Circle actions on Symplectic Manifolds

Ana Alexandra Reis

September 13, 2018

Abstract

M_{μ,c_1,c_2,c_3,c_4} is the symplectic manifold resulting from performing 4 symplectic blow-ups on $(S^2 \times S^2, \sigma \oplus \mu\sigma)$. Its group of symplectomorphisms, G_{μ,c_1,c_2,c_3,c_4}, is a topological group. The purpose of this work is to compute a set of generators of $\pi_1(G_{\mu,c_1,c_2,c_3,c_4}) \otimes \mathbb{Q}$.

Introduction

Symplectic geometry is the study of symplectic manifolds. A symplectic manifold is a differentiable manifold equiped with a closed and non-degenerate 2-form, called the symplectic form. It is not difficult to see that the non degeneracy condition implies that symplectic manifolds can only exist in even dimensions. In dimension 2 the symplectic form is an area form on the manifold. The simplest example of a symplectic manifold is (S^2, σ), the two dimensional sphere with a symplectic form which gives area 1 to the entire sphere.

In this work we focus on a particular manifold of dimension 4, M_{μ,c_1,c_2,c_3,c_4}. Its construction is as follows: we start with $(S^2 \times S^2, \sigma \oplus \mu\sigma)$, where $\mu \geq 1$. Then, we perform 4 symplectic blow-ups of capacities $0 < c_4 < c_3 < c_2 < c_1 < c_1 + c_2 < 1$. A symplectic blow-up is a rather complex construction, but it can be looked at from a very geometric perspective. It amounts to extract an open ball from the manifold and then collapse its boundary in a particular way. The capacity of the blow-up is the size of the extracted ball given by the symplectic form.

Now that we have the manifold, we wish to focus on its group of symplectomorphisms. A symplectomorphism is a diffeomorphism $\phi : M \longrightarrow M$ which preserves the symplectic form.

It is known that the group of symplectomorphisms of M_{μ,c_1,c_2,c_3,c_4}, G_{μ,c_1,c_2,c_3,c_4}, is a topological group, therefore it is a topological space. One important feature of a topological space is its fundamental group.

The fundamental group of G_{μ,c_1,c_2,c_3,c_4} is denoted by $\pi_1(G_{\mu,c_1,c_2,c_3,c_4})$. It is known that, when $\mu = 1$, $\pi_1(G_{\mu,c_1,c_2,c_3,c_4}) \otimes \mathbb{Q}$ is generated by 14 circle actions on M_{μ,c_1,c_2,c_3,c_4}.

Circle Actions

Definition 1. *(Fundamental Group)*
Consider a topological space X and a point $x_0 \in X$. The fundamental group of X based at x_0 is the group of homotopy classes of loops based at x_0, where two loops are homotopic if they can be continuously deformed into each other.

Note that a loop is a path with the same start and finish point, so the elements of the fundamental group can be seen as homotopy classes of functions $\phi : S^1 \to M$.

Definition 2. *(Group action)*
The action of a group G, on a set X, is a map

$$\begin{gathered} G \times X \to X \\ (g, x) \mapsto gx \end{gathered}$$

so that $e_G x = x$ and $(gh)x = g(hx)$, $x \in X$, $g, h \in G$.

We say we have a **circle action** when $G = S^1$.

For the actions considered in this work, if we fix a $g \in S_1$ we get a symplectomorphism $\varphi_g : M \to M$. Therefore, each action induces a function $\phi : S^1 \to G_{\mu,c_1,c_2,c_3,c_4}$, that is, it induces an element of the fundamental group. Furthermore, it can be proved that 14 of theses elements induced by the circle actions generate the entire fundamental group and our purpose is to find a set of generators.

Polytopes

Fortunately, there is an important classification result in symplectic geometry which allows us to turn our problem of computing the set of generators into a linear algebra problem.

It is Delzant's theorem which, in dimension 4 says that there is a one-to-one correspondence between symplectic manifolds equipped with the action of a torus, T^2, and certain two dimensional polytopes (Delzant Polytopes: convex polytopes in $\mathbb{R}^2$ such that the 2 edges meeting at each vertex are given by vectors which form a basis of $\mathbb{Z}^2$), up to transformations by elements of $GL(2, \mathbb{Z})$ (However, in this work only transformations by matrices of $SL(2, \mathbb{Z})$ will be considered).

