can be traced on the circle action corresponding to the projection to y-axis, as can be seen in Figure 3.16.

[1] This is a simplified version of the actual configuration 5.8, which further has a J-holomorphic representative the class $B - E_3$, a curve that intersects $F - E_1$, E_3 and $F - E_2$ exactly once. We utilize the simplified version as it makes it clearer to follow the symplectic blow-downs.

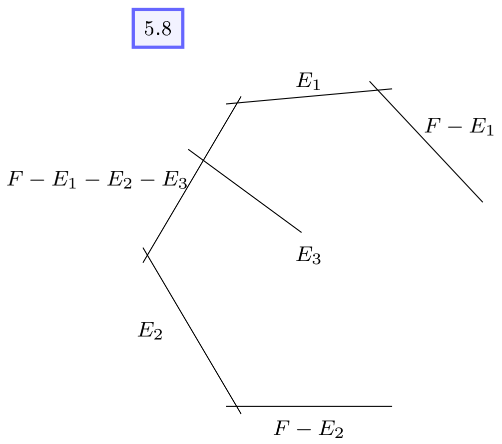


Figure 3.15: Configuration 5.8

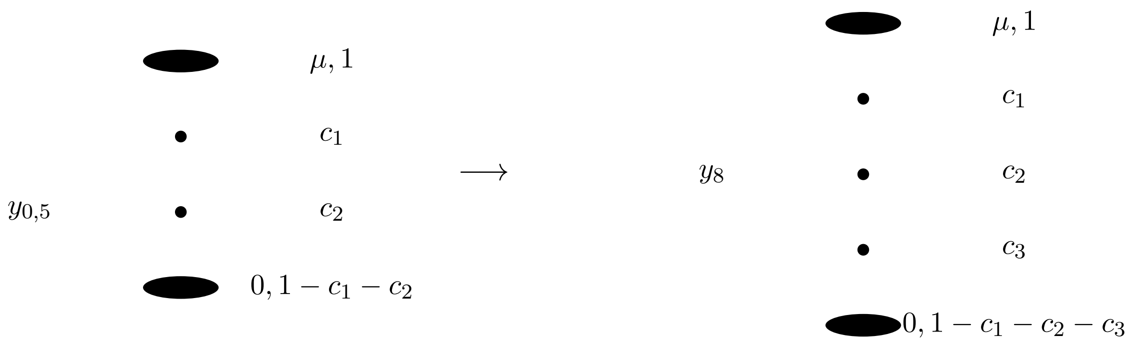


Figure 3.16: The circle action corresponding to Configuration 5.8

Hence the circle action corresponding to Configuration 5.8., seen in $\pi_1(G_{c_1,c_2,c_3})$, is equal to y_8.

Finally, we note that not all configurations arise from S^1-actions, and thereby cannot be read from Delzant polytopes. As mentioned before, there is exactly one configuration that does not correspond to any S^1-action, and it is when $B - E_1$, $B - E_2$ and $B - E_3$ are represented by J-holomorphic spheres. This is Configuration 1.13.

J.D. Evans showed in [8] that in the monotone case (i.e. when $\mu = 1$ and $c_1 = c_2 = c_3 = 1/2$), the symplectomorphism group is contractible. In this case, the

automorphism group of the generic structure J_0 is trivial as it is isomorphic to the stabilizer of four generic points in $\mathbb{CP}^2$. The only possible configuration is 1.13, that is, the space of almost complex structures contains a single stratum, which is therefore contractible. In the non-monotone case, more J-holomorphic curves are allowed to exist and that stratum corresponds to the open stratum of $\mathcal{J}_{\mu=1,c_1,c_2,c_3}$. So, the open stratum corresponding to this configuration does not yield new generators in π_1.

3.4 Proof of the Main Theorem

Theorem 1.3.2: *Let $1 > c_1 + c_2 > c_1 + c_3 > c_1 > c_2 > c_3$. Let $\widetilde{M}_{c_1,c_2,c_3}$ denote the symplectic manifold $(S^2 \times S^2, \sigma \oplus \sigma)$, blown up at three balls of capacities of c_1, c_2 and c_3. Let G_{c_1,c_2,c_3} denote the group of symplectomorphisms of $\widetilde{M}_{c_1,c_2,c_3}$ that act trivially on homology. Define $\widetilde{\Lambda}$ as the algebra over $\mathbb{Q}$ generated by x_0, x_1, x_2, x_3, x_4, y_0, y_2, y_7, z, where all generators have degree 1 and the Samelson products between them are as in Table 3.2.*

<table>
<tr><td colspan="2">$[x_0, y_0] = [x_0, y_7] = [x_1, y_0] = [x_2, y_2] = 0$</td></tr>
<tr><td colspan="2">$[x_2, y_7] = [x_3, y_2] = [x_4, y_2] = [x_4, z] = [y_7, z] = [x_4, y_0] = 0$</td></tr>
<tr><td colspan="2">$[x_1, x_4] = -[x_0, x_1] + [x_0, x_2] + [x_2, x_4]$</td></tr>
<tr><td colspan="2">$[x_1, z] = [x_0, x_2] + [x_0, z] - [x_1, x_2] + [x_1, x_3] + [x_1, y_7] - [x_2, x_3]$</td></tr>
<tr><td colspan="2">$[x_3, y_0] = [x_3, y_7] = -[x_0, x_2] - [x_0, z] + [x_1, x_2] + [x_2, x_3] + [x_3, z]$</td></tr>
<tr><td>$[x_0, y_2] = [x_0, x_3]$</td><td>$[x_1, y_2] = -[x_1, y_7]$</td></tr>
<tr><td>$[x_2, z] = [x_0, x_2] + [x_0, z]$</td><td>$[x_2, y_0] = [x_1, x_2]$</td></tr>
<tr><td>$[x_3, x_4] = [x_2, x_3] + [x_2, x_4]$</td><td>$[y_2, z] = [y_0, y_2] + [y_0, z]$</td></tr>
<tr><td>$[x_0, x_4] = [x_0, x_2] + [x_2, x_4]$</td><td>$[y_2, y_7] = [y_0, y_2] + [y_0, y_7]$</td></tr>
</table>

Table 3.2: The relations between the Samelson products

Then there is an isomorphism between $\widetilde{\Lambda}$ and the homotopy graded Lie algebra

$$\pi_*(G_{\mu=1,c_1,c_2,c_3}) \otimes \mathbb{Q}.$$

Remark 3.4.1. *Note that the Lie algebra $\pi_*(G_{c_1,c_2,c_3})$ contains a subalgebra, generated by x_2, x_4, y_2, y_7 and z, that is isomorphic to the rational homotopy Lie graded algebra of the group $G_{\mu=1,c_1,c_2}$ (described in [5]).*

Proof. In the proof, we will use the results from the previous sections to first compute the relations in $\pi_*(G_{c_1,c_2,c_3})$ between the Samelson products of the generators as listed in the theorem. This means that we can construct a homomorphism from $\widetilde{\Lambda}$ into $\pi_*(G_{c_1,c_2,c_3})$. Next, we will show that these two algebras have the same rank in each dimension so that this homomorphism is in fact an isomorphism.

We start by recalling that in previous sections, we studied the circle actions $x_{i,j}, y_{i,j}$ for $i = 1, .., 5$ and $j = 1, ...6$, which are embedded into $\pi_1(G_{c_1,c_2,c_3})$, and established the type #0 relations, as well as the relations 3.3.1 and 3.3.2. Note that we also have $[x_{i,j}, y_{i,j}] = 0$, since the actions $x_{i,j}$ and $y_{i,j}$ have commuting representatives in the torus action $\widetilde{T}_{i,j}(0)$. Picking generators as in the statement of the theorem and writing down the afore-mentioned relations yield the linearly independent relations in Table 3.3.

- $[x_0, y_0] = [x_0, y_7] = [x_1, y_0] = [x_2, y_2] = [x_2, y_7] = [x_3, y_2] = [x_4, y_2] = [x_4, z] = [y_7, z] = [x_4, y_0] = 0$
- $[x_0, y_2] = [x_0, x_3]$
- $[x_1, x_4] = -[x_0, x_1] + [x_0, x_2] + [x_2, x_4]$
- $[x_1, y_2] = -[x_1, y_7]$
- $[x_1, z] = [x_0, x_4] + [x_0, z] - [x_1, x_2] + [x_1, x_3] + [x_1, y_7] - [x_2, x_3] - [x_2, x_4]$
- $[x_2, z] = [x_0, x_4] + [x_0, z] - [x_2, x_4]$
- $[x_2, y_0] = [x_1, x_2]$
- $[x_3, x_4] = -[x_0, x_2] + [x_0, x_4] + [x_2, x_3]$
- $[x_3, y_0] = [x_3, y_7] = -[x_0, x_2] - [x_0, z] + [x_1, x_2] + [x_2, x_3] + [x_3, z]$
- $[y_2, z] = -[y_0, y_7] + [y_0, z] + [y_2, y_7]$

Table 3.3: The Samelson products derived from the toric pictures

For example, since $x_1 = x_{1,2}$ and $y_0 = y_{1,2}$ have commuting representatives in the torus action $\widetilde{T}_{1,2}(0)$, we obtain

$$[x_1, y_0] = 0.$$

Similarly, using $\widetilde{T}_{5,3}(0)$ we get

$$\begin{aligned} 0 &= [x_1, y_8] \\ &= [x_1, -y_0 + y_2 + y_7] \text{ (by Table 3.1)} \\ &= [x_1, y_2 + y_7] \text{ (because } [x_1, y_0] = 0), \end{aligned}$$

so that $[x_1, y_2] = -[x_1, y_7]$. Writing all these relations and simplifying them, we obtain the resulting list of 20 equalities.

In addition to these relations coming from the toric pictures, there are two extra relations, namely

$$[x_0, x_4] = [x_0, x_2] + [x_2, x_4] \tag{3.4.1}$$

$$[y_2, y_7] = [y_0, y_2] + [y_0, y_7]. \tag{3.4.2}$$

To prove these relations, we will need, as in Section 3.3.2, auxiliary poltyopes and relations which arise more naturally in the setup for $\mu > 1$. However, we would need to introduce 12 new polytopes and demonstrate a total of 22 relations using Karshon's classification. We postpone this construction in full generality to Appendix C. We will confine ourselves to collect the required polytopes and relations, to then move on to the proof of the relations 3.4.1 and 3.4.2.

From Figures C.1, C.2, C.5 and C.4, we get the polytopes 10: (i), (iii); 12: (i), (iii); 13: (i), (ii); 16 (iii), (iv), (vi); 18: (i), (iv), (vi). For simplicity, we will consider the case $k = 1$. The following relations will be useful for our purposes:

(1.1) $x_{1,1,1} = y_{0,1,5} - x_{0,1,5}$	(3.4) $x_{2,1,4} + y_{2,1,4} = 2x_{1,1,1} + y_{1,1,1}$
(1.2) $x_{1,1,3} = y_{0,1,4} - x_{0,1,4}$	(3.6) $x_{2,1,6} + y_{2,1,6} = 2x_{1,1,3} + y_{1,1,3}$
(1.5) $x_{1,2,1} = y_{0,2,3} - x_{0,2,3}$	(3.10) $x_{2,2,4} + y_{2,2,4} = 2x_{1,2,1} + y_{1,2,1}$
(1.6) $x_{1,2,3} = y_{0,2,2} - x_{0,2,2}$	(3.12) $x_{2,2,6} + y_{2,2,6} = 2x_{1,2,3} + y_{1,2,3}$

(4.1) $x_{2,1,6} = x_{2,5,1}$

(4.3) $x_{2,1,3} = x_{2,1,4} = x_{2,5,2}$

(4.5) $x_{2,2,1} = x_{2,2,6}$

(5.3) $y_{2,1,3} = x_{0,2,1} + y_{0,2,1}$

(5.4) $y_{2,1,4} = x_{0,2,4} + y_{0,2,4}$

(5.6) $y_{2,1,6} = x_{0,2,2} + y_{0,2,2}$

(5.7) $y_{2,2,1} = x_{0,1,1} + y_{0,1,1}$

(5.10) $y_{2,2,4} = x_{0,1,6} + y_{0,1,6}$

(5.12) $y_{2,2,6} = x_{0,1,4} + y_{0,1,4}$

(5.25) $y_{2,5,1} = x_{0,5,6} + y_{0,5,6}$

(5.26) $y_{2,5,2} = x_{0,5,1} + y_{0,5,1}$

(6.3) $x_{2,1,3} - y_{2,1,3} = x_{2,2,1} - y_{2,2,1}$

(6.4) $x_{2,1,4} - y_{2,1,4} = x_{2,2,4} - y_{2,2,4}$

(6.17) $x_{2,5,1} - y_{2,5,1} = x_{2,5,2} - y_{2,5,2}$

Set $y = y_{1,1,1}$. The relations above together with Figures 3.5-3.10 and Table 3.1 yield:

- $x_{1,1,1} = y_2 - x_3$
- $x_{1,1,3} = y_2 - x_2$
- $x_{1,2,1} = y_2 - x_5 = y_2 + x_5 - x_3 - x_4$
- $x_{1,2,3} = y_2 - x_4$
- $y_{2,1,3} = x_4 + y_0$
- $y_{2,1,4} = x_6 + y_3 = 2x_0 - x_2 - x_3 + x_4 + y_2$
- $y_{2,1,6} = x_4 + y_2$
- $y_{2,2,1} = x_0 + y_0$
- $y_{2,2,4} = x_0 + y_3 = 2x_0 - x_3 + y_2$
- $y_{2,2,6} = x_2 + y_2$
- $y_{2,5,1} = x_0 + y_7$
- $y_{2,5,2} = x_2 + y_7$

We calculate three more elements that will be useful in our next step.

