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A Combinatorial Approach to the HOMFLY n-Specializations

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Resumo

Neste trabalho apresentamos uma definição das n-especializações do polinómio de HOMFLY por um método puramente combinatório, que permite sistematizar o cálculo das mesmas. A abordagem utilizada é análoga à feita por Kauffman ao polinómio de Jones, mas necessita também de um parêntese (função) numa classe de grafos trivalentes com dois tipos de arestas. A nossa definição segue uma via alternativa ao caminho seguido por H. Murakami, T. Ohtsuki and S. Yamada - MOY (que obtiveram uma fórmula explícita mas pouco prática para o parêntese), definindo o parêntese nos grafos através de uma lista de propriedades. Mostramos que estas propriedades permitem construir um algoritmo para avaliar qualquer grafo na classe, e obtemos unicidade através da fórmula de MOY, que satisfaz as propriedades requeridas. A demonstração é feita de forma ligeiramente diferente à presente no artigo de MOY. Os nossos resultados podem ser úteis na entendimento da categorificação, obtida por Khovanov e Rozansky, das n-especializações do polinómio de HOMFLY, visto que a categorificação deles se baseia precisamente nas relações de MOY.

Palavras-chave: Invariante de Nós, Relação Skein, n-especialização do polinómio HOMFLY, grafo trivalente, fórmula MOY

Abstract

In this work we give a definition of the HOMFLY n-specializations by purely combinatorial methods which allows for the systematical computation of these invariants. The approach taken is analogous to Kauffman's use of his bracket for defining the Jones polynomial, and requires the evaluation of a bracket on a class of 3-valent graphs with two types of edges. Our definition follows an alternative route to that taken by H. Murakami, T. Ohtsuki and S. Yamada - MOY (who give an explicit but impractical formula for the bracket), by defining the bracket on graphs by means of a list of properties. We show that these properties can be used to construct an algorithm for evaluating any graph, and then uniqueness follows from the MOY formula, which satisfies these properties, as we show in a slightly different way to the MOY paper. Our results may be useful for a better understanding of the categorification of the HOMFLY n-specializations due to Khovanov and Rozansky since their categorification was based precisely on the MOY relations.

Keywords: Knot invariant, Skein-relation, HOMFLY n-specialization, trivalent graph, MOY formula

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Introduction

In the 1980's, V. Jones, while working in the seemingly unrelated area of operator algebras, discovered a polynomial invariant of knots, which came to be known as the Jones polynomial. Despite of its unexpected and theoretically sophisticated origin, Kauffman managed to provide a combinatorial definition of it by introducing a bracket, the Kauffman bracket, and showed that it satisfies a skein-relation. This combinatorial version of the Jones polynomial is an extremely powerful and computable invariant, giving rise to an explosive development of knot theory which still continues. After Jones, in particular, mathematicians searched for more general two variable polynomial invariants and the HOMFLY polynomial was discovered.

In 2000, Khovanov found a remarkable connection between knots and homological algebra. Khovanov "categorifies" the Jones polynomial, in the following sense: he associates to each knot diagram, in a purely combinatorial way, a chain complex such that up to chain homotopy this association is a link invariant, and, moreover, the graded Euler characteristic of the chain complex is the unnormalized Jones polynomial of the diagram. Khovanov's work also raised mathematicians' hopes of reaching a better topological understanding of the Jones polynomial, which despite its usefulness is not yet completely understood from this perspective. In comparison, the Alexander polynomial, an earlier knot invariant, has topological origins which are clearly understood.

Following the same line of thought as Kauffman and Khovanov, mathematicians started to search for a combinatorial definition of the HOMFLY polynomial and a categorification for it. To get a result in this direction, it is possible to consider a special family of particular cases of the HOMFLY polynomial, called the HOMFLY n-specializations. For each $n \in \mathbb{N}$ the specialization can be defined by combinatorial methods, and the power to distinguish between knots of the whole family is the same as that of the original polynomial. Regarding the categorification, despite the fact that the HOMFLY polynomial has been categorified in work of Khovanov and Rozansky and Khovanov [12], there is interest in categorifying the n-specializations, because this is simpler and there is a better understanding of these polynomials than of the two variable HOMFLY polynomial. The categorification of the n-specializations was carried out by Khovanov and Rozansky in [6].