For example, the manifold $(S^2 \times S^2, \sigma \oplus \mu\sigma)$ is represented by a rectangle of lengths 1 and μ, $\mu \geq 1$.

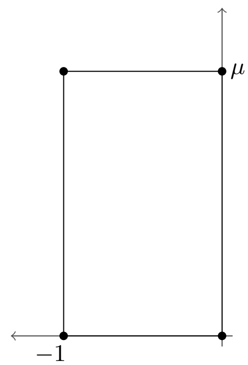


Figure 1: Polytope representing $(S^2 \times S^2, \sigma \oplus \mu\sigma)$

The polytope representation is very useful, in particular to deal with blow-ups. If we have a polytope which represents a toric manifold, to perform a blow-up of size c_1 is simply to cut a corner as in the example below:

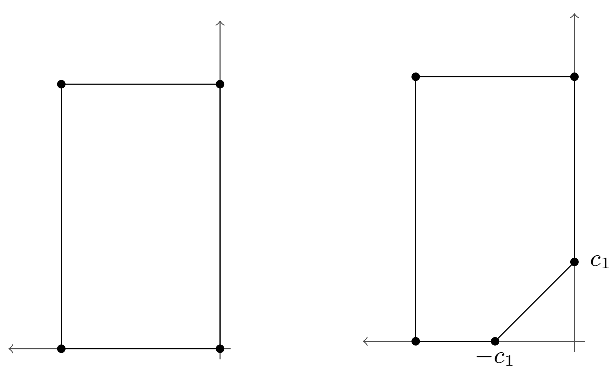


Figure 2: Blow-up

The toric actions are generated by two circle actions ($T^2 = S^1 \times S^1$). Like toric symplectic manifolds have a correspondence with polytopes, symplectic manifolds equipped with circle actions have a one-to-one correspondence with graphs originated by the projections to the axes. The following picture shows how the graphs are constructed.

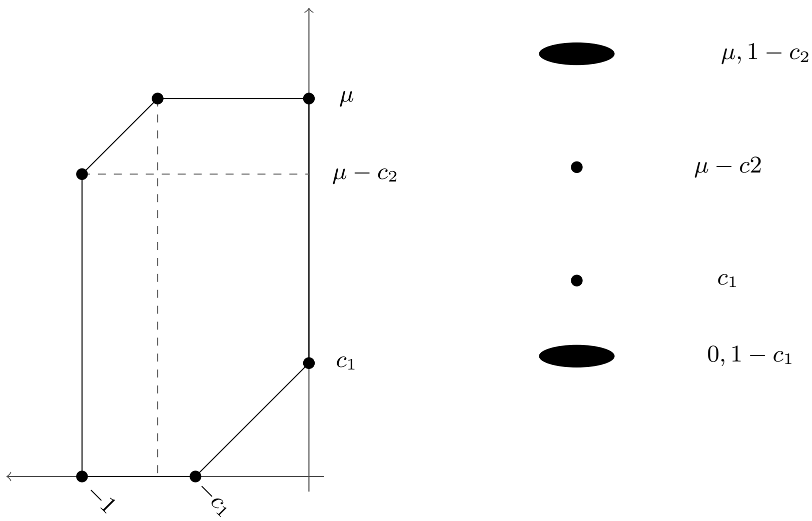


Figure 3: Projection to the y-axis

There is a one-to-one correspondence between the graphs and the symplectic manifolds equipped with circle actions. This fact will allow us to find the relations between the circle actions that will give us the set of generators.

From here the strategy to find the generators will be as follows:

- Start with all the polytopes representing the manifold $(S^2 \times S^2, \sigma \oplus \mu\sigma)$ with 3 symplectic blow-ups with all possible toric actions. There are 30 of these polytopes. Note that each polytope has 7 vertices, since for each blow-up a vertex is created.

