THE FUNDAMENTAL GROUP OF Symp$(S^2 \times S^2 \# 4\overline{\mathbb{CP}^2})$

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Contents

1. Introduction

This report concerns the work done under the guidance of professor Sílvia Anjos for the 2019 edition of the Gulbenkian project *Novos Talentos em Matemática*.

In a succint way, the starting point was the result in [9] which states that the rank of the fundamental group of the group of symplectomorphisms $\text{Symp}(S^2 \times S^2 \# 4\overline{\mathbb{CP}^2}, \tilde{\omega}_\mu)$ is 5, where $\tilde{\omega}_\mu$ is the symplectic form obtained from the form $\sigma \oplus \mu\sigma$, $\mu > 1$, on $S^2 \times S^2$ after 4 blow-ups of size $\frac{1}{2}$. Hence, two main questions were posed:

(1) Can we find a set of generators of the fundamental group that correspond to hamiltonian circle actions?
(2) Can we find explicit formulas for how to write the other circle actions in terms of these set of generators?

These are the two questions that guided the work done throughout the 10 months that encapsulated this project.

In the first part of the report, we give a general overview of some basic terminology and definitions in symplectic geometry and we state Delzant's theorem on the classification of toric manifolds by a certain class of polytopes, which is one of the main tools used in the next sections. On Section 3 we give the basic mathematical framework for the tools used to study the fundamental group of the group of symplectomorphisms of $S^2 \times S^2 \# 4\overline{\mathbb{CP}^2}$. In particular, we give a brief exposition of Karshon's classification of hamiltonian circle actions on toric manifolds by decorated graphs [7].

In Sections 4-6 we begin exploring the answers to the questions posed above. We begin by proving that the set of circle actions $\{z_1, z_{0,12}, z_{0,13}, z_{0,14}, z_{0,2}\}$ are linearly independent in the fundamental group, by mapping to the ring of quantum homology $QH_*(S^2 \times S^2 \# 4\overline{\mathbb{CP}^2})$ by the Seidel morphism, S. To do this, the algebraic results in [3] on the the structure of the quantum homology ring were used to arrive at a simple ring presentation. After this, we dedicate Section 6 to compute expressions for the other circle actions in terms of this set of generators.

Finally, in the last section we propose some further steps that can be taken that came up naturally in the course of this project.

I want to thank the Gulbenkian foundation for once again giving me the opportunity of developing this kind of project.

I would also like to thank my tutor, professor Sílvia Anjos, for all the priceless help given during the course of these 10 months. Last but not least, I thank Ana Reis for helping me with some of the computations and finding the best way to do them, especially in Sections 5 and 6.

2. Background

2.1. Symplectic manifolds and symplectomorphisms.

In this section we review some basic results on symplectic geometry which will be useful in the sequel. As such, no proofs will be given in this report and the main reference used (where all the proofs are provided) is [2].

symplectic geometry is a branch of differential geometry which is concerned with the study of the class of *symplectic manifolds*.

Definition 2.1. *A symplectic manifold is a pair (M, ω) such that M is a smooth manifold and ω is a 2-form on M such that:*

- *ω is closed, i.e., $d\omega = 0$;*
- *ω is non-degenerate, i.e., for all $p \in M$, the map $T_pM \to T_p^*M$ defined by $v \mapsto \iota_v\omega$ is an isomorphism.*

Example 2.2. *(1) The most important example of a symplectic manifold is $M = \mathbb{R}^{2n}$ with symplectic form*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$$

where $x_1, \dots, x_n, y_1, \dots, y_n$ are the coordinates on $\mathbb{R}^{2n}$.

(2) A similar construction also works for $\mathbb{C}^n$ with symplectic form

$$\omega = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\overline{z_j}$$

where $z_1, \dots, z_n$ are coordinates on $\mathbb{C}^n$.

(3) If M is a smooth surface, then the existence of a symplectic structure is equivalent to the orientability of M. Indeed, ω being symplectic is equivalent to ω being a volume form.

Hence, we get an interesting family of symplectic manifolds (S^2, $S^1 \times S^1$, $S^1 \times \mathbb{R}$, ...) and the first example of manifolds that are not symplectic, such as $\mathbb{RP}^2$.

From the example presented, some patterns of what kinds of manifolds are symplectic start to emmerge: for instance, all the examples have even real dimension and the only non-example was non-orientable. The next result states simple general properties of symplectic manifolds, which impose some quite stringent restrictions on the structure of the M as a smooth manifold.

Proposition 2.3. *Let (M, ω) be a symplectic manifold. Then*

- *M has even real dimension;*
- *M is an orientable manifold;*
- *$H^2_{dR}(M) \neq 0$, where H^k_{dR} denotes the k^{th} de Rham cohomology group.*

As we can see, only even-dimensional real euclidean spaces have a symplectic structure and, furthermore, by the last property, the only n-sphere with a symplectic structure is S^2, with symplectic form given by some volume form, as in Example 2.2. This also shows that not all orientable manifolds are symplectic.

As it is usual with mathematical constructions, there are ways of building new symplectic manifolds out of old ones. For the sake of simplicity, we'll describe only the case of the 4-fold symplectic blow-up of $S^2 \times S^2$, which is the manifold we'll focus on in the forthcoming sections.

From the discussion on Example 2.2 on surfaces, we know S^2 is a symplectic manifold with symplectic form given by a volume form σ. Consider now $S^2 \times S^2$, with the usual projection maps $p_i : S^2 \times S^2 \to S^2$ onto the i^{th} component, and suppose that we have volume forms σ_1, σ_2 on each of the S^2 components. Then the following 2-form on $S^2 \times S^2$

$$\omega = p_1^*(\sigma_1) + p_2^*(\sigma_2) \tag{1}$$

is a symplectic form. Upon normalizing the volume of each S^2 so that $Vol_{\sigma_1}(S^2) = 1$ and $Vol_{\sigma_2}(S^2) = \mu \geq 1$, we'll use the notation $\omega_\mu = \sigma \oplus \mu\sigma$ for the form (1).

After equipping $S^2 \times S^2$ with the symplectic form $\omega_\mu = \sigma \oplus \mu\sigma$, we want to do a *symplectic blow-up*. The exact definition of a blow-up is a more technical then what we really need for the rest of this report (see [5]), so we'll consider ourselves satisfied by only presenting the idea of the construction. Starting with a point $p \in S^2 \times S^2$, we would like to substitute it by a copy of $\mathbb{CP}^2$, which would correspond to the lines which pass by p. Although this procedure works in a complex setting, it doesn't work in the symplectic category because of the symplectic structure. Instead, we replace the role of p by a ball around p: we remove this ball and collapse its boundary using the Hopf fibration. This gives us a new manifold, which we'll represent by $S^2 \times S^2 \# \overline{\mathbb{CP}^2}$, and a new symplectic form, $\tilde{\omega}_\mu$, which is a deformation of ω_μ and depends only on the size of the ball removed.

We are interested in blowing-up $S^2 \times S^2$ four times, which in our notation we'll write as $S^2 \times S^2 \# 4\overline{\mathbb{CP}^2}$, and we'll denote its symplectic form by $\tilde{\omega}_\mu$. Following the reasoning above, we see that $\tilde{\omega}_\mu$ is completely described by:

- the size of the original 2-spheres S^2, which in our case is the same as a real number $\mu \geq 1$;
- the size of each ball removed at each blow-up. We'll represent the size of the i^{th} blow-up by c_i.

We would like now to compare symplectic manifolds by considering smooth maps between the manifolds involved while at the same time taking into account their symplectic structure. This idea allows us to define what we mean by equivalence of symplectic structures, using the notion of *symplectomorphism*.

Definition 2.4. *Let (M, ω), (N, ν) be symplectic manifolds. A symplectomorphism is the data $f : (M, \omega) \to (N, \nu)$, where $f : M \to N$ is a diffeomorphism such that $f^*\nu = \omega$. If such a f exists, we say (M, ω) and (N, ν) are sympletomorphic.*

If $M = N$, then the symplectomorphisms of M into itself form a group with respect to composition which we denote by $\mathrm{Symp}(M)$.