The first is $y_{1,2,1} = y$: by relation (3.4), we have

$$x_{2,1,4} = 2x_{1,1,1} + y_{1,1,1} - y_{2,1,4} = -2x_0 + x_2 - x_3 - x_4 + y_2 + y. \tag{3.4.3}$$

Then by (6.4), $x_{2,2,4} = x_{2,1,4} - y_{2,1,4} + y_{2,2,4} = -2x_0 + 2x_2 - x_3 - 2x_4 + y_2 + y$. Hence, using (3.10), we obtain $y_{1,2,1} = x_{2,2,4} + y_{2,2,4} - 2x_{1,2,1} = y$.

Secondly, we have $y_{1,2,3} = -x_0 + 2x_2 - x_3 + y$: combining the relation 3.4.3 above with relation (4.4), we get $x_{2,1,3} = x_{2,1,4} = -2x_0 + x_2 - x_3 - x_4 + y_2 + y$. Then by (6.4), $x_{2,2,1} = x_{2,1,3} - y_{2,1,3} + y_{2,2,1} = -x_0 + x_2 - x_3 - 2x_4 + y_2 + y$, so that (4.5) yields $x_{2,2,6} = x_{2,2,1} = -x_0 + x_2 - x_3 - 2x_4 + y_2 + y$. Finally, by (3.12), we obtain $y_{1,2,3} = x_{2,2,6} + y_{2,2,6} - 2x_{1,2,3} = -x_0 + 2x_2 - x_3 + y$.

Thirdly, we have $y_{1,1,3} = -x_0 + 2x_2 - x_3 + y$: we start by combining the relation 3.4.3 above with relation (4.3) to get $x_{2,5,2} = x_{2,1,4} = -2x_0 + x_2 - x_3 - x_4 + y_2 + y$. Then, by (6.17), $x_{2,5,1} = x_{2,5,2} - y_{2,5,2} + y_{2,5,1} = -x_0 - x_3 - x_4 + y_2 + y$, so that (4.1) yields $x_{2,1,6} = x_{2,5,1} = -x_0 - x_3 - x_4 + y_2 + y$. Finally, by (3.16), we get $y_{1,1,3} = x_{2,1,6} + y_{2,1,6} - 2x_{1,1,3} = -x_0 + 2x_2 - x_3 + y$.

We now move on to utilize some of the toric pictures and insert these values.

By 10 (i), we have $0 = [x_{1,1,1}, y_{1,1,1}] = [y_2 - x_3, y]$, so that

$$[x_3, y] = [y_2, y]. \tag{3.4.4}$$

Then, 12 (i) yields $0 = [x_{1,2,1}, y_{1,2,1}] = [x_2 - x_3 - x_4 + y_2, y]$, which, using the relation 3.4.4 yields

$$[x_2, y] = [x_4, y]. \tag{3.4.5}$$

Using (10) (iii), we obtain

$$
\begin{aligned}
0 &= [x_{1,1,3}, y_{1,1,3}] \\
&= [-x_2 + y_2, -x_0 + 2x_2 - x_3 + y] \\
&= [x_0, x_2] + [x_2, x_3] - [x_2, y] - [x_0, y_2] + [y_2, y] \\
&= [x_0, x_2] + [x_2, x_3] - [x_4, y] - [x_0, x_3] + [y_2, y],
\end{aligned}
$$

where we used $[x_0, y_2] = [x_0, x_3]$ from Table 3.3 and the relation 3.4.5. This yields

$$
[x_4, y] = [x_0, x_2] - [x_0, x_3] + [x_2, x_3] + [y_2, y]. \tag{3.4.6}
$$

Finally, the toric picture (12) (iii) gives

$$
\begin{aligned}
0 &= [x_{1,2,3}, y_{1,2,3}] \\
&= [-x_4 + y_2, -x_0 + 2x_2 - x_3 + y] \\
&= [x_0, x_4] - 2[x_2, x_4] + [x_3, x_4] - [x_4, y] - [x_0, y_2] + [y_2, y]
\end{aligned}
$$

Using relations $[x_0, y_2] = [x_0, x_3]$ and $[x_3, x_4] = -[x_0, x_2] + [x_0, x_4] + [x_2, x_3]$ from Table 3.3 and relation 3.4.6, this becomes

$$
\begin{aligned}
0 &= [x_0, x_4] - 2[x_2, x_4] + -[x_0, x_2] + [x_0, x_4] + [x_2, x_3] \\
&\quad - ([x_0, x_2] - [x_0, x_3] + [x_2, x_3] + [y_2, y]) - [x_0, x_3] + [y_2, y] \\
&= 2[x_0, x_4] - 2[x_2, x_4] - 2[x_0, x_2]
\end{aligned}
$$

We thus obtain the relation 3.4.1: $[x_0, x_4] = [x_0, x_2] + [x_2, x_4]$

To understand this new relation geometrically, we recall the circle actions (Figures 3.5 and 3.6).

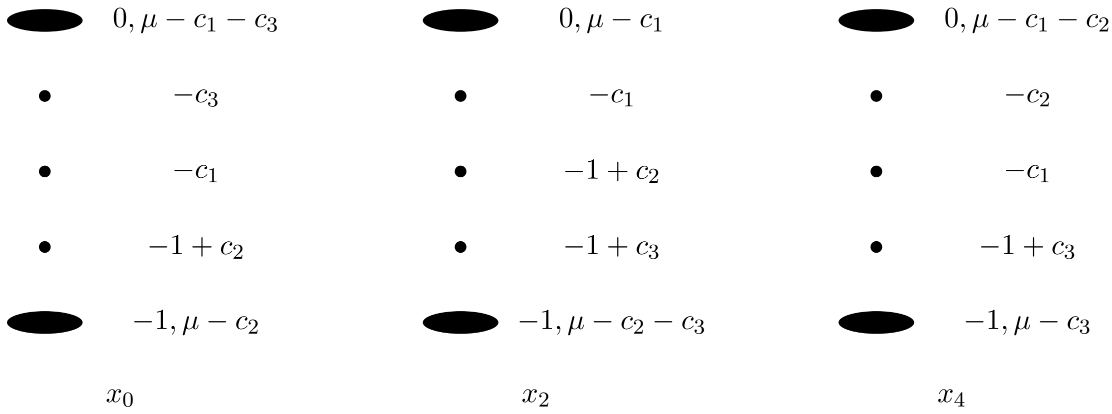


Figure 3.17: The S^1-actions x_0, x_2 and x_4

Looking at the Figure 3.17, we see that the roles of c_2 and c_3 are interchanged in the actions x_0 and x_4, and relation 3.4.1 tells us that these "commute" up to x_2. As in Section 3.3.2, looking at the graphs, we note that this relation does not depend on the value of μ. Since the map $T_{i,j} \to G_{\mu,c_1,c_2,c_3}$ allowing us to see these actions as elements of $\pi_1(G_{\mu,c_1,c_2,c_3})$ is injective, the relation must hold for $\mu = 1$ too.

Interchanging the roles of B and F (and multiplying the graphs of x_0, x_2 and x_4 by -1) we see that we obtain the graphs of y_2, y_0 and y_7, respectively.

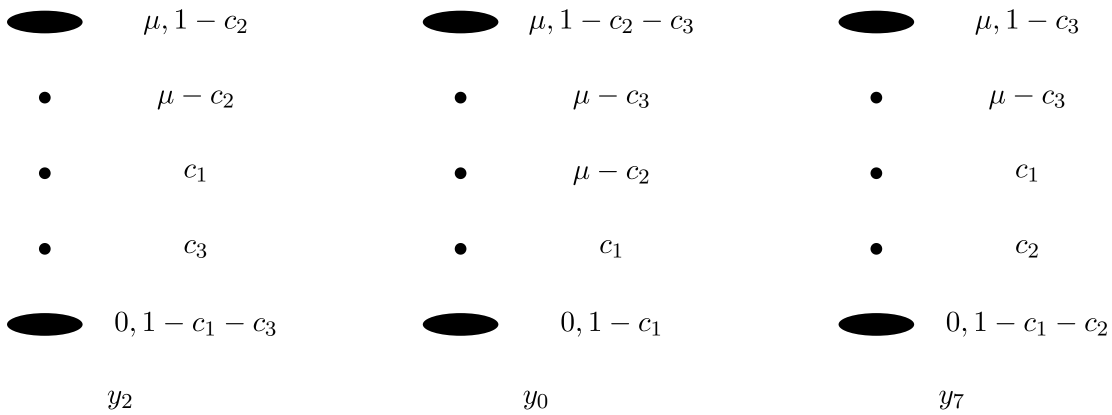


Figure 3.18: The S^1-actions y_2, y_0 and y_7

Hence, by symmetry, a similar result has to hold for these actions, namely: $[y_2, y_7] = [y_0, y_2] + [y_0, y_7]$.

Therefore, we can update Table 3.3 adding these two relations to reach the relations in the statement of the theorem. The following elements form a basis for $\pi_2(G_{c_1,c_2,c_3})$ as a vector space: $[x_0, x_1]$, $[x_0, x_2]$, $[x_0, x_3]$, $[x_0, z]$, $[x_1, x_2]$, $[x_1, x_3]$, $[x_1, y_7]$, $[x_2, x_3]$, $[x_2, x_4]$, $[x_3, z]$, $[x_4, y_7]$, $[y_0, y_2]$, $[y_0, y_7]$, $[y_0, z]$.

Let $\widetilde{\Lambda}$ denote the algebra defined in the statement of the theorem, that is, the Lie graded algebra generated by x_0, x_1, x_2, x_3, x_4, y_0, y_2, y_7, z, where all generators have degree 1 and the relations between the Samelson products of the generators hold as listed in the theorem. Let $\widetilde{\lambda}_n$ be the rank of $\widetilde{\Lambda}$ in dimension n. We now show that indeed $\widetilde{\Lambda}$ is isomorphic as an algebra to $\pi_*(G_{c_1,c_2,c_3}) \otimes \mathbb{Q}$. First we will show that they have the same rank in each dimension.

We need to recall a theorem due to Milnor and Moore. Let L be a graded Lie algebra and TL the tensor algebra on the graded vector space L. Take the ideal

$$I := \{x \otimes y \ - (-1)^{\deg x \deg y} y \otimes x - [x, y] \ : \ x, y \in L\},$$

where $[\cdot, \cdot]$ denotes the Lie bracket on L. We define the *universal enveloping algebra* $\mathcal{U}L$ of L as TL/I.

Theorem 3.4.2. *(Milnor-Moore, [26]) If X is a simply-connected topological space then*

(1) $\pi_*(\Omega X) \otimes \mathbb{Q}$ is a graded Lie algebra, L_X;

(2) The Hurewicz homomorphism extends to an isomorphism of graded algebras

$$\mathcal{U}L_X \xrightarrow{\cong} H_*(\Omega X; \mathbb{Q})$$

Then, combining this result with the Poincaré-Birkoff-Witt Theorem (see [9] Section 33, as applied to topological spaces) yields that $\widetilde{\lambda}_n$ satisfies

$$\sum_{n=0}^{+\infty} \widetilde{h'}_n z^n = \frac{\prod_{n=0}^{\infty}(1 + z^{2n+1})^{\widetilde{\lambda}_{2n+1}}}{\prod_{n=1}^{\infty}(1 - z^{2n})^{\widetilde{\lambda}_{2n}}} \tag{3.4.7}$$

where the left hand side of the equation is the Poincaré series of the universal enveloping algebra of $\widetilde{\Lambda}$, as described in the Milnor-Moore theorem.

In order to calculate $\widetilde{\lambda}_n$, we first establish some notation. Let

$$\widetilde{r}_n = \dim \pi_n(\Omega(\widetilde{M}_{c_1,c_2}) \otimes \mathbb{Q} = \dim \pi_{n+1}(\widetilde{M}_{c_1,c_2}) \otimes \mathbb{Q},$$
$$\widetilde{h}_n = \dim H_n(\Omega(\widetilde{M}_{c_1,c_2}), \mathbb{Q}),$$

the latter of which, as explained earlier in Section 3.2, satisfies $\widetilde{h}_0 = 1$, $\widetilde{h}_1 = 4$ and $\widetilde{h}_n = 4\widetilde{h}_{n-1} - \widetilde{h}_{n-2}$ for $n \geq 2$.

We also need to recall some notation and results from [5]. Let

$$h_n = \dim H_n(\Omega(\widetilde{M}_{c_1}), \mathbb{Q}).$$

Recall from [5] Section 4 that $h_0 = 1$, $h_1 = 3$, and $h_n = 3h_{n-1} - h_{n-2}$ for $n \geq 2$. Let Λ denote the Lie graded algebra $\pi_*(G_{\mu,c_1,c_2}) \otimes \mathbb{Q}$, where G_{μ,c_1,c_2}, for $\mu \geq 1$, is the symplectomorphism group of the 2-point blow-up M_{μ,c_1,c_2} of $S^2 \times S^2$ with capacities c_1, c_2. Let h'_n denote the coefficient of z^n of the Poincaré series of the universal enveloping algebra of Λ. As also explained in Section 4 of [5], these satisfy $h'_0 = 1$ and $h'_n = 5h_{n-1}$ for $n \geq 1$. Finally, let λ_n denote the rank of the algebra Λ in dimension n.