- Then, if we perform a blow-up on each vertex of all the polytopes with 3 blow-ups we will obtain the collection of all polytopes representing M_{μ,c_1,c_2,c_3,c_4} with all possible toric actions. That is $30 \times 7 = 210$ different polytopes. The projections to the x and y-axis will yield the graphs of all possible circle actions on M_{μ,c_1,c_2,c_3,c_4}. Fortunately, some graphs come as the projections of more than one polytope. So, in fact there are only 120 different graphs. It is not difficult to see why this is so. When we do the blow-up in two vertices with the same x-coordinate (resp. y-coordinate), the resulting polytopes will have the same x-projection (resp. y-projection).

- Besides, as we saw before if we apply matrices from $SL(2,\mathbb{Z})$ to a polytope, the resulting polytope will represent the same manifold with the same toric action. However, the projections to the axes will change. The following picture shows the transformation by the matrix $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ of two polytopes.

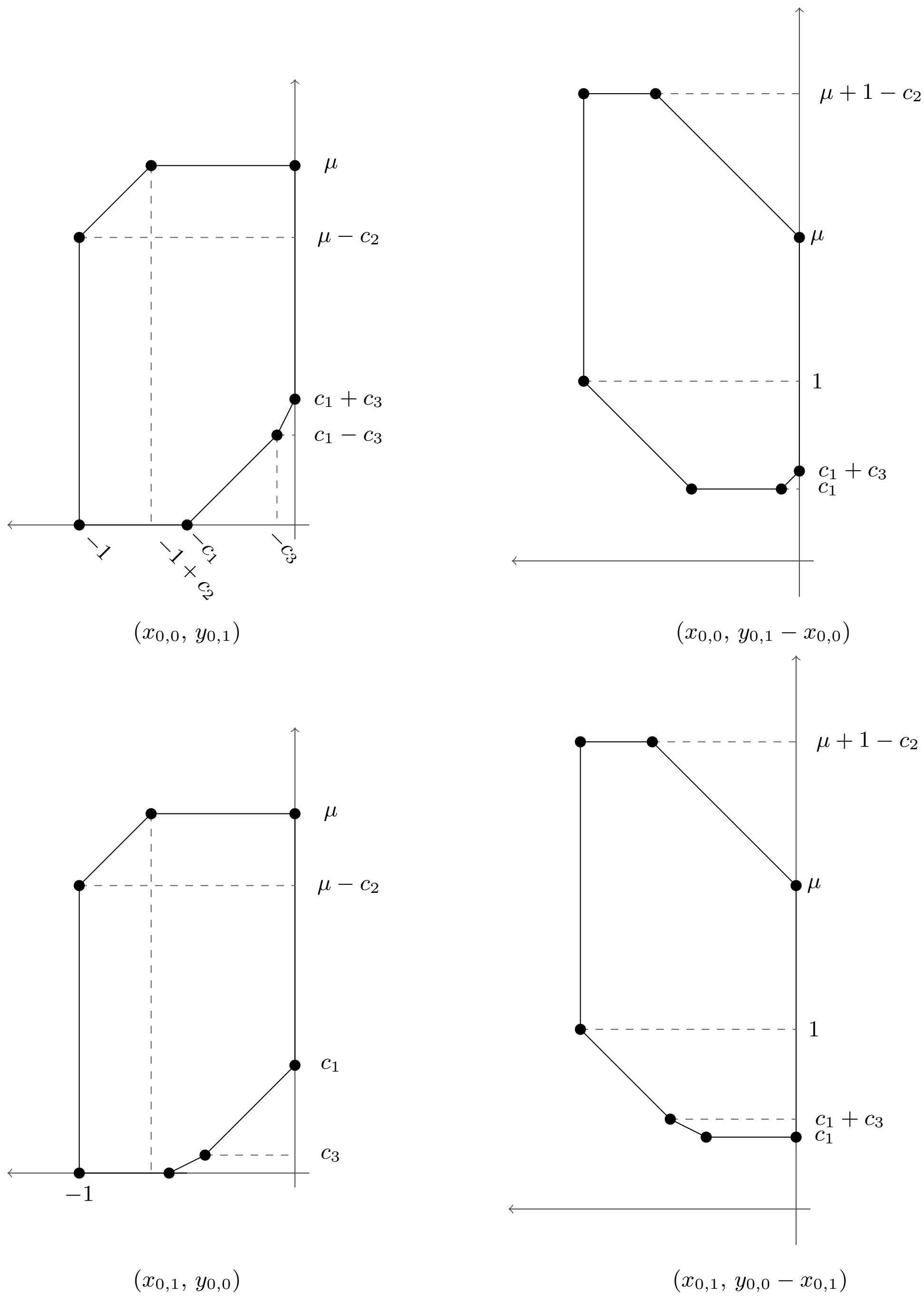


Figure 4: Transformation of polytopes by matrix in $SL(2, \mathbb{Z})$

From the picture it is easily seen that the polytopes on the right have the same projection to the y-axis. So we have the following equation relating the circle actions: $y_{0,1} - x_{0,0} = y_{0,0} - x_{0,1}$

- Since there are 14 generators and 120 different circle actions we need to find 106 linearly independent equations relating the circle actions. We will do so by using a reasoning similar to the one exemplified above.

At first, it may seem a rather cumbersome and almost humanly impossible task. However, a closer look at the problem will show that we can divide the 30 polytopes with 3 blow-ups in three different subsets according to their type,described below:

Definition 3. *We say that two polytopes, A and B* ***are related*** *if there is a matrix $M \in SL(2, \mathbb{Z})$, $M \neq Id$ so that when we apply M to the vertices of A and B, the resultant polytopes have the same y-projection.*

- **Type I**: Polytopes which are not related with any other polytope. It is easy to see that these are the polytopes with no successive blow ups, that is, all blow-ups are in different corners (See the picture below).There are 6 of these.

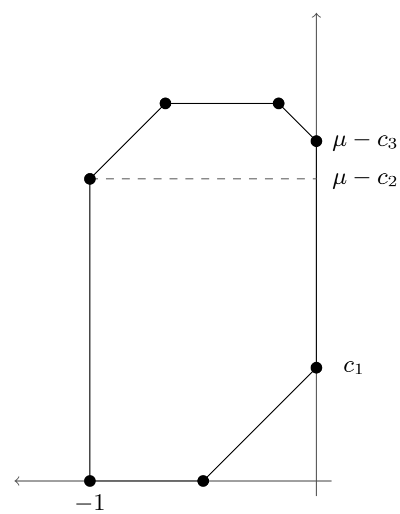


Figure 5: Type I polytope

- **Type II**: Polytopes which are pairwise related. These are the polytopes related with one and only one other polytope. We have 18 polytopes of this type.

- **Type III**: Polytopes related with more that one other polytope. There are 6 of these polytopes. This type only appears when we have the three blow-ups all in the same corner.

Now, we can analyse each type of polytope and see what equations appear when we do blow-ups on them. To do that we first need a few definitions to simplify our language.

Definition 4. *(Vertex originated by a blow-up)*
We say a vertex is originated by a blow-up if it is one of the two vertices which appear when we perform a blow-up on a particular vertex.

Note that the only vertices not originated by a blow-up are the four original vertices of the rectangle representing $(S^2 \times S^2, \sigma \oplus \mu\sigma)$.

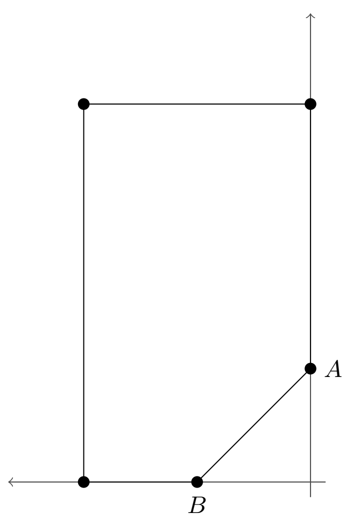
Definition 5. *We say a* ***polytope*** A^* ***comes from another polytope*** A *if it can be obtained from* A *by performing a blow-up on some of its vertices.*

Type I

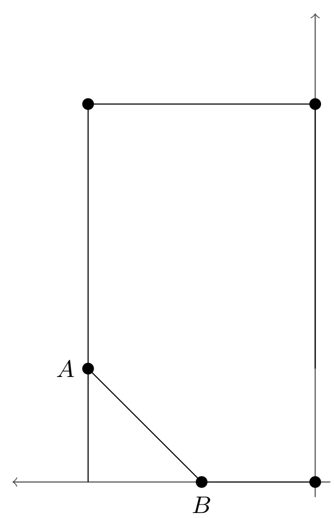
There are 6 polytopes of this type, so when performing the blow ups on all vertices we get a total of $6 \times 7 = 42$ polytopes.