The notion of symplectomorphism gives us the correct language to a question which might have arisen when considering $S^2 \times S^2$: are all symplectic structures on $S^2 \times S^2$ sympletomorphic to $(S^2 \times S^2, \omega_\mu)$? This is a relevant question for us: since we're only considering this kind of symplectic forms, it would tell us that we're actually dealing with the general case of a symplectic structure on $S^2 \times S^2$. Thankfully, this is actually the case (see [4]).

We can try to relate the usal symplectic structure on $\mathbb{R}^{2n}$ to the symplectic structure on a $2n$-manifold M. Indeed, this relation is very simple and is given explicitly by the following theorem.

Theorem 2.5 (Darboux). *Let (M, ω) be an $2n$-dimensional symplectic manifold. Then for each $p \in M$ there exists a chart (U, φ) centered at p such that locally ω is of the form*

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$$

where $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ are local coordinates on U.

Darboux's theorem is a surprising and elegant result in symplectic geometry. For instance, it tells us that symplectic manifolds don't have local invariants since locally all symplectic manifolds are symplectomorphic.

2.2. **Toric manifolds and Delzant's theorem.** One way of studying symplectic manifolds is by having connected Lie groups act smoothly on them. Recall that given a connected Lie group G, a *smooth action of* G on a smooth manifold M is a group homomorphism $\Phi : G \to \text{Diff}(M)$. We usually represent the action by a dot, i.e., $g \cdot p := \Phi(g)p$.

Lie group actions give rise to vector fields through the infinitesimal action of the Lie algebra, $\mathfrak{g}$: given $v \in \mathfrak{g}$ and $p \in M$, we define $X_p^v \in T_pM$ as

$$X_p^v = \frac{d}{dt}_{|t=0}\Big(\exp(tv)\cdot p\Big) \tag{2}$$

This defines a vector field on M, X^v, by using (2) as the definition of its value at p. If (M,ω) is a symplectic manifold then we can consider the contracted form $\iota_{X^v}\omega$.

Definition 2.6. *In the notation above, we say Φ is a symplectic action if for $v \in \mathfrak{g}$ the form $\iota_{X^v}\omega$ is closed.*

We say the action is hamiltonian if each form $\iota_{X^v}\omega$ is exact, i.e., there exists $H_v \in C^\infty(M)$ such that

$$\iota_{X^v}\omega = dH_v$$

and the map $\mu^* : \mathfrak{g} \to C^\infty(M)$ defined by $v \mapsto H_v$ is an algebra homomorphism, where we endow $C^\infty(M)$ with the Poisson bracket.

We usually call μ^* the *comoment map* of the action. We can also restate the hamiltonian condition in terms of a *moment map*, which is much more useful in some situations: if $\mathfrak{g}^*$ is the dual Lie algebra of $\mathfrak{g}$, the an action is hamiltonian if there exists a map $\mu : M \to \mathfrak{g}^*$ such that

(1) if $\mu^v : M \to \mathbb{R}$ is determined by the condition

$$\mu^v(p) := \langle \mu(p), v\rangle$$

for $v \in \mathfrak{g}$, then

$$d\mu^v = \iota_{X^v}\omega$$

(2) the action Φ is equivariant with respect to the coadjoint action of $\mathfrak{g}^*$, i.e.,

$$\mu \circ \Phi(g) = \text{Ad}_g^* \circ \mu$$

holds for all $g \in G$.

Example 2.7. *(1) Consider $M = S^2$ with $\omega = d\theta \wedge dh$ in cylindrical coordinates and consider the action of $G = S^1$ by rotation around the h-axis, given by the flow of $X = \frac{\partial}{\partial\theta}$.*

For an action of S^1, being hamiltonian is equivalent to the existence of $H : M \to \mathbb{R}$ smooth such that $dH = \iota_X\omega$, where X is the vector field generated by the action[1]*. But clearly the map $H(h) = h$ satisfies these conditions and hence this is a hamiltonian action of S^1 on S^2 (the moment map is also H in this case).*

(2) Consider $M = S^1 \times S^1$. This is a symplectic manifold since $\omega = d\theta_1 \wedge d\theta_2$ is symplectic, with (θ_1, θ_2) being the usual circle coordinates on the torus. Then it can be proven that the action given by rotations along the θ_1-direction isn't a hamiltonian S^1-action on M, although it's symplectic.

As the examples chosen portray, S^1-actions are important examples of symplectic actions and they are also the ones we're interested in the next sections. They're actually part of a more general class of actions called *toric actions*.

[1]This is also applicable to $\mathbb{R}$. For torus actions $(S^1)^n$, this is also true for each component of the action.

Definition 2.8. *A toric manifold is a compact connected symplectic manifold (M, ω) with an effective*[2] *hamiltonian action by a torus T such that*

$$\dim T = \frac{1}{2} \dim M$$

and some choice of moment map μ. We'll represent a toric manifold by the data (M, ω, T, μ).

Given a toric manifold (M, ω, T, μ), we call $\mu(M)$ its *moment polytope*.

Toric manifolds are an interesting research topic not only of symplectic geometry but also of algebraic geometry. One of the most amazing results on toric manifolds is a classification theorem by Delzant of all toric manifold by a class of polytopes, which are known as *Delzant polytopes*.

Definition 2.9. *A Delzant polytope $\Delta \subset \mathbb{R}^n$ is a convex polytope satisfying:*

- *Δ is simple, i.e., n edges meet at each vertex;*
- *Δ is rational, i.e., each edge is of the form $p + tu_i$, $0 \leq t < \infty$ for $u_i \in \mathbb{R}^n$;*
- *Δ is smooth, i.e., each $u_1, \dots, u_n$ is a basis of $\mathbb{Z}^n$.*

Theorem 2.10 (Delzant). *There is a bijective correspondence between toric manifolds and Delzant polytopes up to multiplication by an element of $GL_n(\mathbb{Z})$ in the following way: to a toric manifold (M, ω, T, μ) we correspond its moment polytope $\mu(M)$; in the other direction, every Delzant polytope is the moment polytope for some toric manifold.*

We'll be studying toric manifolds $(S^2 \times S^2 \# 4\mathbb{CP}^2, \tilde{\omega_\mu}, T^2, \mu)$, where $T^2 = S^1 \times S^1$ is the 2-torus. By the proof of Delzant's theorem on [2], we see that the corresponding moment polytope is 2-dimensional, which simplifies our analysis. In particular, we would like to understand how, starting from the polytope of $S^2 \times S^2$, we can represent the blow-ups on the Delzant polytope of $S^2 \times S^2$ and of the subsequent blown-up manifolds. From now on, we'll also represent toric manifolds by (M, ω) when the missing data is clear from the context.

Firstly, as we saw before, S^1 has a hamiltonian action on S^2 given by rotations along some axis. This gives a hamiltonian action of T^2 on $S^2 \times S^2$ which gives $(S^2 \times S^2, \omega_\mu)$ the structure of a toric manifold. The corresponding Delzant polytope is a rectangle with sides 1 and μ, which in the Cartesian plane has vertices

$$(0,0),\ (-1,0),\ (-1,\mu), (0,\mu).$$

The top and bottom sides represent a fixed 2-sphere of size 1, represented by $F \in H_2(S^2 \times S^2 \# 4\overline{\mathbb{CP}^2}, \mathbb{Z})$ and will correspond to the *fibre class*, and the left and right sides correspond to a fixed 2-sphere of size μ, which we'll represent by $BF \in H_2(S^2 \times S^2 \# 4\overline{\mathbb{CP}^2}, \mathbb{Z})$ and will correspond to the *base class*.

Suppose now we want to do a blow-up to $S^2 \times S^2$. Conider a general toric manifold (M, ω, T^n, μ). By a theorem of Atiyah, Guillemin and Sternberg (see [2]), the fixed points of the action correspond to the vertices of the moment polytope (for instance, before the blow-up of $S^2 \times S^2$ we have 4 vertices since the S^1-action has 2 fixed points). Hence, let p be such a vertex.

Proposition 2.11. *The blow-up of M (of size ε) at the fixed point p is the toric manifold corresponding to the Delzant polytope where we substitute the vertex by the vertices of the form*

$$p + \varepsilon u_i$$

where $u_1, \dots, u_n \in \mathbb{Z}^n$ are primitive vectors[3].