Proposition 3.4.3. *We have* $\dim \pi_1(G_{c_1,c_2,c_3}) = 9$*, and*

$$\dim \pi_n(G_{c_1,c_2,c_3}) = \dim \pi_n(G_{c_1,c_2}) + \dim \pi_{n+1}(\widetilde{M}_{c_1,c_2}), \text{ for } n > 1.$$

Proof. We recall the long exact sequence in the proof of Proposition 3.2.2:

$$\ldots \xrightarrow{\beta_{*+1}} \pi_{*+1}(\widetilde{M}_{c_1,c_2}) \xrightarrow{\gamma_{*+1}} \pi_*(G_{c_1,c_2,c_3}) \xrightarrow{\alpha_*} \pi_*(G_{c_1,c_2}) \xrightarrow{\beta_*} \pi_*(\widetilde{M}_{c_1,c_2}) \xrightarrow{\gamma_*} \ldots$$

We now revisit Remark 3.4.1. Indeed the Lie algebra $\pi_*(G_{c_1,c_2,c_3})$ contains a subalgebra, generated by x_2, x_4, y_2, y_7 and z, that is isomorphic to the Lie graded algebra of $G_{\mu=1,c_1,c_2}$ (described in [5]). The underlying reason for this is that the S^1-actions

on the manifold $\widetilde{M}_{\mu=1,c_1,c_2,c_3}$ are lifts of S^1-actions on $\widetilde{M}_{\mu=1,c_1,c_2}$. One easy way to see this is to consider the configurations #1.4, #2.2, #3.1, #4.1 and #5.1 in Section 2.2. These configurations correspond to toric pictures $\widetilde{T}_{i,j}(0)$ with the same labelling and thus yield, respectively,

$$[x_2, y_2] = [x_4, y_2] = [x_4, z] = [y_7, z] = [x_2, y_7] = 0.$$

Moreover, if we blow-down the exceptional curve E_3 in those configurations, we observe that the ones obtained downstairs, in $\widetilde{M}_{\mu=1,c_1,c_2}$, give the circle actions $\bar{x}_0$, $\bar{y}_0$, $\bar{x}_1$, $\bar{z}$, $\bar{y}_1$ (in accordance with the notation in [5]) which generate the algebra $\pi_*(G_{\mu=1,c_1,c_2})$. Therefore, we can conclude that the homotopy Lie algebra of $G_{\mu=1,c_1,c_2}$ lifts as a subalgebra of $\pi_*(G_{c_1,c_2,c_3})$.

This shows that the maps α_* are surjective and hence the maps β_* are zero, and this makes the maps γ_{*+1} injective. Hence we can update Proposition 3.2.2 to equality. ■

Lemma 3.4.4. *With the above notation, we have $\widetilde{\lambda}_n = \widetilde{r}_n + \lambda_n$.*

Proof. First we prove that $\widetilde{h'}_n = h'_n + 4\widetilde{h'}_{n-1} - \widetilde{h'}_{n-2}$ for $n \geq 2$: to compute $\widetilde{h'}_n$ for $n \geq 2$, we begin with h'_n elements coming from the enveloping algebra of Λ plus $4\widetilde{h'}_{n-1}$ elements: the number of classes in dimension $n-1$ times the four remaining generators (we should add $9\widetilde{h'}_{n-1}$ elements because, by Proposition 3.4.3, we have 9 generators in total, but $5\widetilde{h'}_{n-1}$ are already counted in h'_n). Then, due to the relations, we can check that we have to remove $\widetilde{h'}_{n-2}$ elements.

Next, we show $\widetilde{h'}_n = \sum_{i+j=n} \widetilde{h}_i h'_j$ by induction. It clearly holds for $n = 0$. For $n = 1$, the afore-mentioned results yield $\widetilde{h}_0 h'_1 + \widetilde{h}_1 h'_0 = 1 \cdot 5 + 4 \cdot 1 = 9$, which is indeed the number $\widetilde{h'}_1$ of generators of π_1. For the inductive step, we use $\widetilde{h'}_n = h'_n + 4\widetilde{h'}_{n-1} - \widetilde{h'}_{n-2}$,

as follows:

$$
\begin{aligned}
\widetilde{h'}_n &= h'_n + 4\widetilde{h'}_{n-1} - \widetilde{h'}_{n-2} \\
&= h'_n + 4(\widetilde{h}_{n-1}h'_0 + \widetilde{h}_{n-2}h'_1 + \widetilde{h}_{n-3}h'_2 + ... + \widetilde{h}_1 h'_{n-2} + \widetilde{h}_0 h'_{n-1}) \\
&\quad - (\widetilde{h}_{n-2}h'_0 + \widetilde{h}_{n-3}h'_1 + \widetilde{h}_{n-4}h'_2 + ... + \widetilde{h}_0 h'_{n-2}) \\
&= \widetilde{h}_0 h'_n + 4h'_{n-1} + (4\widetilde{h}_1 - \widetilde{h}_0)h'_{n-2} + ... \\
&\quad + (4\widetilde{h}_{n-3} - \widetilde{h}_{n-4})h'_2 + (4\widetilde{h}_{n-2} - \widetilde{h}_{n-3})h'_1 + (4\widetilde{h}_{n-1} - \widetilde{h}_{n-2})h'_0 \\
&= \widetilde{h}_0 h'_n + 4h'_{n-1} + \widetilde{h}_2 h'_{n-2} + ... + \widetilde{h}_{n-2}h'_2 + \widetilde{h}_{n-1}h'_1 + \widetilde{h}_n h'_0 \\
&= \widetilde{h}_0 h'_n + \widetilde{h}_1 h'_{n-1} + \widetilde{h}_2 h'_{n-2} + ... + \widetilde{h}_n h'_0 \\
&= \sum_{i+j=n} \widetilde{h}_i h'_j
\end{aligned}
$$

so that

$$
\begin{aligned}
&\frac{\prod_{n=0}^{\infty}(1+z^{2n+1})^{\widetilde{\lambda}_{2n+1}}}{\prod_{n=1}^{\infty}(1-z^{2n})^{\widetilde{\lambda}_{2n}}} = \sum_{n=0}^{+\infty} \widetilde{h'}_n z^n \quad = \sum_{n=0}^{+\infty}(\sum_{i+j=n} \widetilde{h}_i h'_j) z^n \\
&= \left(\sum_{n=0}^{\infty} \widetilde{h}_n z^n\right)\left(\sum_{n=0}^{\infty} h'_n z^n\right) \\
&= \left(\frac{\prod_{n=0}^{\infty}(1+z^{2n+1})^{\widetilde{r}_{2n+1}}}{\prod_{n=1}^{\infty}(1-z^{2n})^{\widetilde{r}_{2n}}}\right)\left(\frac{\prod_{n=0}^{\infty}(1+z^{2n+1})^{\lambda_{2n+1}}}{\prod_{n=1}^{\infty}(1-z^{2n})^{\lambda_{2n}}}\right) \\
&= \left(\frac{\prod_{n=0}^{\infty}(1+z^{2n+1})^{\widetilde{r}_{2n+1}+\lambda_{2n+1}}}{\prod_{n=1}^{\infty}(1-z^{2n})^{\widetilde{r}_{2n}+\lambda_{2n}}}\right).
\end{aligned}
$$

This gives the desired result. ■

Combining Proposition 3.4.3 and Lemma 3.4.4, we obtain

$$\dim \pi_n(G_{c_1,c_2,c_3}) = \dim \pi_n(G_{c_1,c_2}) + \dim \pi_{n+1}(\widetilde{M}_{c_1,c_2}) = \lambda_n + \widetilde{r}_n = \widetilde{\lambda}_n.$$

So we proved that the algebras $\pi_n(G_{c_1,c_2,c_3})$ and $\widetilde{\Lambda}$ have the same rank in each dimen-

sion.

In order to see that they are indeed isomorphic, it suffices to note that we already computed the relations in $\pi_*(G_{c_1,c_2,c_3})$ so that there is a homomorphism $\widetilde{\Lambda} \to \pi_*(G_{c_1,c_2,c_3})$. As the ranks of these algebras match, this homomorphism is in fact an isomorphism.

■

Remark 3.4.5. *We note that the generators and the relations stated in Theorem 1.3.2 do not depend on assumptions as $1 > c_1 + c_2 + c_3$ and $c_1 > c_2 + c_3$, although there are some S^1-actions and therefore some toric pictures that do not exist in those cases. They hold also when $1 \leq c_1 + c_2 + c_3$ and $c_1 \leq c_2 + c_3$, so the fundamental group and the rational homotopy Lie algebra of the group G_{c_1,c_2,c_3} do not depend on these conditions.*

Remark 3.4.6. *When we allow equalities in $1 > c_1+c_2 > c_1+c_3 > c_1 > c_2 > c_3$, some of our calculations work while others fail. For instance, in the case $c_1 = c_2 = 1/2$, the evaluation fibration*

$$Symp_p(\widetilde{M}_{1/2,1/2}) \to Symp(\widetilde{M}_{1/2,1/2}) \to \widetilde{M}_{1/2,1/2}$$

can be still used as the space $G_{1/2,1/2,c_3}$ is homotopy equivalent to $Symp_p(\widetilde{M}_{1/2,1/2})$. Since $Symp(\widetilde{M}_{1/2,1/2})$ is a torus, the long exact sequence yields that the rank of the fundamental group of $G_{1/2,1/2,c_3}$ is 6 in this case.

However, in other cases the computations are more complicated. In particular, when $c_2 = c_3$, the symplectomorphism group is not the stabilizer of a single point so we cannot retrieve information using the previous calculations.

The analysis of these special cases is still a work in progress.

3.5 Some applications

In this section, we will give some corolaries of Theorem 1.3.2. The first is the Pontryagin ring of the space G_{c_1,c_2,c_3}. We recall that for a topological group G, the Pontryagin product in $H_*(G;\mathbb{Z})$ is related to the Samelson product in $\pi_*(G)$ by the formula

$$[x,y] = xy - (-1)^{\deg x \deg y} yx, \text{ for } x, y \in \pi_*(G),$$

where we simplify the notation by suppressing the Hurewicz homomorphism $\rho : \pi_*(G) \to H_*(G;\mathbb{Z})$. We denote by $\mathbb{Q}\langle x_1, ..., x_n\rangle$ the free non-commutative algebra over $\mathbb{Q}$ with generators x_i. Then, applying Milnor-Moore theorem (Theorem 3.4.2) to the classifying space BG_{c_1,c_2,c_3} of the space of symplectomorphisms of $\widetilde{M}_{c_1,c_2,c_3}$, we obtain the following consequence of Theorem 1.3.2.

Corollary 3.5.1. *Let $1 > c_1 + c_2 > c_1 + c_3 > c_1 > c_2 > c_3$. Let G_{c_1,c_2,c_3} denote the group of symplectomorphisms of $\widetilde{M}_{c_1,c_2,c_3}$ that act trivially on homology. Then the Pontryagin ring of G_{c_1,c_2,c_3} is given by*

$$H_*(G_{c_1,c_2,c_3};\mathbb{Q}) = \mathbb{Q}\langle x_0, x_1, x_2, x_3, x_4, y_0, y_2, y_7, z\rangle / R$$

where all generators have degree 1, and R consists of the relations in Table 3.2 together with $x_0^2 = x_1^2 = x_2^2 = x_3^2 = x_4^2 = y_0^2 = y_2^2 = y_7^2 = z^2 = 0$.

We can also deduce information about the rational cohomology algebra of G_{c_1,c_2,c_3}.

Corollary 3.5.2. *Let $1 > c_1 + c_2 > c_1 + c_3 > c_1 > c_2 > c_3$. Let G_{c_1,c_2,c_3} denote the group of symplectomorphisms of $\widetilde{M}_{c_1,c_2,c_3}$ that act trivially on homology. Then the rational cohomology algebra of G_{c_1,c_2,c_3} is infinitely generated.*

Proof. By the Cartan-Serre theorem (see [27] Theorem 1.1 and Theorem 1.2), if the rational homology of G_{c_1,c_2,c_3} is finitely generated in each dimension, then its rational cohomology is a Hopf algebra for the cup product and the coproduct induced by the product in G_{c_1,c_2,c_3} and it is generated as an algebra by elements that are dual to the spherical classes in homology. Furhermore, the number of generators of odd dimension d appearing in the anti-symmetric part of the rational cohomology algebra

is equal to the dimension of $\pi_d(G_{c_1,c_2,c_3}) \otimes \mathbb{Q}$, and the number of generators of even dimension d appearing in the symmetric part of the rational cohomology algebra is equal to the dimension of $\pi_d(G_{c_1,c_2,c_3}) \otimes \mathbb{Q}$. Therefore, we conclude that the rational cohomology algebra of G_{c_1,c_2,c_3} is infinitely generated. ■

Chapter 4

Further discussion and some results for the case $\mu > 1$

Many results that we demonstrated for $\mu = 1$ also hold for $\mu > 1$, whereas some of the calculations become significantly cumbersome. Namely, the combinatorics becomes much more complex as μ increases, since the possible configurations of J-holomorphic curves increases rapidly, see Lemma 4.1.4 and Remark 4.2.1. Furthermore, when $\mu > 1$, not all symplectomorphism groups can be seen as stabilizers, see Theorem 4.2.2 and Remark 4.2.3.

In this chapter, we will briefly highlight some of the results that hold for $\mu > 1$ while at the same time pointing out the differences.

4.1 The structure of J-holomorphic curves for $\mu \geq 1$

The proofs of the following statements are similar to their counterparts in Chapter 2.