From these polytopes the relations obtained come from performing blow-ups on vertices that where originated by a blow-up. The following proposition analyses what happens when we perform blow-ups on vertices originated by a blow-up.

Proposition 1. *If there is a polytope with two vertices on the line* $y = x + b$ *or* $y = -x + b$ *for some* $b \in \mathbb{R}$



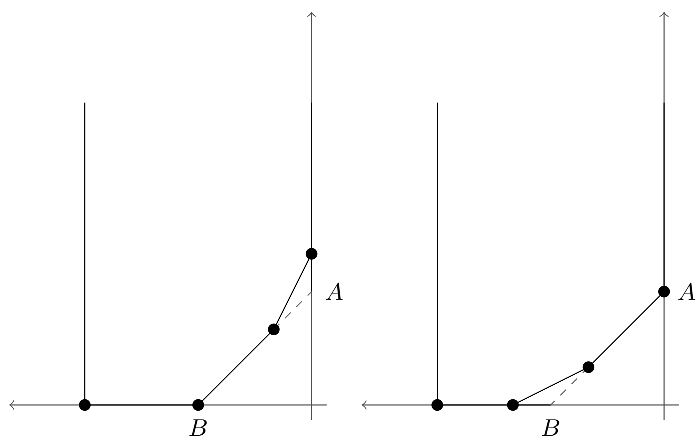
$(x_0,\, y_0)$



$(x_0,\, y_0)$

The two polytopes resulting from the blow up in each of the points A and B will give us a relation of the form $y_{0,A} - x_{0,A} = y_{0,B} - x_{0,B}$ or $y_{0,A} + x_{0,A} = y_{0,B} + x_{0,B}$

Proof. We will present the proof only for the first case, when the vertices lie on the line $y = x + b$ since the other case is analogous.
Without loss of generality, we can assume the vertices where we do the blow up are $A = (0, c_1)$ and $B = (-c_1, 0)$.
In this case, A and B lie on the line $y = x + c_1$. When we do the blow-up, the resulting polytopes will have the same vertices as the original except for the three vertices involved in the blow up.



$(x_{0,A},\, y_{0,A})$ $(x_{0,B},\, y_{0,B})$

By construction the polytopes vary only in three vertices. Let us name polytopes A and B the polytopes resulting from the blow-ups on vertices A and B, respectively.
In polytope A, we have the vertices $(0, c_1 + c_2)$, $(-c_2, c_1 - c_2)$ and $(-c_1, 0)$.
In polytope B, we have the vertices $(-c_1 - c_2, 0)$, $(-c_1 + c_2, c_2)$ and $(0, c_1)$.
If we apply matrix $M = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ to both polytopes, we get:

- Polytope A:
  $$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ c_1 + c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ c_1 + c_2 \end{pmatrix}$$
  $$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -c_2 \\ c_1 - c_2 \end{pmatrix} = \begin{pmatrix} -c_2 \\ c_1 \end{pmatrix}$$
  $$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -c_1 \\ 0 \end{pmatrix} = \begin{pmatrix} -c_1 \\ c_1 \end{pmatrix}$$

- Polytope B:

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -c_1 - c_2 \\ 0 \end{pmatrix} = \begin{pmatrix} -c_1 - c_2 \\ c_1 + c_2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -c_1 + c_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} -c_1 + c_2 \\ c_1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 0 \\ c_1 \end{pmatrix}$$

The remaining vertices are the same in both polytopes, therefore they are sent to the same vertices when the matrix is applied. Consequently, we can conclude that when matrix M is applied to A and B, the resulting polytopes will have the same y-projection, which leads to the relation $y_{0,A} - x_{0,A} = y_{0,B} - x_{0,A}$. □

We have 6 polytopes of this type and each of them has three blow-ups. So for each polytope of type I we find 3 equations. That is $6 \times 3 = 18$