For $M = S^2 \times S^2$, we have $\varepsilon = c_1$ and the proposition says that if we choose $p = (-1, \mu)$, then we substitute this point by the new vertices $(-1, \mu - c_1)$ and $(-1 + c_1, \mu)$ and we chop off the triangle formed by these vertices. Notice this introduces a new homology class, E_1, called the *exceptional class* of the blow-up (which has size c_1) and also alters the other homology classes:

[2]A group action of G on X is effective if for all $g \in G \backslash \{e\}$ there exists $x \in X$ such that $g \cdot x \neq x$, where e is the identity element.

[3]We say $v \in \mathbb{Z}^n$ is a *primitive vector* if we cannot write it in the form $v = ku$ where $k \in \mathbb{Z} \backslash \{\pm 1\}, u \in \mathbb{Z}^n$

the adjacent classes in the polytope turn into $B - E_1$ and $F - E_1$. For the other three blow-ups, the same kind of procedure is used and we get new classes E_2, E_3, E_4.

One more remark should be made regarding the fundamental group we want to compute. In the case $\mu \geq 1$, $c_1 = c_2 = c_3 = c_4 = 1$, $\mathrm{Symp}(S^2 \times S^2 \# 4\overline{\mathbb{CP}^2}, \tilde{\omega}_\mu)$ isn't a connected space and so we should include a basepoint when computing π_1. Instead of doing this, we fix one of the connected components, $\mathrm{Symp}_h(S^2 \times S^2 \# 4\overline{\mathbb{CP}^2}, \tilde{\omega}_\mu)$, which are the symplectomorphisms which are the identity on homology. The choice of connected component is irrelevant for computing the fundamental group since, as the group of symplectomorphisms is a topological group, all connected components are homeomorphic.

3. The fundamental group of $\mathrm{Symp}_h(S^2 \times S^2 \# 4\overline{\mathbb{CP}^2})$

We want to study $\pi_1(\mathrm{Symp}_h(S^2 \times S^2 \# 4\overline{\mathbb{CP}^2}))$ for the specific case where $\mu > 1$ and $c_1 = c_2 = c_3 = c_4 = \frac{1}{2}$. In particular, we want to know what kind of generators this group has and how they change when we alter μ. From now on, unless otherwise stated, we'll fix these conditions.

As mentioned above, one of the tools that will be used in this analysis will be the Delzant polytope of the blown-up 4-manifold. Moreover, we are looking for a special kind of generators, which are elements of $\pi_1(\mathrm{Symp}_h(S^2 \times S^2 \# 4\overline{\mathbb{CP}^2}))$ that come from hamiltonian actions of S^1 on $S^2 \times S^2 \# 4\overline{\mathbb{CP}^2}$. In [7], a classification of hamiltonian circle actions is given through the use of decorated graphs that in some cases we can associate to the Delzant polytopes, which is the other tool that we will use extensively. These graphs are easily constructed: given a Delzant polytope Δ for a 4-manifold, we know that we have an action by a 2-dimensional torus; thus, $\Delta \subset \mathbb{R}^2$ and so we can project Δ onto both the coordinate axis. Then we define the vertices and edges of the graph as follows:

- To a given vertex p on Δ corresponds a vertex on the graph with the value $\mu(p)$;
- To a fixed 2-dimensional surface Σ (in our case, when we project onto the vertical axis these correspond to horizontal edges on Δ; when we project onto the horizontal axis these correspond to vertical edges on Δ) we associate a fat vertex which is labelled with $\mu(\Sigma)$ and its symplectic area, which is the integral $\frac{1}{2\pi}\int_\Sigma \omega$. Sometimes we'll also include the class corresponding to this fixed surface for clarity. In our case, since the symplectic area is linear on formal linear combinations of B, F, E_i, it's enough to specify the values for these classes, which are $\mu, 1, \frac{1}{2}$, respectively;
- The action on M gives rise to what Karshon calls Z_k-spheres, which are 2-spheres where the action is given by rotations at speed k. We connect two vertices on the decorated graph with a weight k if they are fixed points of some Z_k-sphere of M.

Examples of these graphs are given in Figures 1 and 2 below. Throught the report more examples are shown with both the Delzant polytope and the graphs at the same time.

By a result in [9], we know that

Theorem 3.1.

$$\mathrm{rank}\left(\pi_1(\mathrm{Symp}_h(S^2 \times S^2 \# 4\overline{\mathbb{CP}^2}))\right) = 5 \tag{3}$$

Before beginning to discuss our approach to the first question on the introductory section, we make a quick detour into the decorated graphs that we can obtain in the situation at hands. This will not only be elucidative of the kind of diversity of elements we can get but it will also serve as a way of fixing notation for later.

Since $\mu > 1$, we have the 4 circle actions represented by the graphs in Figure 1, where $i, j, l, m \in \{1, 2, 3, 4\}$ are necessarily distinct elements. Notice these actions only exist as long as the size of the classes corresponding to the fixed surfaces are positive (hence, as we increase μ, more and more classes start appearing).

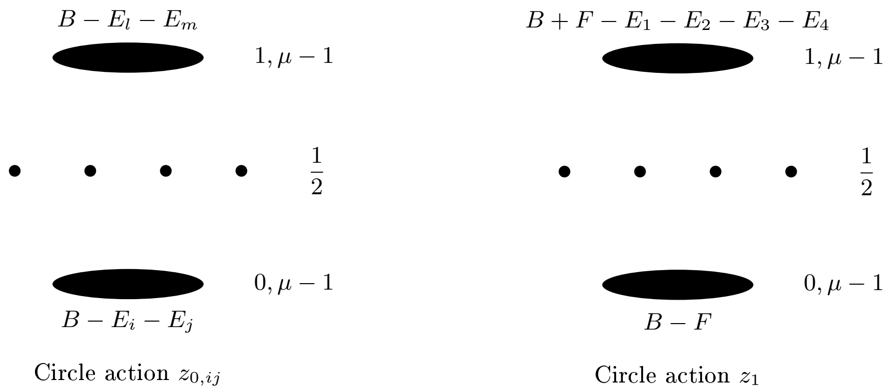


FIGURE 1. Graphs in the case $\mu > 1$

If we now consider $\mu > \frac{3}{2}$, 8 more actions exist, which are given by the graphs on Figure 2.

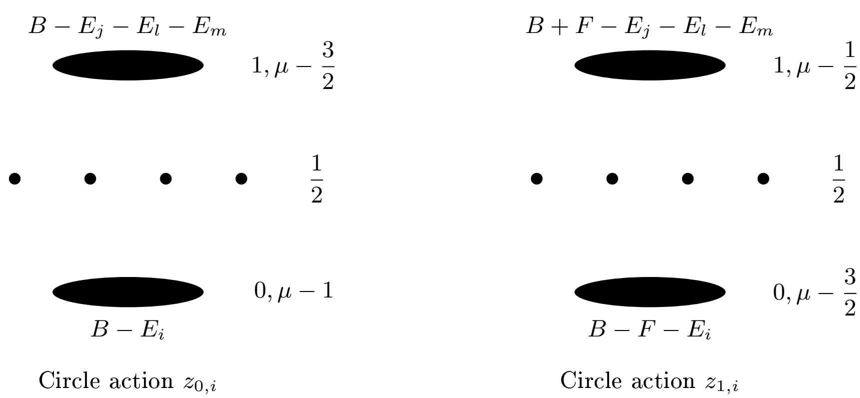


FIGURE 2. Graphs in the case $\mu > \frac{3}{2}$

If we keep on increasing the value of μ, it is an easy computation that the number of circle actions always increases by 8 when μ passes n or $n + \frac{1}{2}$ for $n \in \mathbb{Z}_{>0}$. Therefore, we have the following proposition.

Proposition 3.2. *For $\mu > 1$, $c_1 = c_2 = c_3 = c_4 = \frac{1}{2}$ the only hamiltonian circle actions are the ones represented by the graphs below. In particular, these actions satisfy the following existence conditions:*

- z_k *exists iff* $\mu > k$*;*
- $z_{k,i}$ *exists iff* $\mu > k + \frac{1}{2}$*;*
- $z_{k,ij}$ *exists iff* $\mu > k + 1$*;*
- $z_{k,ijl}$ *exists iff* $\mu > k + \frac{3}{2}$*;*
- $z_{k,1234}$ *exists iff* $\mu > k + 2$*;*

where $k \in \mathbb{Z}_{>0}$ and $i, j, l \in \{1, 2, 3, 4\}$ are distinct elements.