Lemma 4.1.1. *(See Lemma 2.1.1) Every symplectic form on $\widetilde{M}_{\mu,c_1,c_2,c_3}$ is, after rescaling, diffeomorphic to a form Poincaré dual to $\mu B + F - c_1E_1 - c_2E_2 - c_3E_3$ with $0 < c_3 \leq c_2 \leq c_1 < c_1 + c_3 \leq c_1 + c_2 \leq 1 \leq \mu$.*

Lemma 4.1.2. *(See Lemma 2.2.1) Let $J \in \mathcal{J}_{\mu,c_1,c_2,c_3}$. Suppose $A = pB + qF - r_1E_1 - r_2E_2 - r_3E_3 \in H_2(\widetilde{M}_{\mu,c_1,c_2,c_3}, \mathbb{Z})$ has a simple J-holomorphic representative. Then $p \geq 0$. Moreover*

- *if $p = 0$, then A is one of the followings: F, $F - E_i$, $F - E_i - E_j$, E_i, $E_i - E_j$, $E_1 - E_2 - E_3$, $i, j \in \{1, 2, 3\}$ and $i < j$;*
- *if $p = 1$, then $r_1, r_2, r_3 \in \{0, 1\}$.*

Lemma 4.1.3. *(See Lemma 2.2.2) Let $J \in \mathcal{J}_{\mu,c_1,c_2,c_3}$. Then E_3 is represented by a unique embedded J-curve. Hence, if $A = pB + qF - r_1E_1 - r_2E_2 - r_3E_3$ has a simple J-representative, then $r_3 \geq 0$.*

We now define the classes $D_k \in H_2(\widetilde{M}_{\mu,c_1,c_2,c_3}, \mathbb{Z})$, $k \in \mathbb{Z}$, by $D_{2k} = B + kF$, $D_{2k-1,i} = B + kF - E_i$, $D_{2k-2,i,j} = B + kF - E_i - E_j$ and $D_{2k-3} = B + kF - E_1 - E_2 - E_3$. Suppressing the indices i and j, we have $D_k \cdot D_k = k$, $c_1(D_k) = k + 2$ and $k(D_k) = \frac{1}{2}(D_k \cdot D_k + c_1(D_k)) = k + 1$, so that the adjunction formula $g_v(D_k) = 1 + \frac{1}{2}(D_k \cdot D_k - c_1(D_k)) = 0$ implies that any J-holomorphic sphere in class D_k must be embedded.

Lemma 4.1.4. *(Compare with 2.2.5) The set of tamed almost complex structures on $\widetilde{M}_{\mu,c_1,c_2,c_3}$ for which the classes $B - E_i$, $i = 1, 2, 3$ are represented by an embedded J-holomorphic sphere is open in dense in $J \in \mathcal{J}_{\mu,c_1,c_2,c_3}$.*

If for a given J there are no such spheres, then either one of the classes $B - E_i - E_j$ $i < j$ or $B - E_1 - E_2 - E_3$ is represented by a unique embedded J-holomorphic sphere, or there is a unique integer $1 \leq m \leq l$ such that one of the classes $D_{-2m} = B - mF$, $D_{-2m-1} = B - mF - E_i$, $D_{-2m-2} = B - mF - E_i - E_j$, $D_{-2m-3} = B - mF - E_1 - E_2 - E_3$ is represented by a unique embedded J-holomorphic sphere.

We thus get a set of possible configurations of J-holomorphic curves. This gives a lot of configurations, and we will not list them here.

4.2 Homotopy type of symplectomorphism groups for $\mu \geq 1$

The space G_{μ,c_1,c_2,c_3} is connected, by [18].

Remark 4.2.1. *Keeping Theorem 3.1.1 in mind, it would be reasonable to conjecture the following statement:*

Let $\mu = l + \lambda$ *with* $l \in \mathbb{N}$ *and* $0 < \lambda \leq 1$. *Given* $\mu' \in (l, l+1]$, *write* $\mu' = l + \lambda'$ *with* $0 < \lambda' \leq 1$. *Consider* $c_1, c_2, c_3, c_1', c_2', c_3'$ *such that either*

- $\lambda \leq c_3 \leq c_2 \leq c_1 < c_1 + c_3 \leq c_1 + c_2$ *and* $\lambda' \leq c_3' \leq c_2' \leq c_1' < c_1' + c_3' \leq c_1' + c_2'$; *or*
- $c_3 < \lambda \leq c_2 \leq c_1 < c_1 + c_3 \leq c_1 + c_2$ *and* $c_3' < \lambda' \leq c_2' \leq c_1' < c_1' + c_3' \leq c_1' + c_2'$; *or*
- $c_3 \leq c_2 < \lambda \leq c_1 < c_1 + c_3 \leq c_1 + c_2$ *and* $c_3' \leq c_2' < \lambda' \leq c_1' < c_1' + c_3' \leq c_1' + c_2'$; *or*
- $c_3 \leq c_2 \leq c_1 < \lambda \leq c_1 + c_3 \leq c_1 + c_2$ *and* $c_3' \leq c_2' \leq c_1' < \lambda' \leq c_1' + c_3' \leq c_1' + c_2'$; *or*
- $c_3 \leq c_2 \leq c_1 < c_1 + c_3 < \lambda \leq c_1 + c_2$ *and* $c_3' \leq c_2' \leq c_1' < c_1' + c_3' < \lambda' \leq c_1' + c_2'$.
- $c_3 \leq c_2 \leq c_1 < c_1 + c_3 \leq c_1 + c_2 \leq \lambda$ *and* $c_3' \leq c_2' \leq c_1' < c_1' + c_3' \leq c_1' + c_2' \leq \lambda'$.

Then the symplectomorphism groups G_{μ,c_1,c_2,c_3} *and* $G_{\mu',c_1',c_2',c_3'}$ *are homotopy equivalent.*

The proof of such a statement would rely on inflation, as is detailed in Appendix A for the case $\mu = 1$, *which would require the list of all possible configurations of J-holomorphic spheres for each of the cases above and a selection for each configuration of a set of curves to apply the inflation procedure.*

Theorem 4.2.2. *(See Theorem 3.1.4) Consider* $c_1, c_2, c_3 \in (0,1)$ *such that either*

(1) $c_3 < \lambda \leq c_2 < c_1 < c_1 + c_3 < c_1 + c_2 < 1$;

(2) $c_3 < c_2 < \lambda \leq c_1 < c_1 + c_3 < c_1 + c_2 < 1$*;*

(3) $c_3 < c_2 < c_1 < c_1 + c_3 < \lambda \leq c_1 + c_2 < 1$*; or*

(4) $c_3 < c_2 < c_1 < c_1 + c_3 < c_1 + c_2 \leq \lambda \leq 1$*.*

Then G_{μ,c_1,c_2,c_3} *is homotopy equivalent to* $Symp_p(\widetilde{M}_{\mu,c_1,c_2})$*.*

Remark 4.2.3. *The proof of Theorem 3.1.4 does not hold if* $\lambda \leq c_3$*. When* $\lambda \leq c_3$*, the group does not have the homotopy type of a stabilizer, because we cannot use Theorem 3.1.1 to make* c_3 *very small. (This step in the proof is crucial to approximate symplectomorphisms that act linearly in a small neighborhood of the exceptional fiber whose cohomology class is* E_3 *by symplectomorphisms that fix a point* p*). The same is true when* $c_1 < \lambda \leq c_1 + c_3$*. We therefore excluded these cases above.*

As for the homotopy groups of G_{μ,c_1,c_2,c_3}, we can still implement the same tools (Poincaré series, Poincaré-Birkoff-Witt Theorem, spectral sequence) as in Section 3.2 to calculate $\widetilde{r}_n = \dim \pi_n(\Omega \widetilde{M}_{\mu,c_1,c_2,c_3}) \otimes \mathbb{Q} = \dim \pi_{n+1}(\widetilde{M}_{\mu,c_1,c_2,c_3}) \otimes \mathbb{Q}$.

Proposition 4.2 of [5], together with Remark 4.2.1 and the evaluation fibration

$$Symp_p(\widetilde{M}_{\mu,c_1,c_2}) \to Symp(\widetilde{M}_{\mu,c_1,c_2}) \to \widetilde{M}_{\mu,c_1,c_2}$$

would give us information on the homotopy groups of G_{μ,c_1,c_2,c_3}.

The remaining discussion on the "jumping generator" (coined by Abreu-McDuff in [3]) and the calculation of homotopy groups of G_{μ,c_1,c_2,c_3} is beyond the scope of this work. We expect that it requires not only a more profound study but also a more robust approach, for the following reasons. Firstly, each time μ crosses an integer, the number of actions (and strata) to consider increases by about 60. Thus, labels for the configurations and isometry groups produce a rather cumbersume presentation. Secondly, depending on the relation of λ to the c_i, only a selection of these configurations would be allowed. This implies a case study with six cases every time μ crosses an integer value. Thirdly, all these bulky lists make the inflation procedure even harder to follow.

Appendices

Appendix A

Proof of Theorem 3.1.1

This section is dedicated to complement the proof of Theorem 3.1.1. As the calculations are similar in all cases, we will confine ourselves with listing the curves used in inflation.

Step 1: $\mathcal{A}_{c_1',c_2,c_3} = \mathcal{A}_{c_1,c_2,c_3}$ for $c_1 \leq c_1'$.

The inclusion $\mathcal{A}_{c_1',c_2,c_3} \subset \mathcal{A}_{c_1,c_2,c_3}$ was proved previously. To show $\mathcal{A}_{c_1,c_2,c_3} \subset \mathcal{A}_{c_1',c_2,c_3}$, we use Table A.1 to pick the curves for each configuration.

Configurations	Curves
#1: 1-13; #2: 1,2,3,6,7,9,10	$B+F-E_1$, $B+F-E_2$, $2B+2F-E_1-E_2-E_3$
#2: 4,5,8	$B+F-E_1$, $B+F-E_2$, $3B+3F-E_1-E_2-E_3$
#3: 1,2,6; #6: 1,2,7	$B+F-E_1$, $B+F-E_3$, $2B+2F-E_1-E_2-E_3$
#3: 3; #6: 3	$B+F-E_1$, $2B+2F-E_1-E_2$, $F-E_1-E_3$
#3: 4,8; #6: 4,6	$B+F-E_1$, $2B+2F-E_1-E_2$, $E_1-E_2-E_3$
#3: 5,7; #6: 5	$B+F-E_1$, $2B+2F-E_1-E_2$, $3B+3F-E_1-E_2-E_3$

Table A.1: Inflation process to show $\mathcal{A}_{c_1,c_2,c_3} \subset \mathcal{A}_{c_1',c_2,c_3}$

Remark A.0.1. *Note that in the above table the first two rows look quite similar. However, we cannot simply interchange their roles, as the configurations in the second row have $B-E_1-E_2-E_3$ represented by a J-holomorphic curve and indeed the proof in this case uses the inequality $c_1+c_2+c_3<1$. In the first row, on the other hand,*

there is no a priori reason to assume this inequality.

Since interchanging the roles os B and F in the configurations #2 and #3 yield the configurations #5 and #4, respectively, Table A.1 implies all the cases.

Step 2: $\mathcal{A}_{c_1,c_2',c_3} = \mathcal{A}_{c_1,c_2,c_3}$ for $c_2 \leq c_2'$.

Starting with the inclusion $\mathcal{A}_{c_1,c_2',c_3} \subset \mathcal{A}_{c_1,c_2,c_3}$, using negative inflation along the curve E_2 suffices for the following configurations: #1: 1,4-8,10-13; #2: 1-4,7,8,10; #3: 1,2,3,6,8; #6: 1,2,3,6,7. For the remaining cases, we use $B+F+E_1$, $B+F$ and $E_2 - E_3$.

For the reverse inclusion, we use Table A.2. Here too, since interchanging the roles of B and F yield #4 and #5 too, this table covers all the possible configurations.

Configurations		Curves
#1: 1-13; #2: 1-3,6,7,9,10		$B+F-E_1$, $B+F-E_2$, $2B+2F-E_1-E_2-E_3$
#2: 4,5,8		$B+F-E_1$, $B+F-E_2$, $3B+3F-E_1-E_2-E_3$
#3: 1,2,7; #6: 1,2,5,7	$2c_1 < 1$	$B+F$, $2B+2F-E_1-E_2$, $2B+2F-E_1-E_2-E_3$
	$2c_1 \geq 1$	$B+F-E_1$, $2B+2F-E_1-E_2$, $2B+2F-E_1-E_2-E_3$
#3: 3,4,8; #6: 3,4,6	$2c_1+2c_3 < 1$	$B+F$, $2B+2F-E_1-E_2$, $E_1-E_2-E_3$
	$2c_1+2c_3 \geq 1$	$B+F-E_1$, $2B+2F-E_1-E_2$, $E_1-E_2-E_3$
#3: 5,6		$B+F-E_1$, $3B+3F-E_1-E_2-E_3$, E_1-E_2

Table A.2: Inflation process to show $\mathcal{A}_{c_1,c_2,c_3} \subset \mathcal{A}_{c_1,c_2',c_3}$

Step 3: $\mathcal{A}_{c_1,c_2,c_3'} = \mathcal{A}_{c_1,c_2,c_3}$ for $c_3 \leq c_3'$.

To show the inclusion $\mathcal{A}_{c_1,c_2,c_3'} \subset \mathcal{A}_{c_1,c_2,c_3}$, it suffices to use negative inflation over E_3, for, by Lemma 2.2.2, E_3 is represented by an embedded J-curve for all J.

For the reverse inclusion, we pick the curves in Table A.3.