Type II

Polytopes of this type are pair wise related. That means they originate from a polytope with 2 blow-ups of type I. See the following example:

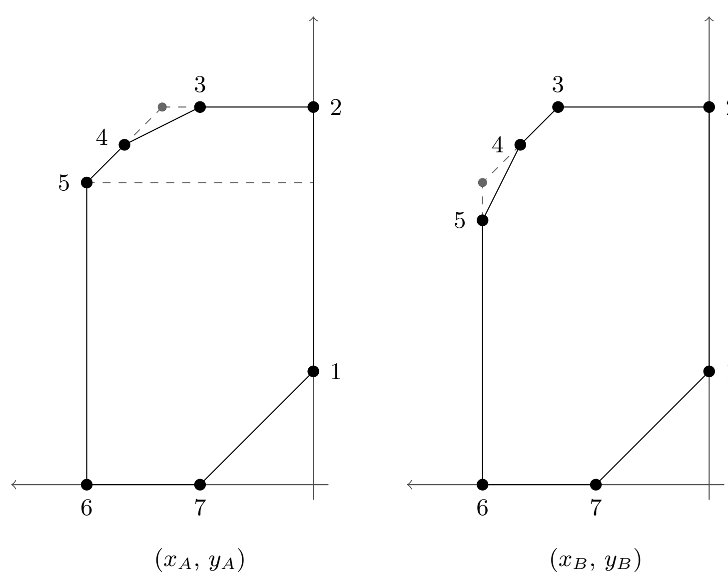


Figure 6: Type II polytopes

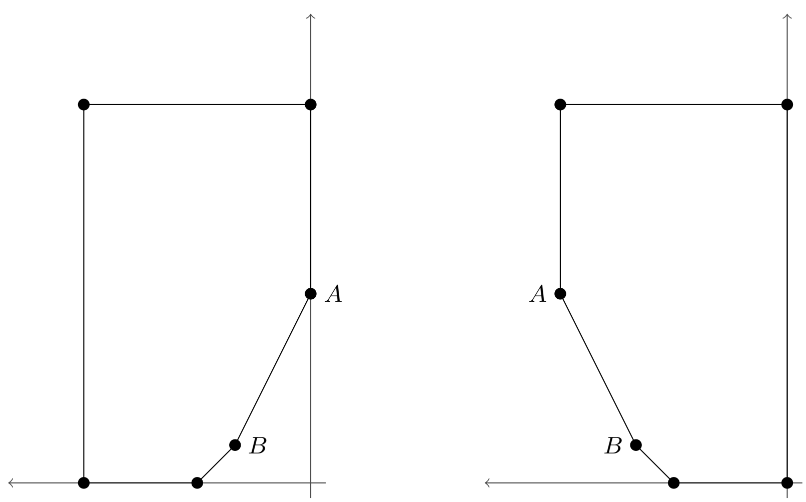
The first feature we see from the picture is that this two polytopes originate from a polytope of type I, then we have the following relation $y_A - x_A = y_B - x_B$.

Polytopes of this type always have two blow-ups in the same corner and another blow-up in a different corner.

As we can see from the picture, vertices 1,2,6 and 7 are the same in both polytopes. Consequently, when we perform the blow-up in vertices 1,2,6 or 7 in each polytope we will preserve the already existent relation. So we get an equation of the form $y_{A,i} - x_{A,i} = y_{B,i} - x_{B,i}$, with $i \in \{1, 2, 6, 7\}$.

Besides, if we look at vertices 4 and 5 of polytope A we conclude they lie on the line $y = x + b$, for some $b \in \mathbb{R}$ and the vertices 1 and 7 lie on the line $y = x + d$, for some $d \in \mathbb{R}$. Therefore, we can use **Proposition 1** to get the equations $y_{A,4} - x_{A,4} = y_{A,5} - x_{A,5}$ and $y_{A,1} - x_{A,1} = y_{A,7} - x_{A,7}$. The same phenomena happens with vertices 3 and 4 of polytope B, then we have $y_{B,4} - x_{B,4} = y_{B,3} - x_{B,3}$ and $y_{B,1} - x_{B,1} = y_{B,7} - x_{B,7}$.