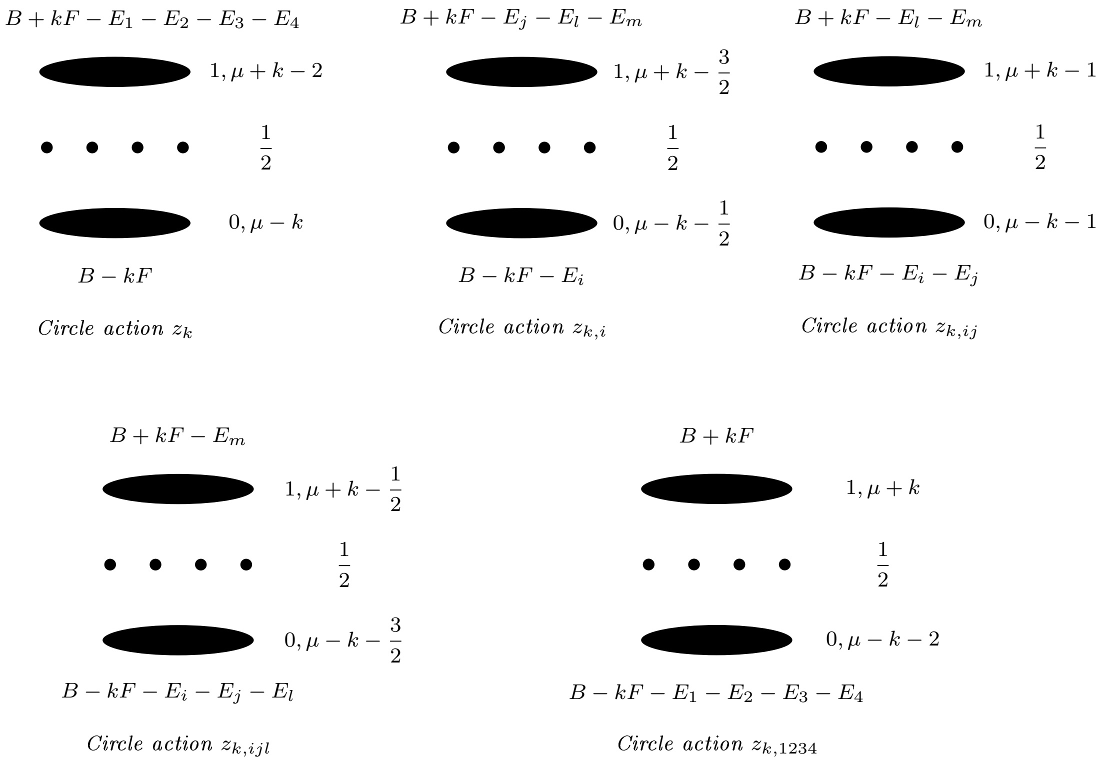


FIGURE 3. Graphs of Hamiltonian circle actions for $\mu > 1,\ c_1 = c_2 = c_3 = c_4 = \frac{1}{2}$

Remark 3.3. *Proposition 3.2 sheds some light on the importance and non-triviality of* (3). *Indeed, although the number of hamiltonian circles actions keeps increasing as* μ *also increases, the rank of* π_1 *remains constant.*

4. The Seidel morphism and quantum homology

We now begin to answer the question posed in the introduction about the possibility of having hamiltonian circle actions generating the fundamental group. Looking at the graphs presented above, it's natural to choose as generators the graphs corresponding to manifolds with small μ since these ones exist for most values of μ. We're particularly interested in the actions $z_{0,12}, z_{0,13}, z_{0,14}, z_{0,2}, z_1$. Upon this choice of candidates, one thing immediately sticks out: the action $z_{0,2}$ exists as long as $\mu > \frac{3}{2}$. Therefore, even if we prove that all these actions are linearly independent, we know that in the range $1 < \mu < \frac{3}{2}$ only 4 of these actions are induced by hamiltonian circle actions. We'll return to this phenomenon in the last section.

For now, lets pursue our attempt of proving linear independence of these actions. One way of doing this is by using the *Seidel morphism*, S (see [10]). In the case of $S^2 \times S^2 \# 4\overline{\mathbb{CP}^2}$, this is a group homomorphism

$$S : \pi_1(\mathrm{Symp}_h(S^2 \times S^2 \# 4\overline{\mathbb{CP}^2})) \to QH_*(S^2 \times S^2 \# 4\overline{\mathbb{CP}^2})$$

where $QH_*(S^2 \times S^2 \# 4\overline{\mathbb{CP}^2})$ is the ring of quantum homology of $S^2 \times S^2 \# 4\overline{\mathbb{CP}^2}$. Then, checking linear independence of the actions becomes a matter of proving that the images of these actions under S are distinct.

As stated in [10], computing $S(\Lambda)$ in a general setting is an extremely difficult task, due partially to understanding the ring structure of the quantum homology ring. However, we'll see that our candidates for generators actually fall into a class of hamiltonian circle actions for which a closed formula is known. This combined with a knowledge of QH_* will allow us to compare the images of S and decide if they are distinct, solving our first question.

We'll begin by defining what we mean by quantum homology, even though we'll use a approach more based on results on algebraic geometry to understand it. Following [10], consider the ring of Laurent polynomials

$$\Lambda := \Lambda^{univ}[q, q^{-1}]$$

where q has grading 2 and Λ^{univ} is a ring of generalised Laurent series on a degree 0 variable t:

$$\Lambda^{univ} = \left\{ \sum_{k \in \mathbb{R}} c_k t^k : c_k \in \mathbb{Q},\ \#\{k > c : c_k \neq 0\} < \infty,\ \forall c \in \mathbb{R} \right\}$$

Then for a symplectic manifold (M, ω) we define the *quantum homology groups* $QH_*(M)$ as

$$QH_*(M) := H_*(M) \otimes \Lambda$$

There is a natural $\mathbb{Z}$-grading on $QH_*(M)$

$$\deg(a \otimes q^d t^k) := \deg(a) + 2d$$

and a product, $* : QH_i(M) \otimes QH_j(M) \to QH_{i+j-\dim M}(M)$, called the *quantum homology product*, which we refrain from defining in this report.

In [3], Crauder and Miranda compute the quantum cohomology[4] of what they call a general rational surfaces, which is a complex surface obtained by blowing up n points and some other conditions on the kind of linear systems and irreducible rational curves it admits. Since we have a diffeomorphism $S^2 \times S^2 \# 4\overline{\mathbb{CP}^2} \cong \mathbb{CP}^2 \# 5\overline{\mathbb{CP}^2}$, we get a correspondence between the homology classes B, F, E_i and the classes H, E'_j [5] on $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}^2}$, given by

$$\begin{cases} H = B + F - E_1 \\ E'_1 = B - E_1 \\ E'_2 = B - E_2 \\ E'_3 = E_2 \\ E'_4 = E_3 \\ E'_5 = E_4 \end{cases} .$$

An explicit formula for the quantum product in terms of the classes H, E'_i is given in Proposition 5.3. of [3], which, in addition to the Tables in Section 4, allows us to compute the desired products. As an example, the product of two classes (different from the class of a single point) in $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}^2}$

[4]Although we want to understand quantum homology, this won't matter too much for us since the formulas also work for homology because we can pass from quantum cohomology to quantum homology using Poincaré duality.

[5]These represent the classes of $\mathbb{CP}^2$ and the exceptional classes of the blow-ups.

is given by[6]

$$
\begin{aligned}
(dH - \sum_i m_i E_i') * (d'H - \sum_i m_i' E_i') = \left(dd' - \sum_i m_i m_i' \right) pt^{[0]} + + \sum_k m_k m_k' E_k' t^{[E_k']} + \\
+ \sum_{j,k} (d - m_j - m_k)(d' - m_j' - m_k')(H - E_j' - E_k') t^{[H - E_j' - E_k']} + \\
+ \left(2d - \sum_i m_i \right) \left(2d' - \sum_i m_i' \right) (2H - E_1' - E_2' - E_3' - E_4' - E_5') t^{[2H - E_1' - E_2' - E_3' - E_4' - E_5']} + \\
+ \sum_j (d - m_j)(d' - m_j') X t^{[H - E_j']} + \\
+ \sum_{j,k,l,n} (2d - m_j - m_k - m_l - m_n)(2d' - m_j' - m_k' - m_l' - m_n') X t^{[2H - E_j' - E_k' - E_l' - E_n']}
\end{aligned}
$$

where i, j, k, l, n always represent different indices, X is the class of the manifold, p is the class of a point, $d \in \mathbb{Z}_{>0}$, $m_i, m_j' \in \mathbb{Z}_{\geq 0}$ and $t^{[A]}$ means t to the power of the symmetric of the symplectic area of the corresponding class (in the symplectic viewpoint).