Configurations		Curves
#1: 1,4,7,8,10,11,13; #2:1,2,10; #3: 1,2; #6: 1,2,7		$B+F-E_1$, $B+F-E_3$, $2B+2F-E_1-E_2-E_3$
#1: 2,3,9; #2: 6,9	$2c_1 < 1$	$B+F-E_2$, $2B+2F-E_1-E_2-E_3$, E_2-E_3
	$2c_1 \geq 1$	$B+F-E_1$, $2B+2F-E_1-E_2-E_3$, E_2-E_3
#1: 5,6,12; #2: 3,7		$B+F-E_1$, $2B+2F-E_1-E_2-E_3$, E_1-E_3
#2: 4,8	$c_1+2c_2 < 1$	$B+F-E_1$, $3B+3F-E_1-E_2-E_3$, E_1-E_3
	$c_1+2c_2 \geq 1$	$B+F-E_1$, $3B+3F-E_1-E_2-E_3$, $2B+2F-E_1-E_2$
#2: 5		$B+F-E_1$, $B+F-E_2$, E_2-E_3
#3: 3,8; #6: 3,6	$2c_1+2c_2 < 1$	$B+F$, $2B+2F-E_1-E_3$, $E_1-E_2-E_3$
	$2c_1+2c_2 \geq 1$	$B+F-E_1$, $2B+2F-E_1-E_3$, $E_1-E_2-E_3$
#3: 4,5,7; #6: 4,5	$2c_1 < 1$	$B+F$, $2B+2F-E_1-E_2$, E_2-E_3
	$2c_1 \geq 1$	$B+F-E_1$, $2B+2F-E_1-E_2$, E_2-E_3
#3: 6	$c_1+2c_2 < 1$	$B+F-E_1$, $3B+3F-E_1-E_2-E_3$, $B+F-E_3$
	$c_1+2c_2 \geq 1$	$B+F-E_1$, $3B+3F-E_1-E_2-E_3$, $2B+2F-E_1-E_2$

Table A.3: Inflation process to show $\mathcal{A}_{c_1,c_2,c_3} \subset \mathcal{A}_{c_1,c_2,c_3'}$

Finally, as noted previously, Table A.3 covers the configurations #4 and #5 as well. This finishes Step 3 and the proof of the theorem.

Appendix B

The correspondence between isometry groups and configurations

In this section, we will state some of the basic results on the differential and topological aspects of the space $\mathcal{J}_\omega = \tilde{\mathcal{J}}_{c_1,c_2,c_3}$ of compatible almost complext structures on $\widetilde{M}_{c_1,c_2,c_3}$. Our main goal is to illustrate the correspondence between isometry groups and configurations, as was used in Section 3.3.

By Chapter 2, we know that the space $\mathcal{J}_\omega$ is a disjoint union of finitely many strata, each of which is characterized by the existence of a unique chain of holomorphic spheres (configurations). Our first remark is that these sets indeed give a stratification of $\mathcal{J}_\omega$ into Fréchet manifolds.

Proposition B.0.1. *Let $U_\mathcal{A} \subset \mathcal{J}_\omega$ be a stratum characterized by the existence of a configuration of J-holomorphic embedded spheres $C_1 \cup C_2 \cup ... \cup C_N$ representing a given set of distinct homology classes, $\mathcal{A} = \{A_1, ..., A_N\}$ of negative self-intersection. Then $U_\mathcal{A}$ is a cooriented Fréchet submanifold of $\mathcal{J}_\omega$ of real codimension $2N - 2c_1(A_1 + ... + A_N)$.*

Proof. The proof of this proposition is identical to the proof of Proposition 7.1 in [5]. ■

Consider the subspace $\mathcal{J}_\omega^{int} \subset \mathcal{J}_\omega$ of compatible, integrable, complex structures. Given a stratum U_i, set $V_i = U_i \cap \mathcal{J}_\omega^{int}$.

Lemma B.0.2. *For any $J \in \mathcal{J}_\omega^{int}$, the complex surface $(\mathbb{X}_4, J)$ is the 3-fold blow-up of a Hirzebruch surface.*

Proof. Given $J \in \mathcal{J}_\omega^{int}$, the class E_3 is always represented by an exceptional curve. Blowing it down yields a surface $(\mathbb{X}_3, J)$, which is a twofold blow-up of a Hirzebruch surface by Lemma 7.2 in [5]. See Chapter 2 for an explanation of how these configurations appear. ■

Remark B.0.3. *As detailed in [2] (Theorem 2.3) and [5] (Lemma 7.5), the space $\mathcal{J}_\omega^{int}$ is in fact a Fréchet submanifold of $\mathcal{J}_\omega$.*

We denote the group of diffeomorphisms of $\mathbb{X}_4$ acting trivially on homology by Diff_h, let $\mathrm{Aut}_h(J) \subset \mathrm{Diff}_h$ be the subgroup of complex automorphisms of (M, J) and $\mathrm{Iso}_h(\omega, J) \subset \mathrm{Aut}_h(J)$ be the Kähler isometry group of (M, ω, J).

Proposition B.0.4. *Let $J_1, J_2 \in V_i$ be two integrable compatible structures in the same stratum. Then there exists $\phi \in Diff_h$ such that $J_2 = \phi_* J_1$. Hence we have*

$$V_i = U_i \cap \mathcal{J}_\omega^{int} = (Diff_h \cdot J_i) \cap \mathcal{J}_\omega, \text{ for any } J_i \in V_i.$$

Proof. The proof is similar to Proposition 7.3 in [5] with the only difference being that in our case the construction of compatible complex structures on $\mathbb{X}_4$ can be made in more numerous ways, resulting in a total of 56 types that yield the configurations listed at the end of Chapter 2.

Briefly, for $\mathbb{X}_3$, the one-point blow-down of our space, we have exactly 6 types of compatible complex structures:

(1) Twofold blow-up of $\mathbb{F}_0$ at two generic points (not lying on the same fiber F nor on the same section B).

(2) Twofold blow-up of $\mathbb{F}_0$ at two distinct points on the same fiber F.

(3) Twofold blow-up of $\mathbb{F}_0$ at two distinct points on the same section B.

(4) Twofold blow-up of $\mathbb{F}_0$ at two "infinitely near" points on a fiber, that is, the blow-up of $\mathbb{F}_0$ at p followed by the blow-up of $\widetilde{\mathbb{F}}_0$ at the line $l_p = T_pF \subset T_p\mathbb{F}_0$ on the exceptional divisor.

(5) Twofold blow-up of $\mathbb{F}_0$ at two "infinitely near" points on a flat section B, that is, at $(p, l_p = T_pB)$.

(6) Twofold blow-up of $\mathbb{F}_0$ at two "infinitely near" points (p, l_p) with the direction l_p transverse to T_pF and T_pB.

To get $\mathbb{X}_4$, we consider a third blow-up on each of these types. This produces the 56 types mentioned above, and we refer the reader to the Hirzebruch surfaces in Section 3.3 to observe where the last blow-up may take place.

For instance, consider the toric picture 1(iv).

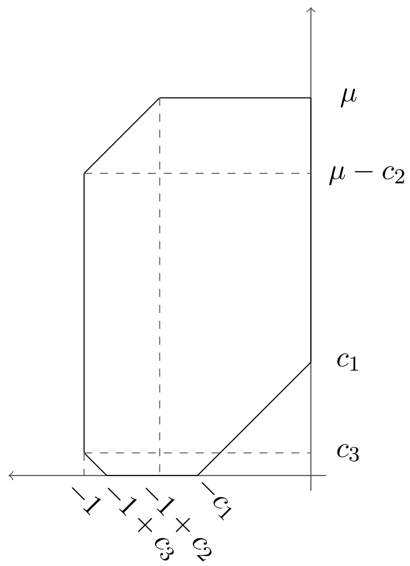


Figure B.1: Toric action 1(iv)

This figure is obtained by blowing up $\mathbb{F}_0$ at three distinct points p_1, p_2 and p_3 such that p_1 and p_2 lie on the same fiber F, and p_2 and p_3 lie on the same section B. Since the complex automorphism group of $\mathbb{F}_0$ is isomorphic to

$$\mathrm{Aut}(\mathbb{F}_0) \simeq (\mathrm{PSL}(2,\mathbb{C}) \times \mathrm{PSL}(2,\mathbb{C})) \ltimes \mathbb{Z}_2,$$

it follows that the automorphism group of the 3-point blow-up must act transitively on the triple (p_1, p_2, p_3) that defines the almost complex structure.

∎

Now, to prove the correspondence between isometry groups and configurations, we start by recalling the following fact due to Abreu-Granja-Kitchloo:

Theorem B.0.5. *([2], Corollary 2.6) If $J \in \mathcal{J}_\omega^{int}$ is such that the inclusion*

$$Iso_h(\omega, J) \hookrightarrow Aut_h(J)$$

is a weak homotopy equivalence, then the inclusion of the $Symp_h(M,\omega)$-orbit of J in $(Diff_h \cdot J) \cap \mathcal{J}_\omega^{int}$

$$Symp_h(M,\omega)/Iso_h(\omega, J) \hookrightarrow (Diff_h \cdot J) \cap \mathcal{J}_\omega^{int}$$

is also a weak homotopy equivalence.

Lemma B.0.6. *For any $J \in \mathcal{J}^{int}(\widetilde{M}_{c_1,c_2,c_3})$, the inclusion $Iso_h(\omega_{c_1,c_2,c_3}, J) \hookrightarrow Aut_h(J)$ is a weak homotopy equivalence.*

Proof. We start by recalling that if $(\widetilde{M}, \widetilde{J})$ is the blow-up of (M, J) at a point p, then the complex automorphism group of $\widetilde{J}$ is isomorphic to the stabilizer subgroup of p in the automorphism group of J.

Going through all possible types of compatible almost complext structures, we can see that the space $\mathrm{Aut}_h(\mathbb{X}_4, J)$ is homotopy equivalent to either a point, S^1 or T^2. More specifically, looking at the configurations in Chapter 2, we get:

$$\mathrm{Aut}_h(\mathbb{X}_4, J) \simeq \begin{cases} * & \text{if } J \text{ corresponds to } \#1{:}13 \\ T^2 & \text{if } J \text{ corresponds to } \#1{:}1\text{-}6,\ \#2{:}1\text{-}6,\ \#3{:}1\text{-}6,\ \#4{:}1\text{-}6,\ \#5{:}1\text{-}6 \\ S^1 & \text{if } J \text{ corresponds to one of the remaining configurations} \end{cases}$$

On the other hand, the isometry groups of the Hirzebruch surfaces are the maximal compact Lie subgroups of their complex automorphism groups:

$$\mathrm{Iso}(\mathbb{F}_i, \omega) \simeq \begin{cases} (SO(3) \times SO(3)) \ltimes \mathbb{Z}_2 & \text{for } i = 0 \\ U(2) & \text{for } i = 1 \end{cases}$$

In particular, they are deformation retracts of $\mathrm{Aut}(\mathbb{F}_m)$, and after blow-up they induce isometry groups $\mathrm{Iso}_h(\mathbb{X}_4, \omega, J)$ that are homotopy equivalent to the cases designated above.

For instance, recalling Figure B in the proof of Proposition B, we see that its automorphism group is homotopy equivalent to the isometry group of the almost complex structure in the stratum characterized by # 1.4:

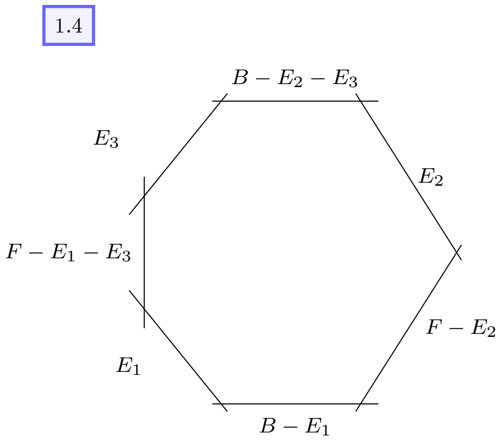


Figure B.2: Configuration 1.4 revisited

■

Combining the last three results, we get:

Corollary B.0.7. *Given $J \in V_i \subset \mathcal{J}_\omega^{int}$, there is a weak homotopy equivalence*

$$Symp_h(\widetilde{M}_{c_1,c_2,c_3}, \omega)/Iso_h(\omega, J) \simeq V_i.$$

Proposition B.0.8. *The action of $Symp(\widetilde{M}_{c_1,c_2,c_3})$ on $\mathcal{J}_\omega$ is homotopy equivalent to*

its restriction to $\mathcal{J}_\omega^{int}$

Proof. As in [5], Lemma 7.11, given a set $\mathcal{A} = \{A_1, ..., A_N\}$ of distinct spherical homology classes of negative self-intersections, $U_\mathcal{A}$ its stratum in $\mathcal{J}_\omega$, C the unique configuration of J-holomorphic spheres of type $\mathcal{A}$, and $u = (u_1, ...u_N)$ a J-holomorphic parametrization of C, it is possible to show that the induced map $u* : H^{0,1}(T\mathbb{X}_4) \to H^{0,1}(u * (T\mathbb{X}_4))$ is surjective. The result then follows from a result due to Abreu-Granja-Kitchloo ([2], Theorem 2.9). ■

Corollary B.0.9. *The space* $\mathcal{J}_\omega^{int}$ *of compatible integrable almost complex structures on* $\widetilde{M}_{c_1,c_2,c_3}$ *is contractible.*

Appendix C

The possible toric actions for $\mu > 1$

Following a similar line of thought to that of Section 3.3, we can draw the possible Delzant polytopes for $\widetilde{M}_{\mu,c_1,c_2,c_3}$. In this chapter, we will briefly summarize this construction and use Karshon's classification of Hamiltonian circle actions to list the relations between these actions.