Proposition 2. *If there is a polytope with two vertices on the line $y = 2x + b$ or $y = -2x + b$ for some $b \in \mathbb{R}$*



$(x_0,\ y_0)$ $(x_0,\ y_0)$

The two polytopes resulting from the blow up in each of the points A and B will be give us a relation of the form $y_{0,A} - 2x_{0,A} = y_{0,B} - 2x_{0,B}$ or $y_{0,A} + 2x_{0,A} = y_{0,B} + 2x_{0,B}$. If the points lie on the line $y = \frac{1}{2}x + b$ or $y = -\frac{1}{2}x + b$ for some $b \in \mathbb{R}$ the equation we obtain is $2y_{0,A} - x_{0,A} = 2y_{0,B} - x_{0,B}$ or $2y_{0,A} + x_{0,A} = 2y_{0,B} + x_{0,B}$, instead.

Proof. The proof of this proposition follows the same reasoning as the proof of **Propostion 1**. We simply need to consider the matrices $\begin{pmatrix} -1 & 3 \\ -1 & 2 \end{pmatrix}$ and $\begin{pmatrix} -3 & 1 \\ -1 & 2 \end{pmatrix}$ instead of $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ □

From this type we get 117 relations. However, not all of them are linearly independent. So, given the number of equations we are dealing with, Mathematica was used to compute the linearly independent equations.

Type III

There are 6 polytopes of this type. We find here many equations similar to the ones described in **Propositons 1 and 2**. Therefore on this section we will only focus on the new kind of equations that appear only in this kind of polytopes.

Since the reasoning that we need to follow is similar to what we saw in previous sections, we will simply give one example of the kind of relation that we find in this polytopes.

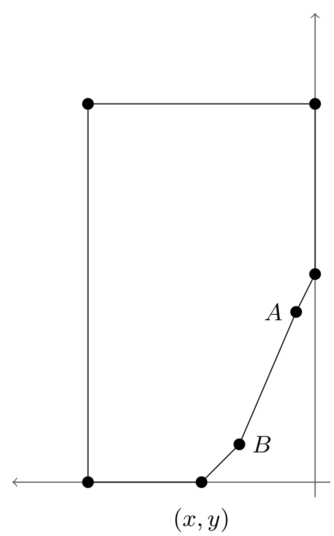


Figure 7: Polytope of type III

In this example, when we perform the polytopes resultant from performing blow-ups in vertices A and B will be related, by the following equation $2y_A - 3x_A = 2y_B - 3x_B$.

To see why this is we have to apply matrix $\begin{pmatrix} -1 & 1 \\ -3 & 2 \end{pmatrix}$

Type IV

To find the generators we need 106 linearly independent equations. However, we only found 102 linearly independent equations so far. This phenomena also appeared when this same study was done for M_{c_1,c_2,c_3}. In [2] the solution was to consider a different kind of polytopes to find the missing equations. In the end, it was also found in [2] that we could see this relation simply by analyzing some graphs. Fortunately, the same technique results here. Below we exemplify how the one of the relations can be found using the graphs. However, in order to prove the result it is necessary to use a different kind of polytopes.

It is relevant to note, that from the graphs on the x's we find 2 relations and the remaining 2 are relations on the y's, which we obtain by symmetry.

The following picture illustrates how we can use the graphs to find relations.

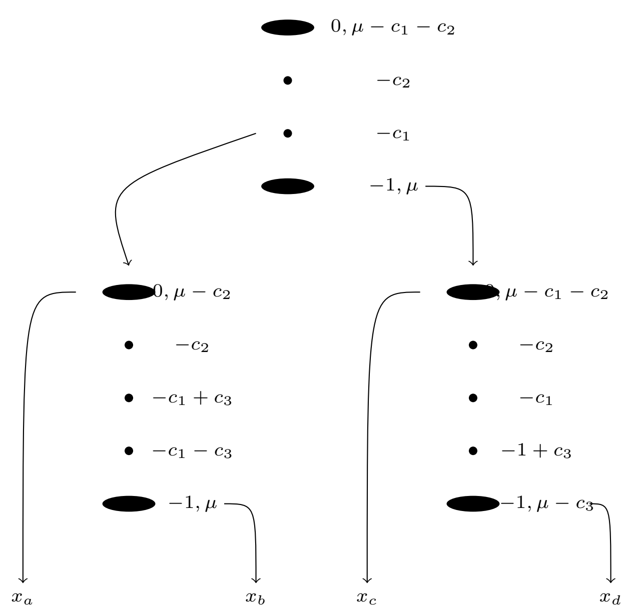


Figure 8: Graphs of x-projections

The relation we find is $x_a + x_b = x_c + x_d$.