The next proposition gives a description of the ring $QH_*(S^2 \times S^2 \# 4\overline{\mathbb{CP}^2})$. For the sake of simpler notation, let $f_{ij} = q(F - E_i - E_j)$, $b_{ij} = q(B - E_i - E_j)$ and $e_i = qE_i$ and, as before, let different letters in the indices correspond to different elements. It follows from Proposition 5.3. in [3] that we have the presentation for $QH_*(S^2 \times S^2 \# 4\overline{\mathbb{CP}^2})$ given below:

Proposition 4.1. *With the notation above, the quantum homology ring $QH_*(S^2 \times S^2 \# 4\overline{\mathbb{CP}^2})$ is generated by the elements $\{t, f_{ij},\ b_{ij},\ e_i\}_{i<j}$ satisfying the relations:*

(1) $f_{ik}(b_{ij} + t^{-1/2}(1 - t^{1-\mu})) = 0$;
(2) $b_{ij} b_{kl} = t^{-1}(1 - t^{1-\mu})^2$;
(3) $f_{ij} f_{kl} = 0$;
(4) $f_{ij}(e_k + t^{1/2-\mu}) = 0$;
(5) $b_{ij}(f_{ij} + e_i + t^{1/2-\mu}) = t^{-\mu}(1 - 1t^{1-\mu})(1 + e_j t^{\mu-1/2})$;
(6) $b_{ij}(e_k + t^{1/2-\mu}) = t^{-\mu}(1 - t^{1-\mu}) \left(1 + (f_{kj} + e_j) t^{\mu-1/2}\right)$;
(7) $f_{ij}(b_{ij} + e_i + y^{-1/2}) = 0$;
(8) $f_{ij}(f_{ik} + e_i + t^{1/2-\mu}) = 0$;
(9) $e_i^2 = f_{ij} f_{ik} + (e_l - e_i) t^{1/2-\mu} + e_i e_l$;
(10) $f_{ij}^2 = f_{ij}(f_{ik} + f_{il})$.

Remark 4.2. *Of course our description doesn't give a minimal set of generators nor is that the intention of Proposition 4.1. The generators were picked with the intent of simplifying the computations of the Seidel morphism and they also give some simple insight into the ring structure: for instance, relation (3) implies that there are zero divisors.*

5. Computing the Seidel morphism

In this section we compute the Seidel morphism for the candidates to linearly independent elements of π_1. As we stated in previous sections, computing Seidel elements (images by the Seidel morphism) can be a very tricky task in general but we'll see that our elements are covered by the formulas in [1]. The toric manifolds we are interested in are endowed with a complex structure which may make them into a class of manifolds[7] for which S is possible to compute, using Theorem 4.5. and Figure 1 of [1].

[6]Using the conversion formulas, it's easy to define similar formulas for the product of the classes in the $S^2 \times S^2$ notation. Although this was used for the computations below, we won't write out here the explicit formula since it's not as elegant.

[7]These are called NEF and Fano manifolds. They are defined by a non-negativity condition on the first Chern number of J-holomorphic curves on the manifold.

We won't explain the reasoning for testing if our manifold fails or not these conditions, but the idea is, given some action Λ, draw a Delzant polytope whose projection on one of the axis gives the intended action. Then consider the fixed sphere for which μ is maximum and check if its first Chern number c_1, is non-negative. This is easy to do using the relations

$$c_1(B) = c_1(F) = 2, c_1(E_i) = 1.$$

If c_1 is non-negative, then we are under the conditions of Theorem 4.5. in [1]. Checking the sign of the first Chern number of the classes adjacent to the maximum fixed sphere in the polytope and comparing with Figure 1 in [1] tells us which formula we should use for $S(\Lambda)$.

Some caution is important in using these formulas: if we substitute immediately the values of μ and c_i, we can get some indeterminate results. Hence, we first use the formulas for the general case ($c_4 < c_3 < c_2 < c_1 < 1,\ c_1 + c_2 < 1$) and after some cancellations we finally substitute the correct values so that our result makes sense.

As a first case, consider $z_{0,12},\ z_{0,13},\ z_{0,14}$. If we draw the polytope for $z_{0,12}$ (Figure 4) notice that the projection on the x-axis is $z_{0,12}$, with fixed spheres $B - E_1 - E_2$ and $B - E_3 - E_4$, the latter being the maximum of the moment map. Since the first Chern number of $B - E_3 - E_4$ is not negative, we fall under the conditions of the results we know and it's easy to check that we get

$$S(z_{0,12}) = [B - E_3 - E_4] \otimes q \frac{t^{\epsilon}}{1 - t^{1-\mu}} \tag{4}$$

where $\epsilon = \frac{1}{2}$ is the value of the moment map on $B - E_3 - E_4$ when the polytope is normalized. In general, the same exact reasoning applies to $z_{0,13}$ and $z_{0,14}$ and similarly to (4) we have

$$S(z_{0,1i}) = [B - E_j - E_k] \otimes q \frac{t^{1/2}}{1 - t^{1-\mu}} = b_{jk} \frac{t^{1/2}}{1 - t^{1-\mu}} \tag{5}$$

All these elements are clearly different, as we wanted to see. The reasoning for z_1 is the same and we get

$$S(z_1) = [B + F - E_1 - E_2 - E_3 - E_4] \otimes q \frac{t^{1/2}}{1 - t^{1-\mu}} = (b_{12} + f_{34}) \frac{t^{1/2}}{1 - t^{1-\mu}} \tag{6}$$

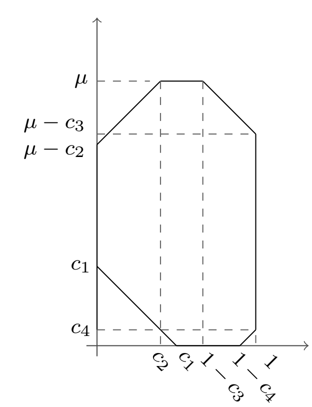


FIGURE 4. Toric action corresponding to $z_{0,12}$

To compute $S(z_{0,2})$, we first consider the toric action T_1, whose Delzant polytope is given in Figure 5.

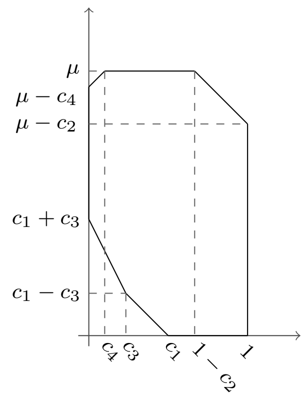


FIGURE 5. Auxiliary toric action T_1

The action projected on the x-axis is $z_{0,2}$ and let y_1 be the action on the y-axis. This polytope can't be used to compute the Seidel morphism since

$$c_1(B - E_1 - E_3 - E_4) = 2 - 1 - 1 - 1 = -1 < 0$$

To fix this situation, consider the toric action T_2 in Figure 6. Denote by x_2, y_2 the circle actions on the x and y-axis respectively. Notice that in this case $c_1 \geq 0$ so we're in the desired condition and we can compute $S(x_2)$ and $S(y_2)$.

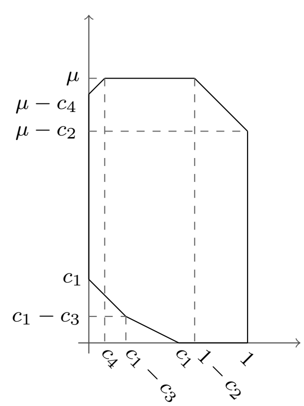


FIGURE 6. Auxiliary toric action T_2

If we transform both Delzant polytopes by the matrix $A \in GL_2(\mathbb{Z})$

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

then it's easily verified that the actions on the y-axis are the same. Hence, the relations

$$y_2 - x_2 = y_1 - z_{0,2} \implies z_{0,2} = y_1 - y_2 + x_1 \tag{7}$$

hold. Thus, if we compute $S(y_1)$ then we're done.