In this work we aim to give a definition of the HOMFLY n-specializations by purely combinatorial methods. Besides the interest of a construction by such elementary methods, this is useful as regards the use of computers to systematically and efficiently compute more knot invariants. The approach we will be describing here also provides a natural route for the categorification of the HOMFLY n-specializations. It involves relations due to H. Murakami, T. Ohtsuki and S. Yamada (see [8]), called the MOY relations, for a bracket defined on a certain type of graphs, and it was these relations which were categorified by Khovanov and Rozansky in [6]

Throughout the text we will proceed inspired by the ideas that Kauffman used to define the Jones polynomial via the Kauffman bracket. Briefly, after discussing the ideas behind the Kauffman bracket, we will introduce a bracket for knot diagrams, the n-bracket, which will play the role of the Kauffman bracket in this work. To circumvent a problem with orientations we need to use a special kind of trivalent planar graphs. We will define a bracket for these graphs, the Γ-bracket, and show that the n-bracket will be almost a link invariant if the Γ-bracket satisfies a certain list of properties. Then, we define a link invariant $I(L)$ using the n-bracket and a trick of Kauffman's. Along the way, we will discuss how the above-mentioned list of properties allows us to compute the Γ-bracket and how uniqueness is established via the MOY formula.

The organization of this text is as follows. In chapter 1, we first introduce some fundamental objects and ideas in knot theory and we use a theorem of Conway's to introduce the notion of skein-relation; in section 1.2 we discuss in some detail how Kauffman defined the Jones polynomial via the Kauffman bracket. Chapter 2 starts with a brief discussion of the HOMFLY polynomial and its n-specializations; in section 2.2 we introduce both the n- and Γ-bracket, we define $I(L)$ and search for properties that will turn $I(L)$ into a link invariant. In chapter 3, we start with a discussion of how to compute the Γ-bracket in section 3.1; in section 3.2 we present an explicit formula for the Γ-bracket and we prove that this formula satisfies the properties obtained in 2.2.

Chapter 1

The Kauffman Bracket as a Route to the Jones Polynomial

In section 1 we summarize some fundamental ideas which are extensively used throughout the text. There are comments and examples, but familiarity with the basic concepts of knot theory is recommended. In the rest of the chapter, we introduce the skein-relation of the Jones polynomial and we carry out some calculations to obtain the Kauffman bracket. These calculations are an important motivation for the work in later chapters.

1.1 Preliminaries

1.1.1 Basic Definitions

A knot is a knotted loop of string. We should think of it as a closed curve in space with no self-intersections. Figure 1.1 illustrates some of the simplest knots. Putting together a finite number of knots we obtain a link. To proceed rigorously, we have to introduce some differentiability conditions, and we have the following definitions:

Definition 1.1.1 *A* ***knot****, K, is the image of a smooth embedding of the circle, S^1, in the three dimensional space, $\mathbb{R}^3$, with non-zero derivative at each point.*

Definition 1.1.2 *A* ***link****, $L \subset \mathbb{R}^3$, is a finite union of disjoint knots.*

Figure 1.1: Three knots: the trefoil, the figure eight, and the unknot

We consider two links to be the same if they are related by an isotopy in $\mathbb{R}^3$, i.e. they can be deformed into each other in space without cutting and gluing back the string. The main problem in this field is to find an effective algorithm for deciding whether two links are the same or not.

Given any knot we can represent it by a planar curve, by taking a projection onto a plane. We need to be careful not to choose a plane for which 3 different points in space are sent to the same point by the projection. We call this projection a **knot diagram**. We call each point of the diagram which is the image of two points in space, a **crossing**. In the diagrams, we interrupt the line which is closest to the plane of projection at each crossing. This has already been done in figure 1.1. Obviously, there are many diagrams of the same knot. Furthermore there are diagrams of the same knot with different numbers of crossings.

We will say that two diagrams are the same if we can obtain one from the other by a planar isotopy. Intuitively, we should imagine that we can deform the plane as if it were made of rubber (figure 1.2). Knots and links can be oriented (figure 1.3).

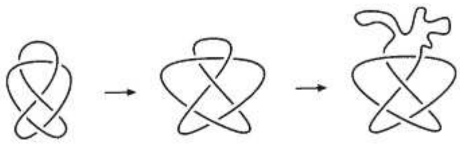


Figure 1.2: Planar isotopy

Definition 1.1.3 *An **orientation** in a knot is defined by picking a direction to travel along the knot. To orientate a link we have to orientate each component. We speak of an **oriented knot** or an **oriented link**.*

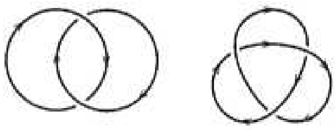


Figure 1.3: Oriented Hopf-link and trefoil

Definition 1.1.4 *The **writhe**, ω, of an oriented knot or link diagram is defined by*

$$\omega(K) = \Sigma_i \epsilon_i$$

where i runs over all crossings and the sign of the crossing ϵ_i is given by:

Figure 1.4: Positive and negative crossings.