The arrows in the picture indicate where the blow-up is perform. The blow-up is performed at the vertex corresponding to the point or surface in the tail of arrow.

A set of generators

With the 106 linearly independent equations we can choose a set of generators among the many possible sets. We chose the set:

$-1, \mu - c_2 - c_4$
$-1 + c_4$
$-1 + c_2$
$-c_1$
$-c_3$
$0, \mu - c_1 - c_3$

$-1, \mu - c_2$
$-1 + c_2$
$-c1$
$-c_3$
$-c_4$
$0, \mu - c_1 - c_3 - c_4$

$-1, \mu - c_2 - c_3$
$-1 + c_3$
$-1 + c_2$
$-c_1$
$-c_4$
$0, \mu - c_1 - c_4$

$-1, \mu - c_3 - c_4$
$-1 + c_4$
$-1 + c_3$
$-c_1$
$-c_2$
$0, \mu - c_1 - c_2$

$\mu, 1 - c_2 - c_3 - c_4$
$\mu - c_4$
$\mu - c_3$
$\mu - c_2$
c_1
$0, 1 - c_1$

$\mu, 1 - c_2 - c_3$
$\mu - c_3$
$\mu - c_2$
c_1
c_4
$0, 1 - c_1 - c_4$

$\mu, 1 - c_2 - c_4$
$\mu - c_4$
$\mu - c_2$
c_1
c_3
$0, 1 - c_1 - c_3$

$\mu, 1 - c_3 - c_4$
$\mu - c_4$
$\mu - c_3$
c_1
c_2
$0, 1 - c_1 - c_2$

$\mu+1$	$\mu+1$	$\mu+1$
$\mu+c_4$	$\mu+c_3$	$\mu+c_2$
$\mu-c_4$	$\mu-c_3$	$\mu-c_2$
$1+c_2$	$1+c_4$	$1+c_3$
$1-c_2$	$1-c_4$	$1-c_3$
c_1+c_3	c_1+c_2	c_1+c_4
c_1, c_1-c_3	c_1, c_1-c_2	c_1, c_1-c_4
μ	μ	$\mu-c_3, c_3-c_4$
$\mu-1+c_4$	$\mu-1+c_3$	$\mu-c_3-c_4$
$\mu-1-c_4$	$\mu-1-c_3$	$\mu-1+c_2$
c_1	c_1	$\mu-1-c_3$
$-c_1$	$-c_1$	c_1
$-1+c_2+c_3$	$-1+c_2+c_4$	$-c_1$
$-1+c_2, c_2-c_3$	$-1+c_2, c_2-c_4$	-1

Conclusion

The concept of fundamental group of a topological space can be generalized if we consider the set homotopy classes of functions $\phi : S^n \to X$, π_n. For a fixed n this set is called the homotopy group of order n.

The set of all rational homotopy groups, π_*, of a topological group is an algebra (it comes equipped with a product induced by the product in the group). It is an open question whether this algebra is generated by elements in $\pi_1(G_{\mu,c_1,c_2,c_3,c_4}) \otimes \mathbb{Q}$ or not. In the previous cases, that is when the manifold in question is $(S^2 \times S^2, \sigma_{std} \oplus \sigma_{std})$ with 1,2 or 3 blow-ups the full rational homotopy algebra is generated by elements in $\pi_1 \otimes \mathbb{Q}$.

Hopefully, this work will be a step towards proving the same fact for the case with 4 blow-ups.

References

[1] *What is Symplectic Geometry?* `http://math.columbia.edu/~dusa/`

[2] S. Anjos and S Eden. *The homotopy Lie algebra of symplectomorphism groups of 3-fold blow-ups of $(S^2 \times S^2, \sigma_{std} \oplus \sigma_{std})$.* to appear in Michigan Math. J.