For that, let T_3 be the toric action represented in Figure 7. This polytope satisfies the non-negativity conditions we need and its circle action on the y-axis, y_3, is the same as y_1. Therefore, we can compute $S(y_1) = S(y_3)$.

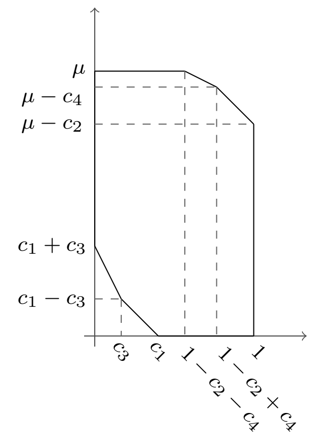


FIGURE 7. Auxiliary toric action T_3

With all this, the formulas in [1] very easily give us the expressions (for the general case):

(8)
$$S(y_3) = \frac{1}{1 - t^{c_1+c_2+c_3+c_4-\mu-1}} \left([F - E_2 - E_4] q \frac{t^{\alpha}}{1 - t^{c_2+c_4-1}} - [B - E_1 - E_3] q \frac{t^{\alpha+c_1+c_3-\mu}}{1 - t^{c_1+c_3-\mu}} \right)$$

(9)
$$S(y_2)^{-1} = [F - E_1 - E_3] q \frac{t^{\beta}}{1 - t^{c_1+c_3-1}}$$

(10)
$$S(x_2) = \frac{1}{1 - t^{c_3+c_4-\mu-1}} \left([B - E_1 - E_4] q \frac{t^{\gamma}}{1 - t^{c_1+c_4-\mu}} - [E_1 - E_3] q \frac{t^{\beta-c_1+c_3}}{1 - t^{c_3-c_1}} \right)$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ are constants which can be determined by normalizing each Delzant polytope.

By (7),

$$S(z_{0,2}) = S(y_1)S(-y_2)S(x_1) = S(y_3)S(y_2)^{-1}S(x_1)$$

and after some very tedious quantum products (and setting $c_i = \frac{1}{2}$), we get

(11)
$$S(z_{0,2}) = \frac{t^{\delta}}{(1 - t^{1-\mu})^4} \Big([B - E_1 - E_3 - E_4] q + [E_3 + E_4] q t^{1-\mu} - [B - E_1] q t^{2-2\mu} + ([B - E_1 - E_3] q)([B - E_1 - E_4] q) t^{3/2-\mu} \Big)$$

which in the notation of Section 4 is

$$S(z_{0,2}) = \frac{t^{\delta}}{(1 - t^{1-\mu})^4} \left((b_{14} - e_3) + (e_3 + e_4) t^{1-\mu} - (b_{13} + e_3) t^{2-2\mu} + b_{13} b_{14} t^{3/2-\mu} \right) \qquad (12)$$

where

$$\delta = \frac{1 - 3\mu}{3(1 - 2\mu)}$$

We gather all the result in this section in the following

Proposition 5.1. *The Seidel elements of* $z_{0,12}$, $z_{0,13}$, $z_{0,14}$, $z_{0,2}$ *and* z_1 *are given by* (5), (6) *and* (11). *In particular, all these elements have different images in* $QH_*(S^2 \times S^2 \# 4\overline{\mathbb{CP}^2})$. *Therefore they represent linearly independent elements in* $\pi_1(\mathrm{Symp}_h(S^2 \times S^2 \# 4\overline{\mathbb{CP}^2}))$ *and they generate* $\pi_1(\mathrm{Symp}_h(S^2 \times S^2 \# 4\overline{\mathbb{CP}^2}))$.

6. Circle actions on $(S^2 \times S^2 \# 4\overline{\mathbb{CP}^2}, \tilde{\omega}_\mu)$

In this section, we write the elements of $\pi_1(\mathrm{Symp}(S^2 \times S^2 \# \overline{\mathbb{CP}^2}))$ induced by circle actions in terms of the previous generators. The idea for doing this is similar to the one used to compute the Seidel elements in the previous section: by applying elements of $GL_2(\mathbb{Z})$ to the Delzant polytopes, we can transform them into other polytopes whose circle actions are either the generators or other elements which we already know how to write in terms of them.

To include the new circle actions, we need to consider *Hirzebruch polytopes*, which are Delzant polytopes for $S^2 \times S^2$ that give different toric actions. The base Hirzebruch trapezoid has coordinates

$$(0,0),\ (0,\mu-k),\ (1,\mu),\ (1,-k) \tag{13}$$

where, as usual, μ represents the symplectic area of the class B and k is the coefficient of F. The circle actions will then be obtained by blowing up (13). Since the case where $c_i = \frac{1}{2}$ is a degenerate case compared to the general case, we'll represent the polytopes assuming that $c_4 < c_3 < c_2 < c_1$ since this simplifies the figures.

Consider the elements z_k and let T_k be the toric action whose polytope is represented in Figure 8) and the auxiliary action w_k defined in Figure 9. Notice that z_k is the action projected on the x-axis and w_k is the action projected on the y-axis.

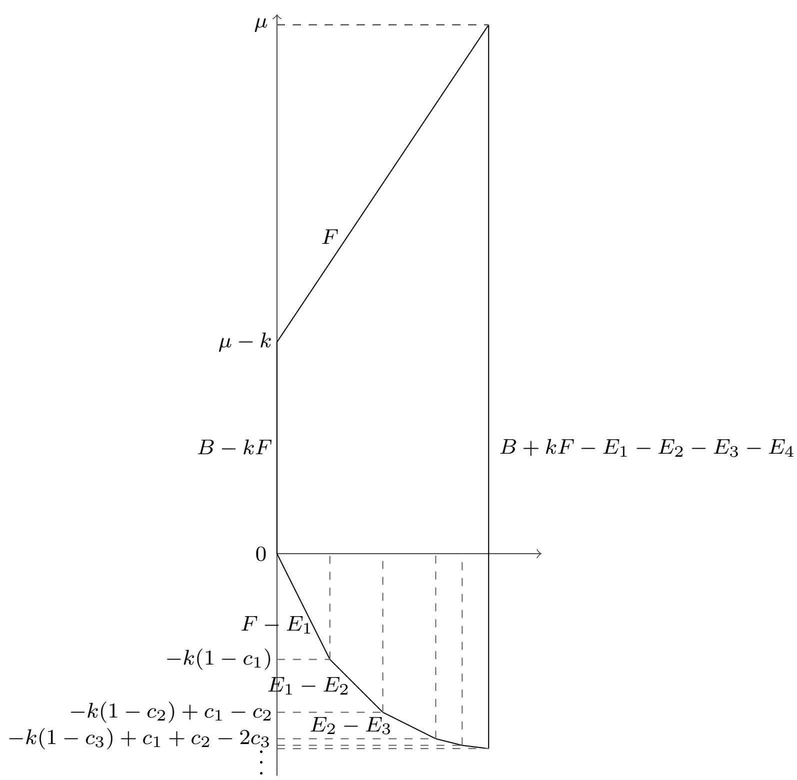


Figure 8. Toric action T_k

If we apply the transformation associated to the invertible matrix

$$\begin{bmatrix} 1 & 0 \\ j & -1 \end{bmatrix}$$

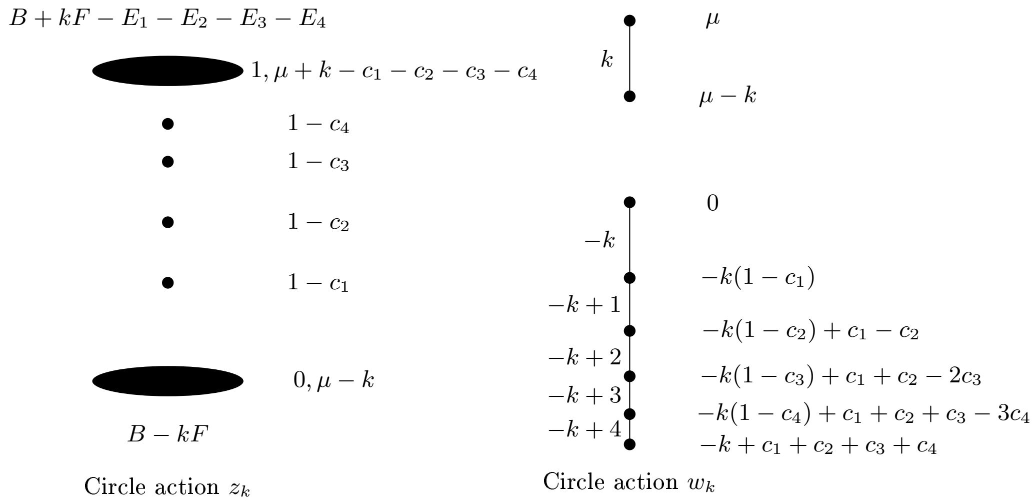


FIGURE 9. Graphs of the actions (z_k, w_k)