Let $T^4 \subset U(4)$ act in the standard way on $\mathbb{C}^4$. Given an integer $n \geq 0$, the action of the subtorus $T^2_n := (ns+t, t, s, s)$ is Hamiltonian with moment map

$$(z_1, ..., z_4) \mapsto (n|z_1|^2 + |z_3|^2 + |z_4|^2, |z_1|^2 + |z_2|^2).$$

We identify $(S^2 \times S^2, \mu\sigma \oplus \sigma)$ with each of the toric Hirzebruch surfaces $\mathbb{F}^{2k}_\mu$, $0 \leq k \leq l$, defined as the symplectic quotient $\mathbb{C}^4//T^2_{2k}$ at the regular value $(\mu + k, 1)$ endowed with the residual action of the torus $T(2k) := (0, u, v, 0) \subset T^4$. The image $\Delta(2k)$ of the moment map is the convex hull of

$$\{(0,0), (1,0), (1, \mu + k), (0, \mu - k)\}.$$

Similarly, we identify $(S^2 \tilde{\times} S^2, \omega_\mu)$ with the toric Hirzebruch surface $\mathbb{F}^\mu_{2k-1}$, $1 \leq k \leq l$, defined as the symplectic quotient $\mathbb{C}^4//T^2_{2k-1}$ at the regular value $(\mu + k, 1)$. The image $\Delta(2k-1)$ of the moment map of the residual action of the torus $(0, u, v, 0)$ is the convex hull of

$$\{(0,0),(1,0),(1,\mu+k),(0,\mu-k+1)\}.$$

Since the group $\mathrm{Symp}_h(M_\mu)$ of symplectomorphisms acting trivially on homology is connected, any two identifications of $\mathbb{F}_n^\mu$ with the spaces $S^2\times S^2$ and $S^2\tilde{\times}S^2$ are isotopic and lead to isotopic identifications of $\mathrm{Symp}_h(\mathbb{F}_n^\mu)$ with the respective symplectomorphism groups.

We identify the symplectic blow-up $\widetilde{M}_{\mu,c_1}$ at a ball of capacity c_1 with the equivariant blow-up of the Hirzebruch surfaces $\mathbb{F}_n^\mu$.

We define the even torus action $\widetilde{T}(2k)$ as the equivariant blow-up of the toric action of $T(2k)$ on $\mathbb{F}_{2k}^\mu$ at the fixed point $(0,0)$ with capacity c_1. The image of the moment map then is the convex hull of

$$\{(1,\mu+k),(0,\mu-k),(0,c_1),(c_1,0),(1,0)\}.$$

Similarly, we define the odd torus action $\widetilde{T}(2k-1)$ as the equivariant blow-up of the toric action of $T(2k-1)$ on $\mathbb{F}_{2k-1}^\mu$ at the fixed point $(0,0)$ with capacity $1-c_1$. The image of the moment map then is the convex hull of

$$\{(1,\mu-c_1+k),(0,\mu-c_1-k+1),(0,1-c_1),(1-c_1,0),(1,0)\}.$$

Note that when $c_1<c_{crit}:=\mu-l$, $\widetilde{M}_{\mu,c_1}$ admits exactly $2l+1$ inequivalent toric structures $\widetilde{T}(0),...,\widetilde{T}(2l)$, while when $c_1\geq c_{crit}$, it admits $2l$ of those, namely $\widetilde{T}(0),...,\widetilde{T}(2l-1)$.

The Kähler isometry group of $\mathbb{F}_n^\mu$ is $N(T_n^2)/T_n^2$ where $N(T_n^2)$ is the normalizer of T_n^2 in $U(4)$. There is a natural isomorphism $N(T_0^2)/T_0^2\simeq SO(3)\times SO(3):=K(0)$, while for $k\geq 1$, we have $N(T_{2k}^2)/T_{2k}^2\simeq S^1\times SO(3):=K(2k)$ and $N(T_{2k-1}^2)/T_{2-1}^2\simeq U(2):=K(2k-1)$ The restrictions of these isomorphisms to the maximal tori are given in coordinates by

$(u,v)\mapsto(-u,v)\in T(0):=S^1\times S^1\subset K(0)$

$(u,v)\mapsto(u,ku+v)\in T(2k):=S^1\times S^1\subset K(2k)$

$(u,v)\mapsto(u+v,ku+(k-1)v)\in T(2k-1):=S^1\times S^1\subset K(2k-1)$.

These identifications imply that the moment polygon associated to the maximal tori $T(n) = S^1 \times S^1 \subset K(n)$ and $\widetilde{T}(n)$ are the images of $\Delta(n)$ and $\widetilde{\Delta}(n)$, respectively, under the transformations $C_n \in GL(2,\mathbb{Z})$ given by

$$C_0 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \ C_{2k} = \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} \text{ and } C_{2k-1} = \begin{pmatrix} 1-k & k \\ 1 & -1 \end{pmatrix}.$$

Under the blow-down map, $\widetilde{T}(n)$ is sent to the maximal torus of $K(n)$ for all $n \geq 0$. By [17] $\mathrm{Symp}(\widetilde{M}_{\mu,c_1})$ is connected, hence the choices involved in these identifications give the same maps up to homotopy.

We identify the symplectic blow-up $\widetilde{M}_{\mu,c_1,c_2}$ with the equivariant two blow-up of the Hirzebruch surfaces $\mathbb{F}_n^{\mu}$ and obtain inequivalent toric structures. We define the torus actions $\widetilde{T}_i(2k)$, $\widetilde{T}_i(2k-1)$, $i = 1, ..., 5$ as the equivariant blow-ups of the toric action of $T(n)$ on $\widetilde{\mathbb{F}}_{2k}^{\mu}$ and $\widetilde{\mathbb{F}}_{2k-1}^{\mu-c_1}$ respectively, with capacity c_2, at each one of the five fixed points, which correspond to the vertices of the moment polygon $\widetilde{\Delta}(n)$.

We blow-up each of the resulting toric actions in their six fixed points and obtain $\widetilde{T}_{i,j}(2k)$, $\widetilde{T}_{i,j}(2k-1)$, $i = 1, ..., 5$, $j = 1, ..., 6$. These toric pictures arise from the ones described in Section 4.2 of [5], by blowing up once more at a ball of capacity c_3. We will draw the resulting figures.[1] In each case, we will further pick two Hamiltonian S^1-actions, $(x_{2k-1,i,j}, y_{2k-1,i,j})$ or $(x_{2k,i,j}, y_{2k,i,j})$, as we did in Section 3.3.

[1] The numbering of the figures will register the corresponding configurations of J-holomorphic spheres, as depicted in Figures 3-6 in [5].

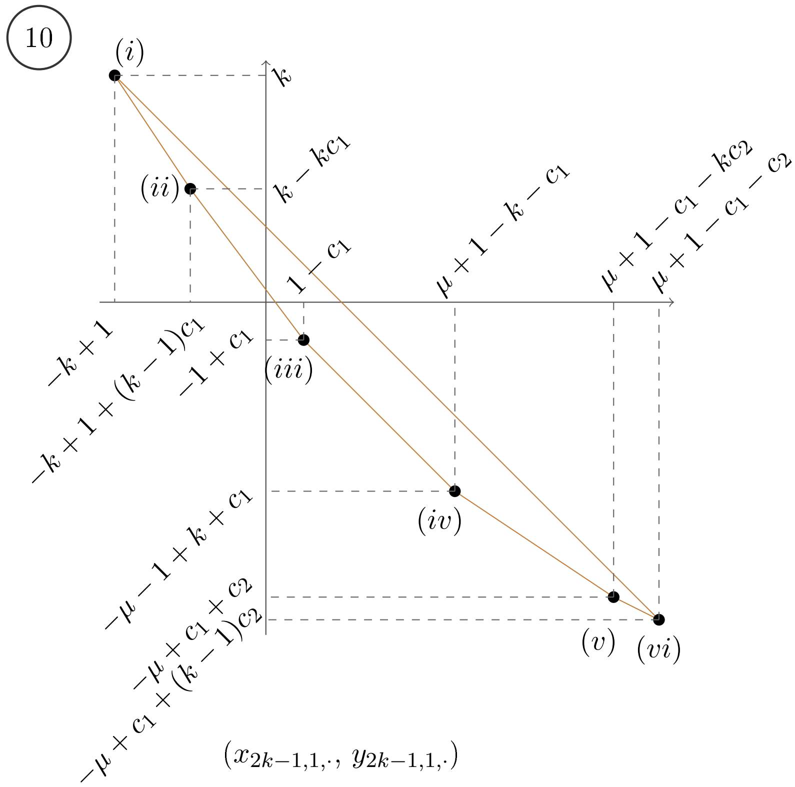
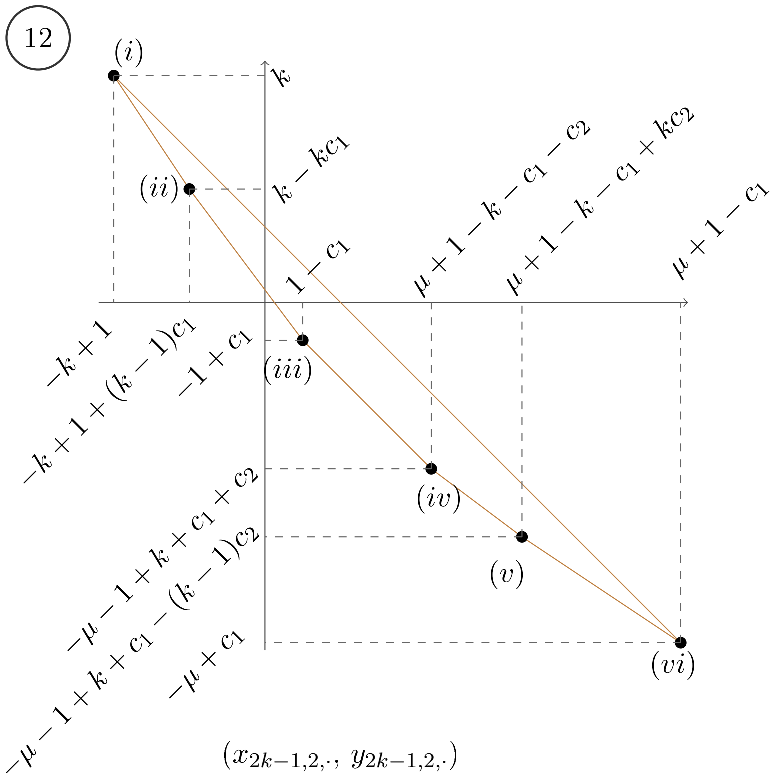


Figure C.1: The Delzant polytope corresponding to Configuration (10) of [5] for $\mathbb{X}_3$, with the next blow-ups plotted



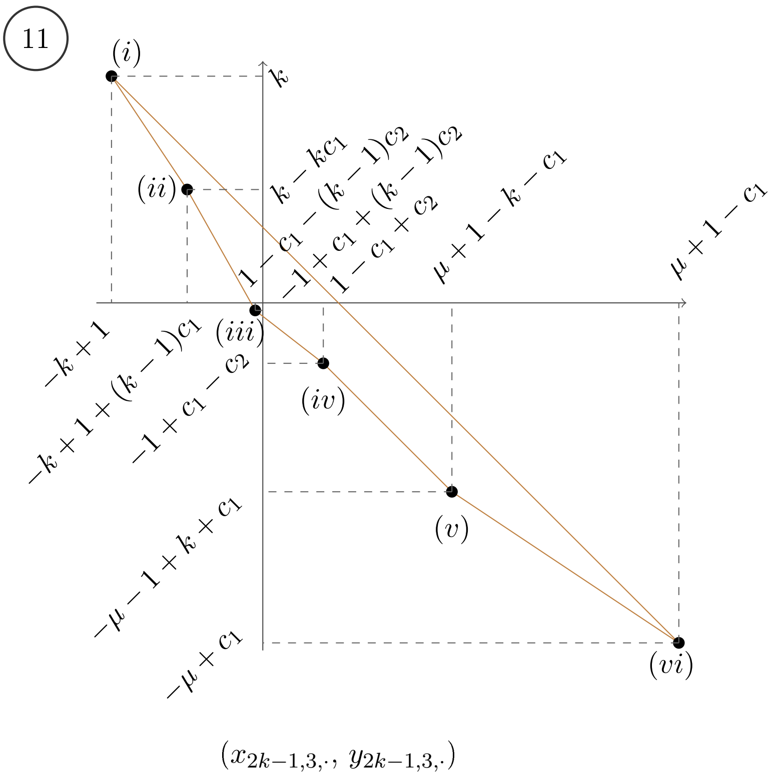
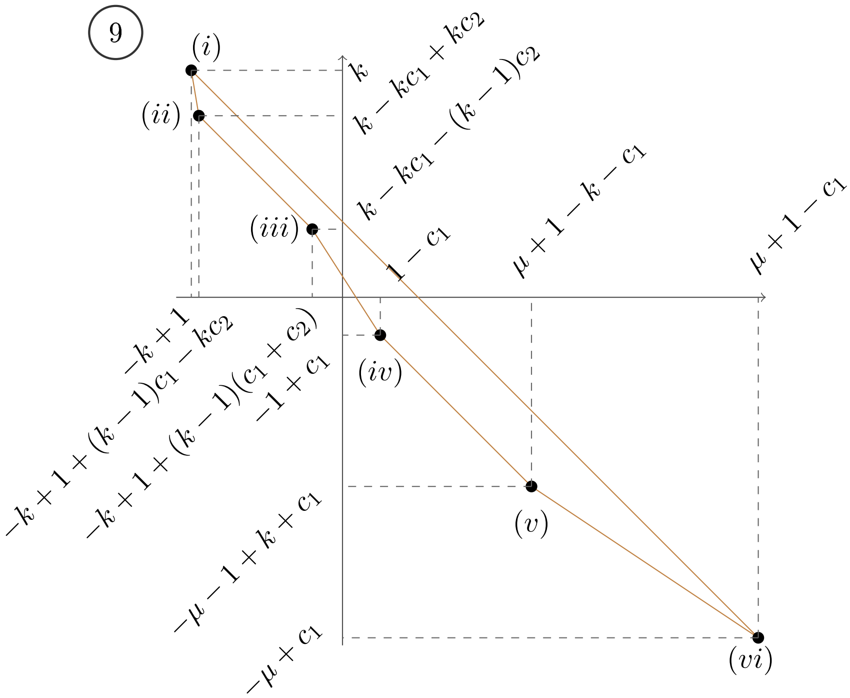
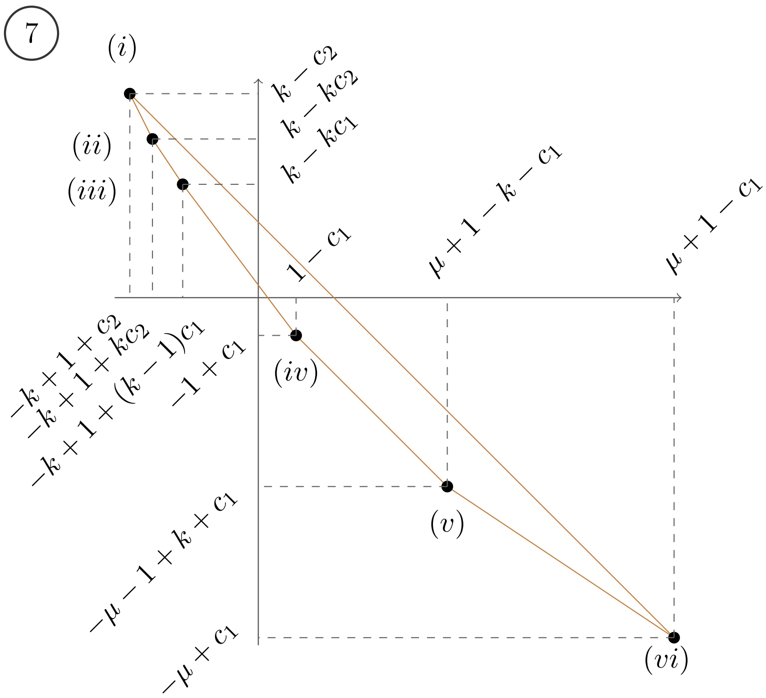