we get a new polytope which corresponds to the same toric action and whose vertices are

$$(1, j - \mu),\ (0, k - \mu),\ (0, 0),\ (1 - c_1, (j + k)(1 - c_1)),\ (1 - c_2, (j + k)(1 - c_2) - c_1 + c_2),$$
$$(1 - c_3, (j + k)(1 - c_3) - c_1 - c_2 + 2c_3),\ (1 - c_4, (j + k)(1 - c_4) - c_1 - c_2 - c_3 + 3c_4),\ \text{and}$$
$$(1, j + k - c_1 - c_2 - c_3 - c_4).$$

Moreover, if we were to apply the matrix

$$\begin{bmatrix} 1 & 0 \\ k & -1 \end{bmatrix}$$

then, by representing the corresponding polytope, it's easy to check that the actions obtained by projecting on the y-axis after multiplying by both of these matrices are the same. Hence, the pair (z_k, w_k) must satisfy the relation

$$jz_k - w_k = kz_j - w_j \tag{14}$$

for $k, j \geq 1$.

To get a relation for the w_j, consider the toric action T_0 represented in Figure 10 and the circle actions (z_0, y_0) in Figure 11, which correspond to the ones obtained by projecting the polytope of T_0 onto the x-axis and the y-axis, respectively.

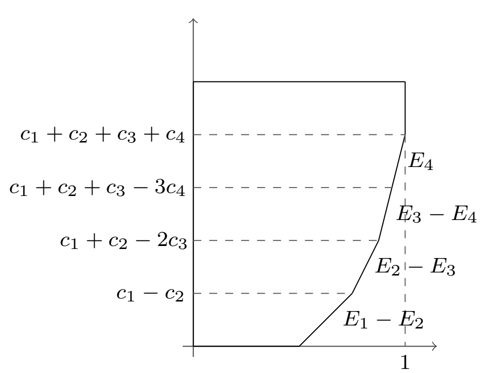


FIGURE 10. Toric action T_0

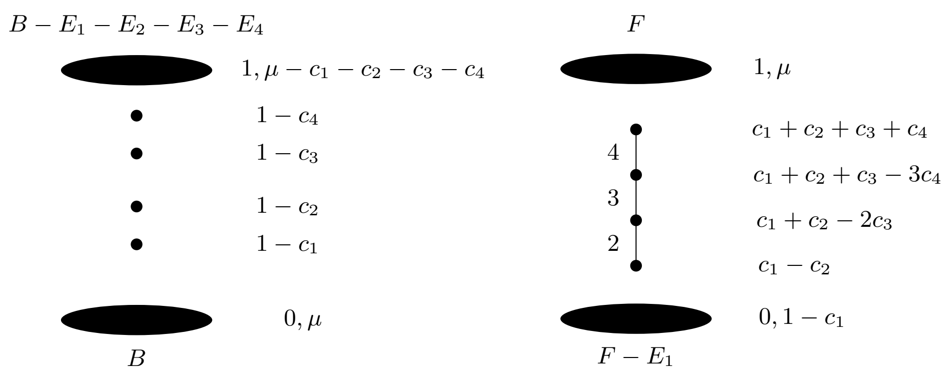


FIGURE 11. Graphs of circle actions z_0 and y_0, respectively.

If we now perform the transformation

$$\begin{bmatrix} -1 & 0 \\ -k & 1 \end{bmatrix}$$

to Figure 10, we get a new polytope representing the same toric action with vertices

$$(-1, \mu - k), (0, \mu), (0, 0), (-1 + c_1, -k(1 - c_1)), (-1 + c_2, -k(1 - c_2) + c_1 - c_2),$$
$$(-1 + c_3, -k(1 - c_3) + c_1 + c_2 - 2c_3), (-1 + c_4, -k(1 - c_4) + c_1 + c_2 + c_3 - 3c_4), \text{ and}$$
$$(-1, -k + c_1 + c_2 + c_3 + c_4).$$

If we project the new polytope onto the y-axis we now obtain w_k and so the relation

$$w_k = -kz_0 + y_0 \tag{15}$$

holds for $k \geq 1$. Substituting into (14) gives

$$jz_k + kz_0 - y_0 = kz_j + jz_0 - y_0 \implies jz_k + kz_0 = kz_j + jz_0$$

and setting $j = 1$ gives us

$$z_k = kz_1 + (1 - k)z_0$$

which only leaves us with the case of writing z_1 in term of the other generators.

The other circle actions lead to very similar results using the same kind of reasoning. Just as in the previous case, our result is dependent on us being able to write some actions in terms of the generators, namely

$$z_1, z_{0,k}, \ z_{1,4}$$

for $k \in \{1, 3, 4\}$. To do this, we need the following lemma.

Lemma 6.1. *The following relations hold:*

$$z_0 = z_{0,13} + z_{0,2} + z_{0,4} \tag{16}$$

$$z_1 + z_{0,4} = z_0 + z_{1,4} \tag{17}$$

Proof. We only prove statement (17). Consider the toric actions represented by the polytopes in Figure 12.

For the leftmost polytope, the action on the x-axis is z_1 and on the y-axis we will denote it by s_1. As for the right polytope, the action on the x-axis is $z_{1,4}$ and on the y-axis we will denote it by $s_{1,4}$. The graphs for the actions $s_1, s_{1,4}$ are represented in Figure 13.

Applying the transformation

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

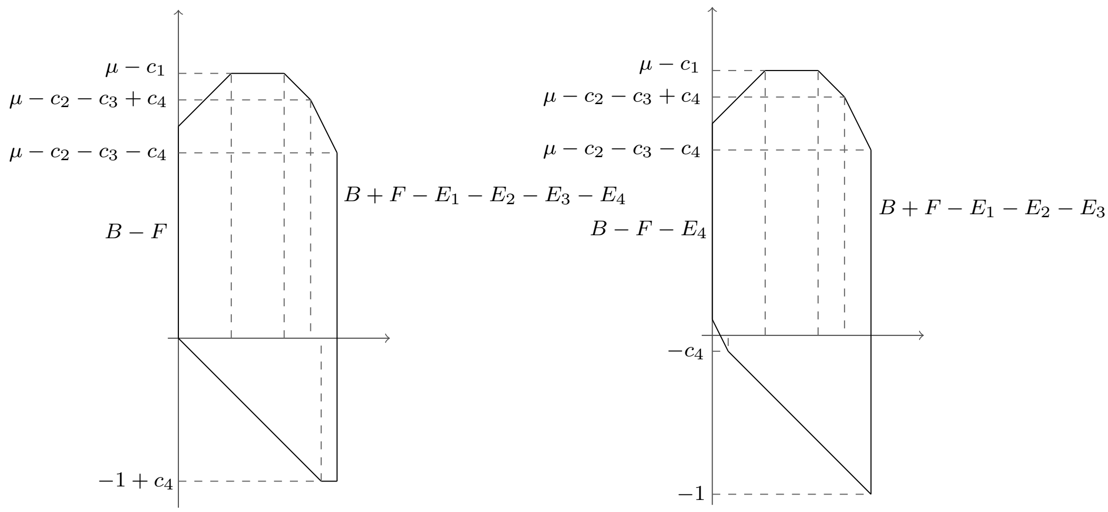
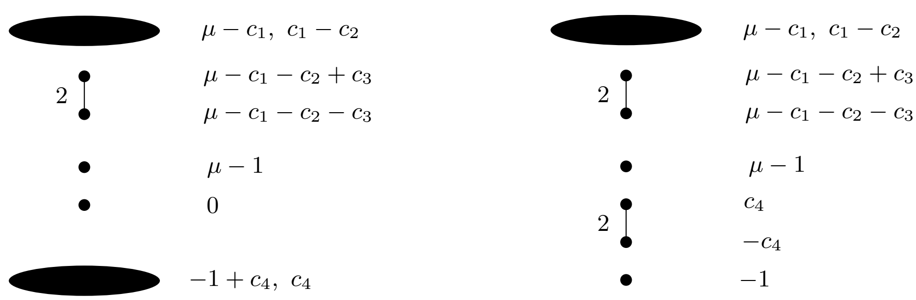


Figure 12. Auxiliary toric actions



Circle action s_1 Circle action $s_{1,4}$

Figure 13. Graphs of the actions s_1 and $s_{1,4}$

to both polytopes yields the same action on the y-axis and thus we have

$$z_1 + s_1 = z_{1,4} + s_{1,4} \tag{18}$$

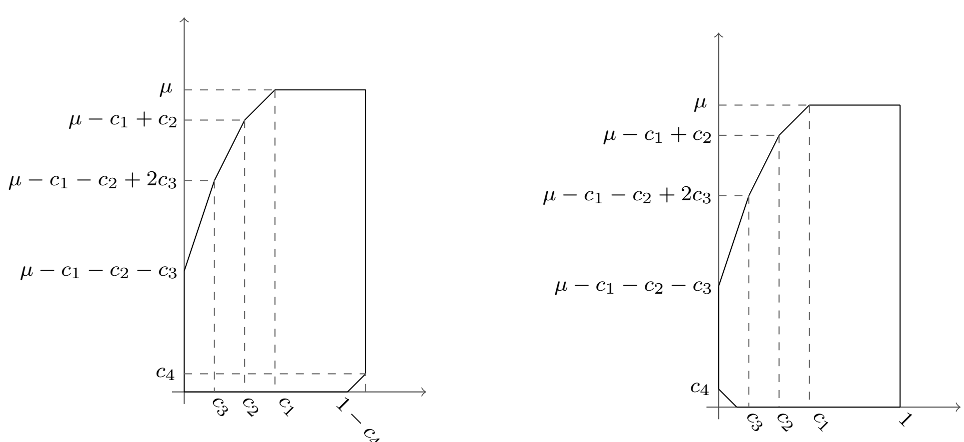


Figure 14. Toric actions (x_1, y_1) and $(x_{1,4}, y_{1,4})$

Next consider the toric actions represented in Figure 14. The circle actions are, from left to right, $(-z_{0,4}, y_1)$ and $(-z_0, y_{1,4})$. In particular, note that the graphs on the y-axis are the same so $y_1 = y_{1,4}$. If we now use the transformation

$$\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}$$

then, by projecting on the y-axis, we see that the following holds:

$$y_1 + z_{0,4} = s_1, y_{1,4} + z_0 = s_{1,4} \tag{19}$$

Substituting into (18) and using that $y_1 = y_{1,4}$ we have the desired relation. □

Proposition 6.2. *The following relations hold:*

$$z_{0,1} = z_1 - z_{1,4} + z_{0,14} \tag{20}$$
$$z_{0,2} = z_1 - z_{1,4} - z_{0,13} \tag{21}$$
$$z_{0,3} = z_1 - z_{1,4} - z_{0,12} \tag{22}$$
$$z_{0,4} = z_1 - z_{1,4} - z_{0,12} + z_{0,14} \tag{23}$$
$$z_0 = 2z_1 - 2z_{1,4} - z_{0,12} - z_{0,13} + z_{0,14} \tag{24}$$

Proof. Relation (21) is just a restatement of the first relation of the previous lemma. The other identities can be deduced from the following relations, whose proof is similar to the previous one:

$$\begin{aligned} z_{0,1} &= z_{0,14} - z_{0,12} + z_{0,2} \\ z_{0,3} &= z_{0,13} - z_{0,12} + z_{0,2} \\ z_{0,4} &= z_{0,14} - z_{0,12} + z_{0,2} \end{aligned}$$

Using the relations obtained in the lemma, we get

$$\begin{aligned} z_{0,1} &= z_1 - z_{1,4} + z_{0,14} \\ z_{0,3} &= z_1 - z_{1,4} - z_{0,12} \\ z_{0,4} &= z_1 - z_{1,4} - z_{0,12} - z_{0,13} + z_{0,14}. \end{aligned}$$

which accounts for identities (20), (21) and (23). Finally, combining the last equation for $z_{0,4}$ with identities (17) in Lemma (6.1) we arrive at

$$\begin{aligned} z_0 &\overset{(17)}{=} z_1 + z_{0,4} - z_{1,4} = z_1 + (z_1 - z_{1,4} - z_{0,12} - z_{0,13} + z_{0,14}) - z_{1,4} \\ &= 2z_1 - 2z_{1,4} - z_{0,12} - z_{0,13} + z_{0,14} \end{aligned}$$

which is (24), as we wanted. □

Taking into account all of the discussion in this section, we can now state our main result.

Proposition 6.3. *Let $k \geq 1$ such that it corresponds to an admissible element of $\pi_1(\mathrm{Symp}(S^2 \times S^2 \# 4\overline{\mathbb{CP}^2})$, $i, j \in \{1,2,3,4\}$ such that $i \neq j$ and t be the element $t = z_1 + z_{0,12} - 2z_{0,2} - z_{0,13} - z_{0,14}$. Then we have the following identities*

$$\begin{aligned} z_k &= kt + 2z_{0,2} + z_{0,14} + z_{0,13} - z_{0,12} & z_{k,ij} &= kt + z_{0,ij} \\ z_{k,1} &= kt + z_{0,2} + z_{0,13} + z_{0,14} & z_{k,124} &= kt - z_{0,2} - z_{0,13} \\ z_{k,2} &= kt + z_{0,2} & z_{k,134} &= kt - z_{0,2} \\ z_{k,3} &= kt + z_{0,2} + z_{0,13} - z_{0,12} & z_{k,234} &= kt - z_{0,2} - z_{0,13} - z_{0,14} \\ z_{k,4} &= kt + z_{0,2} + z_{0,14} - z_{0,12} & z_{k,123} &= kt + z_{0,12} - z_{0,2} - z_{0,14} \end{aligned}$$

and $z_{k,1234} = kt + z_{0,12} - 2z_{0,2} - z_{0,13} - z_{0,14}$.

Remark 6.4. *(1) Multiplying by the matrix $-I_2$, where I_2 is the 2×2 identity matrix, gives us the relations*

$$z_{0,ij} = -z_{0,kl}$$

Thus, we get

$$z_{0,23} = -z_{0,14}, \qquad z_{0,24} = -z_{0,13}, \qquad z_{0,34} = -z_{0,12}$$

which allows us to write $z_{k,ij}$ completely in term of the chosen generators.

(2) Since we already know how to compute the Seidel elements of z_1, $z_{0,2}$ and $z_{0,1j}$ for $j \in \{2, 3, 4\}$, Proposition 6.3 allows us to compute the Seidel elements of all circle actions.

7. Further Steps

As stated at the beginning, we are dealing with the case $\mu > 1$ and $c_i = \frac{1}{2}$ for all $i \in \{1, 2, 3, 4\}$. A very natural follow-up problem would be to do the same kind of computations we did, but under some other conditions. Using Table 5.1. of [8], we can suppose the parameters of our problem determine a cone (with vertices $MOABCD$) where each point corresponds to a choice $(\mu, c_1, c_2, c_3, c_4)$. For instance in the notation of [8], M corresponds to the monotone case $\left(1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ where as the cases we studied corresponds to edge MA.

Along MA the rank of $\pi_1(\mathrm{Symp}(S^2 \times S^2 \# 4\overline{\mathbb{CP}^2})$ is 5 and we saw that on the interval $\mu \in \left]1, \frac{3}{2}\right]$ one of the generators is not a circle action. This considerations led us to the following enquiries:

- Is the same true for the other edges?
- Can we identify a neighbourhood of M, U, such that all manifolds $(S^2 \times S^2 \# 4\overline{\mathbb{CP}^2})$ corresponding to parameters on U don't have $\pi_1(\mathrm{Symp}_h(S^2 \times S^2 \times 4\overline{\mathbb{CP}^2}))$ only generated by circle actions?

These questions were not answered during the project; nonetheless, they seem to be a useful generalization of the work done that might have some meaningful implications.

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