Figure C.2: The Delzant polytopes with the next blow-ups plotted, continued



$(x_{2k-1,4,\cdot},\, y_{2k-1,4,\cdot})$



$(x_{2k-1,5,\cdot},\, y_{2k-1,5,\cdot})$

Figure C.3: The Delzant polytopes with the next blow-ups plotted, continued

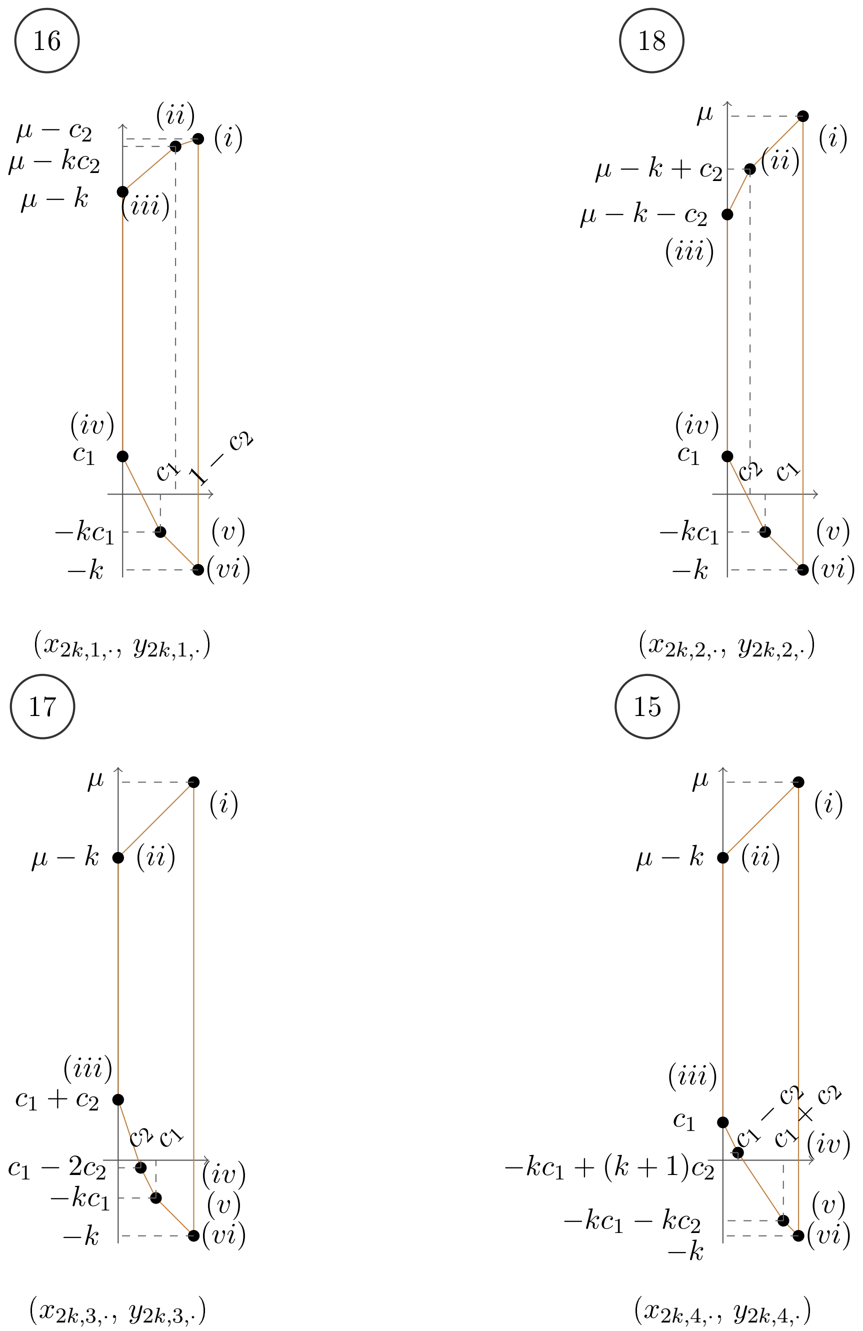


Figure C.4: The Delzant polytopes with the next blow-ups plotted, continued

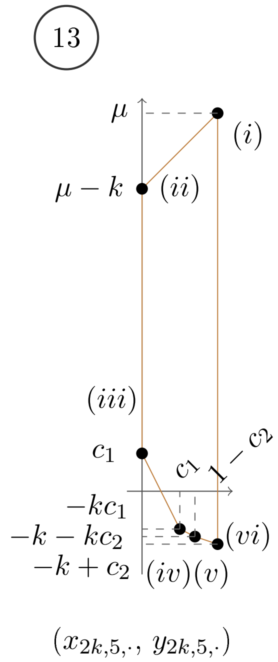


Figure C.5: The Delzant polytopes with the next blow-ups plotted, continued

Since the maps $T_{i,j}(n) \to G_{\mu,c_1,c_2,c_3}$ induce injective maps of fundamental groups ([25]), we can see the actions $x_{n,i,j}$, $y_{n,i,j}$ as elements of $\pi_1(G_{\mu,c_1,c_2,c_3})$. Then, using Karshon's classification, we can find several relations between these elements, as listed below. In fact, an easy but long calculation shows that exactly one more generator, for instance $y_{1,1,1}$, is necessary and sufficient to produce all the other elements.

Remark C.0.1. *With the above notation, the auxiliary polytopes used in Section 3.3.2 for the proof of the new relations should be clear.*

1. Type #1 relations

   1.1. $x_{1,1,1} = x_{1,1,2} = y_{0,1,5} - x_{0,1,5}$

   1.2. $x_{1,1,3} = y_{0,1,4} - x_{0,1,4}$

   1.3. $x_{1,1,4} = y_{0,1,1} - x_{0,1,1}$

   1.4. $x_{1,1,5} = x_{1,1,6} = y_{0,1,2} - x_{0,1,2}$

   1.5. $x_{1,2,1} = x_{1,2,2} = y_{0,2,3} - x_{0,2,3}$

   1.6. $x_{1,2,3} = y_{0,2,2} - x_{0,2,2}$

   1.7. $x_{1,2,4} = y_{0,2,5} - x_{0,2,5}$

   1.8. $x_{1,2,5} = y_{0,2,6} - x_{0,2,6}$

1.9. $x_{1,2,6} = y_{0,2,1} - x_{0,2,1}$

1.10. $x_{1,3,1} = x_{1,3,2} = y_{0,5,4} - x_{0,5,4}$

1.11. $x_{1,3,3} = y_{0,5,3} - x_{0,5,3}$

1.12. $x_{1,3,4} = y_{0,5,2} - x_{0,5,2}$

1.13. $x_{1,3,5} = y_{0,5,6} - x_{0,5,6}$

1.14. $x_{1,3,6} = y_{0,5,1} - x_{0,5,1}$

1.15. $x_{1,4,1} = x_{1,4,2} = x_{1,5,2} = x_{1,5,3} = y_{0,3,3} - x_{0,3,3}$

1.16. $x_{1,4,3} = x_{1,5,1} = y_{0,3,5} - x_{0,3,5}$

1.17. $x_{1,4,4} = x_{1,5,4} = y_{0,3,2} - x_{0,3,2}$

1.18. $x_{1,4,5} = x_{1,5,5} = y_{0,3,6} - x_{0,3,6}$

1.19. $x_{1,4,6} = x_{1,5,6} = y_{0,3,1} - x_{0,3,1}$

2. Type #2 relations

2.1. $(j-1)x_{2k-1,3,1} + jy_{2k-1,3,1} = (k-1)x_{2j-1,3,1} + ky_{2j-1,3,1}$

2.2. $(j-1)x_{2k-1,3,2} + jy_{2k-1,3,2} = (k-1)x_{2j-1,3,2} + ky_{2j-1,3,2}$

2.3. $(j-1)x_{2k-1,3,3} + jy_{2k-1,3,3} = (k-1)x_{2j-1,3,3} + ky_{2j-1,3,3}$

2.4. $(j-1)x_{2k-1,3,4} + jy_{2k-1,3,4} = (k-1)x_{2j-1,3,4} + ky_{2j-1,3,4}$

2.5. $(j-1)x_{2k-1,4,1} + jy_{2k-1,4,1} = (k-1)x_{2j-1,4,1} + ky_{2j-1,4,1}$

2.6. $(j-1)x_{2k-1,4,2} + jy_{2k-1,4,2} = (k-1)x_{2j-1,4,2} + ky_{2j-1,4,2}$

2.7. $(j-1)x_{2k-1,4,3} + jy_{2k-1,4,3} = (k-1)x_{2j-1,4,3} + ky_{2j-1,4,3}$

2.8. $(j-1)x_{2k-1,4,4} + jy_{2k-1,4,4} = (k-1)x_{2j-1,4,4} + ky_{2j-1,4,4}$

2.9. $(j-1)x_{2k-1,5,1} + jy_{2k-1,5,1} = (k-1)x_{2j-1,5,1} + ky_{2j-1,5,1}$

2.10. $(j-1)x_{2k-1,5,2} + jy_{2k-1,5,2} = (k-1)x_{2j-1,5,2} + ky_{2j-1,5,2}$

2.11. $(j-1)x_{2k-1,5,3} + jy_{2k-1,5,3} = (k-1)x_{2j-1,5,3} + ky_{2j-1,5,3}$

2.12. $(j-1)x_{2k-1,5,4} + jy_{2k-1,5,4} = (k-1)x_{2j-1,5,4} + ky_{2j-1,5,4}$

2.13. $(j-1)x_{2k-1,3,5} + jy_{2k-1,3,5} = (k-1)x_{2j-1,3,6} + ky_{2j-1,3,6}$

2.14. $(j-1)x_{2k-1,4,5} + jy_{2k-1,4,5} = (k-1)x_{2j-1,4,6} + ky_{2j-1,4,6}$

2.15. $(j-1)x_{2k-1,5,5} + jy_{2k-1,5,5} = (k-1)x_{2j-1,5,6} + ky_{2j-1,5,6}$

2.16. $(j-1)x_{2k-1,1,1} + jy_{2k-1,1,1} = (k-1)x_{2j-1,2,1} + ky_{2j-1,2,1}$

2.17. $(j-1)x_{2k-1,1,2} + jy_{2k-1,1,2} = (k-1)x_{2j-1,2,2} + ky_{2j-1,2,2}$

2.18. $(j-1)x_{2k-1,1,3} + jy_{2k-1,1,3} = (k-1)x_{2j-1,2,3} + ky_{2j-1,2,3}$

2.19. $(j-1)x_{2k-1,1,5} + jy_{2k-1,1,5} = (k-1)x_{2j-1,2,5} + ky_{2j-1,2,5}$

2.20. $(j-1)x_{2k-1,1,4} + jy_{2k-1,1,4} = (k-1)x_{2j-1,2,6} + ky_{2j-1,2,6}$

2.21. $(j-1)x_{2k-1,1,6} + jy_{2k-1,1,6} = (k-1)x_{2j-1,2,4} + ky_{2j-1,2,4}$

3. Type #3 relations

3.1. $kx_{2k,1,1} + y_{2k,1,1} = (k+1)x_{2k-1,1,6} + ky_{2k-1,1,6}$

3.2. $kx_{2k,1,2} + y_{2k,1,2} = (k+1)x_{2k-1,1,5} + ky_{2k-1,1,5}$

3.3. $kx_{2k,1,3} + y_{2k,1,3} = (k+1)x_{2k-1,1,4} + ky_{2k-1,1,4}$

3.4. $kx_{2k,1,4} + y_{2k,1,4} = (k+1)x_{2k-1,1,1} + ky_{2k-1,1,1}$

3.5. $kx_{2k,1,5} + y_{2k,1,5} = (k+1)x_{2k-1,1,2} + ky_{2k-1,1,2}$

3.6. $kx_{2k,1,6} + y_{2k,1,6} = (k+1)x_{2k-1,1,3} + ky_{2k-1,1,3}$

3.7. $kx_{2k,2,1} + y_{2k,2,1} = (k+1)x_{2k-1,2,6} + ky_{2k-1,2,6}$

3.8. $kx_{2k,2,2} + y_{2k,2,2} = (k+1)x_{2k-1,2,5} + ky_{2k-1,2,5}$

3.9. $kx_{2k,2,3} + y_{2k,2,3} = (k+1)x_{2k-1,2,4} + ky_{2k-1,2,4}$

3.10. $kx_{2k,2,4} + y_{2k,2,4} = (k+1)x_{2k-1,2,1} + ky_{2k-1,2,1}$

3.11. $kx_{2k,2,5} + y_{2k,2,5} = (k+1)x_{2k-1,2,2} + ky_{2k-1,2,2}$

3.12. $kx_{2k,2,6} + y_{2k,2,6} = (k+1)x_{2k-1,2,3} + ky_{2k-1,2,3}$

3.13. $kx_{2k,4,1} + y_{2k,4,1} = (k+1)x_{2k-1,4,6} + ky_{2k-1,4,6}$

3.14. $kx_{2k,4,2} + y_{2k,4,2} = (k+1)x_{2k-1,4,5} + ky_{2k-1,4,5}$

3.15. $kx_{2k,4,3} + y_{2k,4,3} = (k+1)x_{2k-1,4,1} + ky_{2k-1,4,1}$

3.16. $kx_{2k,4,4} + y_{2k,4,4} = (k+1)x_{2k-1,4,2} + ky_{2k-1,4,2}$

3.17. $kx_{2k,4,5} + y_{2k,4,5} = (k+1)x_{2k-1,4,3} + ky_{2k-1,4,3}$

3.18. $kx_{2k,4,6} + y_{2k,4,6} = (k+1)x_{2k-1,4,4} + ky_{2k-1,4,4}$

3.19. $kx_{2k,3,1} + y_{2k,3,1} = (k+1)x_{2k-1,5,6} + ky_{2k-1,5,6}$

3.20. $kx_{2k,3,2} + y_{2k,3,2} = (k+1)x_{2k-1,5,5} + ky_{2k-1,5,5}$

3.21. $kx_{2k,3,3} + y_{2k,3,3} = (k+1)x_{2k-1,5,1} + ky_{2k-1,5,1}$

3.22. $kx_{2k,3,4} + y_{2k,3,4} = (k+1)x_{2k-1,5,2} + ky_{2k-1,5,2}$

3.23. $kx_{2k,3,5} + y_{2k,3,5} = (k+1)x_{2k-1,5,3} + ky_{2k-1,5,3}$

3.24. $kx_{2k,3,6} + y_{2k,3,6} = (k+1)x_{2k-1,5,4} + ky_{2k-1,5,4}$

3.25. $kx_{2k,5,1} + y_{2k,5,1} = (k+1)x_{2k-1,3,6} + ky_{2k-1,3,6}$

3.26. $kx_{2k,5,2} + y_{2k,5,2} = (k+1)x_{2k-1,3,5} + ky_{2k-1,3,5}$

3.27. $kx_{2k,5,3} + y_{2k,5,3} = (k+1)x_{2k-1,3,1} + ky_{2k-1,3,1}$

3.28. $kx_{2k,5,4} + y_{2k,5,4} = (k+1)x_{2k-1,3,2} + ky_{2k-1,3,2}$

3.29. $kx_{2k,5,5} + y_{2k,5,5} = (k+1)x_{2k-1,3,3} + ky_{2k-1,3,3}$

3.30. $kx_{2k,5,6} + y_{2k,5,6} = (k+1)x_{2k-1,3,4} + ky_{2k-1,3,4}$

4. Type #4 relations

4.1. $x_{2k,1,1} = x_{2k,1,6} = x_{2k,5,1} = x_{2k,5,6}$

4.2. $x_{2k,1,2} = x_{2k,5,5}$

4.3. $x_{2k,1,3} = x_{2k,1,4} = x_{2k,5,2} = x_{2k,5,3}$

4.4. $x_{2k,1,5} = x_{2k,5,4}$

4.5. $x_{2k,2,1} = x_{2k,2,6} = x_{2k,3,1} = x_{2k,3,6}$

4.6. $x_{2k,2,2} = x_{2k,3,4}$

4.7. $x_{2k,2,3} = x_{2k,2,4} = x_{2k,3,2} = x_{2k,3,3}$

4.8. $x_{2k,2,5} = x_{2k,3,5}$

4.9. $x_{2k,4,1} = x_{2k,4,6}$

4.10. $x_{2k,4,2} = x_{2k,4,3}$

5. Type #5 relations

5.1. $y_{2k,1,1} = kx_{0,2,5} + y_{0,2,5}$

5.2. $y_{2k,1,2} = kx_{0,2,6} + y_{0,2,6}$

5.3. $y_{2k,1,3} = kx_{0,2,1} + y_{0,2,1}$

5.4. $y_{2k,1,4} = kx_{0,2,4} + y_{0,2,4}$

5.5. $y_{2k,1,5} = kx_{0,2,3} + y_{0,2,3}$

5.6. $y_{2k,1,6} = kx_{0,2,2} + y_{0,2,2}$

5.7. $y_{2k,2,1} = kx_{0,1,1} + y_{0,1,1}$

5.8. $y_{2k,2,2} = kx_{0,1,2} + y_{0,1,2}$

5.9. $y_{2k,2,3} = kx_{0,1,3} + y_{0,1,3}$

5.10. $y_{2k,2,4} = kx_{0,1,6} + y_{0,1,6}$

5.11. $y_{2k,2,5} = kx_{0,1,5} + y_{0,1,5}$

5.12. $y_{2k,2,6} = kx_{0,1,4} + y_{0,1,4}$

5.13. $y_{2k,3,1} = kx_{0,3,6} + y_{0,3,6}$

5.14. $y_{2k,3,2} = kx_{0,3,1} + y_{0,3,1}$

5.15. $y_{2k,3,3} = kx_{0,3,5} + y_{0,3,5}$

5.16. $y_{2k,3,4} = kx_{0,3,4} + y_{0,3,4}$

5.17. $y_{2k,3,5} = kx_{0,3,3} + y_{0,3,3}$

5.18. $y_{2k,3,6} = kx_{0,3,2} + y_{0,3,2}$

5.19. $y_{2k,4,1} = kx_{0,4,6} + y_{0,4,6}$

5.20. $y_{2k,4,2} = kx_{0,4,1} + y_{0,4,1}$

5.21. $y_{2k,4,3} = kx_{0,4,5} + y_{0,4,5}$

5.22. $y_{2k,4,4} = kx_{0,4,4} + y_{0,4,4}$

5.23. $y_{2k,4,5} = kx_{0,4,3} + y_{0,4,3}$

5.24. $y_{2k,4,6} = kx_{0,4,2} + y_{0,4,2}$

5.25. $y_{2k,5,1} = kx_{0,5,6} + y_{0,5,6}$

5.26. $y_{2k,5,2} = kx_{0,5,1} + y_{0,5,1}$

5.27. $y_{2k,5,3} = kx_{0,5,5} + y_{0,5,5}$

5.28. $y_{2k,5,4} = kx_{0,5,4} + y_{0,5,4}$

5.29. $y_{2k,5,5} = kx_{0,5,3} + y_{0,5,3}$

5.30. $y_{2k,5,6} = kx_{0,5,2} + y_{0,5,2}$

6. Type #6 relations

6.1. $jx_{2k,1,1} - y_{2k,1,1} = kx_{2j,2,3} - y_{2j,2,3}$

6.2. $jx_{2k,1,2} - y_{2k,1,2} = kx_{2j,2,2} - y_{2j,2,2}$

6.3. $jx_{2k,1,3} - y_{2k,1,3} = kx_{2j,2,1} - y_{2j,2,1}$

6.4. $jx_{2k,1,4} - y_{2k,1,4} = kx_{2j,2,4} - y_{2j,2,4}$

6.5. $jx_{2k,1,5} - y_{2k,1,5} = kx_{2j,2,5} - y_{2j,2,5}$

6.6. $jx_{2k,1,6} - y_{2k,1,6} = kx_{2j,2,6} - y_{2j,2,6}$

6.7. $jx_{2k,3,1} - y_{2k,3,1} = kx_{2j,3,2} - y_{2j,3,2}$

6.8. $jx_{2k,3,3} - y_{2k,3,3} = kx_{2j,3,3} - y_{2j,3,3}$

6.9. $jx_{2k,3,4} - y_{2k,3,4} = kx_{2j,3,4} - y_{2j,3,4}$

6.10. $jx_{2k,3,5} - y_{2k,3,5} = kx_{2j,3,5} - y_{2j,3,5}$

6.11. $jx_{2k,3,6} - y_{2k,3,6} = kx_{2j,3,6} - y_{2j,3,6}$

6.12. $jx_{2k,4,1} - y_{2k,4,1} = kx_{2j,4,2} - y_{2j,4,2}$

6.13. $jx_{2k,4,3} - y_{2k,4,3} = kx_{2j,4,3} - y_{2j,4,3}$

6.14. $jx_{2k,4,4} - y_{2k,4,4} = kx_{2j,4,4} - y_{2j,4,4}$

6.15. $jx_{2k,4,5} - y_{2k,4,5} = kx_{2j,4,5} - y_{2j,4,5}$

6.16. $jx_{2k,4,6} - y_{2k,4,6} = kx_{2j,4,6} - y_{2j,4,6}$

6.17. $jx_{2k,5,1} - y_{2k,5,1} = kx_{2j,5,2} - y_{2j,5,2}$

6.18. $jx_{2k,5,3} - y_{2k,5,3} = kx_{2j,5,3} - y_{2j,5,3}$

6.19. $jx_{2k,5,4} - y_{2k,5,4} = kx_{2j,5,4} - y_{2j,5,4}$

6.20. $jx_{2k,5,5} - y_{2k,5,5} = kx_{2j,5,5} - y_{2j,5,5}$

6.21. $jx_{2k,5,6} - y_{2k,5,6} = kx_{2j,5,6} - y_{2j,5,6}$

7. Type #7 relations

7.1. $x_{2k-1,1,1} + y_{2k-1,1,1} = x_{2k-1,5,6} + y_{2k-1,5,6} = x_{2k-1,1,6} + y_{2k-1,1,6} = x_{2k-1,5,1} + y_{2k-1,5,1}$

7.2. $x_{2k-1,1,2} + y_{2k-1,1,2} = x_{2k-1,5,3} + y_{2k-1,5,3}$

7.3. $x_{2k-1,1,3} + y_{2k-1,1,3} = x_{2k-1,5,4} + y_{2k-1,5,4} = x_{2k-1,1,4} + y_{2k-1,1,4} = x_{2k-1,5,5} + y_{2k-1,5,5}$

7.4. $x_{2k-1,1,5} + y_{2k-1,1,5} = x_{2k-1,5,2} + y_{2k-1,5,2}$

7.5. $x_{2k-1,2,1} + y_{2k-1,2,1} = x_{2k-1,3,6} + y_{2k-1,3,6} = x_{2k-1,2,6} + y_{2k-1,2,6} = x_{2k-1,3,1} + y_{2k-1,3,1}$

7.6. $x_{2k-1,2,2} + y_{2k-1,2,2} = x_{2k-1,3,2} + y_{2k-1,3,2}$

7.7. $x_{2k-1,2,3} + y_{2k-1,2,3} = x_{2k-1,3,5} + y_{2k-1,3,5} = x_{2k-1,2,4} + y_{2k-1,2,4} = x_{2k-1,3,4} + y_{2k-1,3,4}$

7.8. $x_{2k-1,2,5} + y_{2k-1,2,5} = x_{2k-1,3,3} + y_{2k-1,3,3}$

7.9. $x_{2k-1,4,1} + y_{2k-1,4,1} = x_{2k-1,4,6} + y_{2k-1,4,6}$

7.10. $x_{2k-1,4,4} + y_{2k-1,4,4} = x_{2k-1,4,5} + y_{2k-1,4,5}$

8. Type #8 relations

8.1. $kx_{2k,1,5} + y_{2k,1,5} = kx_{2k,1,6} + y_{2k,1,6}$

8.2. $kx_{2k,2,5} + y_{2k,2,5} = kx_{2k,2,6} + y_{2k,2,6}$

8.3. $kx_{2k,3,5} + y_{2k,3,5} = kx_{2k,3,6} + y_{2k,3,6}$

8.4. $kx_{2k,4,1} + y_{2k,4,1} = kx_{2k,5,1} + y_{2k,5,1}$

8.5. $kx_{2k,4,2} + y_{2k,4,2} = kx_{2k,5,2} + y_{2k,5,2}$

8.6. $kx_{2k,4,3} + y_{2k,4,3} = kx_{2k,5,3} + y_{2k,5,3}$

8.7. $kx_{2k,4,4} + y_{2k,4,4} = kx_{2k,5,6} + y_{2k,5,6}$

8.8. $kx_{2k,4,5} + y_{2k,4,5} = kx_{2k,5,5} + y_{2k,5,5} = kx_{2k,4,6} + y_{2k,4,6} = kx_{2k,5,4} + y_{2k,5,4}$

The relations above can be easily calculated by starting with the Delzant polytopes, transforming them with the appropriate by $SL(2,\mathbb{Z})$-actions and comparing the resulting graphs for the S^1-actions. In Section 3.3.2, we went through this process for some of the relations.

Remark C.0.2. *We further note that the discussion in Appendix B are valid for $\mu > 1$, and thus the correspondence between isometry groups and configurations still holds.*

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