Remark 1.1.5 *For a knot diagram the writhe is independent of the orientation chosen.*

1.1.2 The Reidemeister Theorem

As stated in the previous section, if two knots or links are related by an isotopy in $\mathbb{R}^3$ they are considered to be the same. It is clear that two isotopy equivalent knots may have different knot diagrams (see figure 1.5).

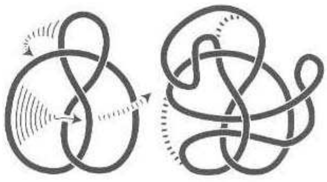
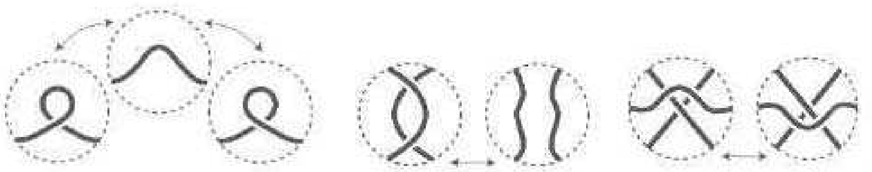


Figure 1.5: Two diagrams of the same knot.

In order to decide if two knots are equivalent, there is an important result by Kurt Reidemeister. In [11] Sossinsky gives a brief but clear description of the main ideas of the proof. For a more complete treatment see [10], a translation to English of the original reference.

Theorem 1.1.6 *(Reidemeister) Given two knots, K_1 and K_2, they are equivalent if and only if a diagram of K_1 can be obtained from a diagram of K_2 by a sequence of the following moves:*



(a) Move 1 (b) Move 2 (c) Move 3

Figure 1.6: Reidemeister moves

All diagrams are considered up to planar isotopy. The moves are local, i.e. only the part of the diagram which is represented changes, and outside this neighbourhood the diagram remains the same.

We now know that in order to decide if two knots are the same we can look at their diagrams. If we have two knots, K_1 and K_2, we could pick a diagram for K_1 and try all sequences of Reidemeister moves. If one of the resulting diagrams is a diagram of K_2, they are equivalent. Unfortunately, if the knots are different this algorithm will never stop. In fact, there is no known algorithm to distinguish between knots. Haken, in 1961, found an algorithm to decide if a knot is the unknot, but as pointed out by Sossinsky [11] it does not involve the Reidemeister moves directly and it is too complicated to put on a computer. Hence, in pratical terms there is not even an algorithm to distinguish any knot from the unknot. Figure 1.7 shows a simple application of the Reidemeister moves.

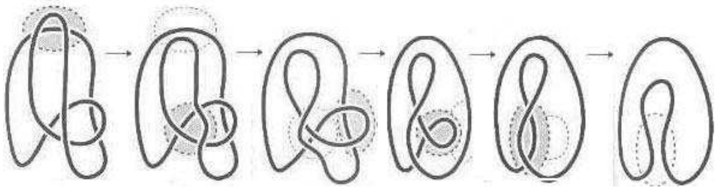


Figure 1.7: Unravelling a knot

Remark 1.1.7 *It is easy to check that the writhe is an invariant of moves 2 and 3. Move 1 changes the writhe by* $+1$ *or* -1*. Both move 1 and 2 can increase or decrease the number of crossings. Move 3 doesn't change this number.*

Fortunately, the applications of the Reidemeister theorem are not limited to this kind of algorithms. It is central in the theory of Jones and Kauffman, and it is very useful to show that some functions are **knot invariants**.

Definition 1.1.8 *We say that a function,* F*, is a* ***knot invariant****, if given any two equivalent knots,* $K_1 \sim K_2$*, we have* $F(K_1) = F(K_2)$*.*

In view of the Reidemeister theorem, if we have a function F defined on knot diagrams we only have to show that the value of F does not change when the Reidemeister moves are performed in order to get a knot invariant. This fact will be extensively used in this text.

A topological notion that has been shown to have an important connection with knots and links are the so-called **braids**. Intuitively, a braid is a set of descending strands which may be wound round each other.

Definition 1.1.9 *A braid with* n *strands,* $B \subset \mathbb{R}^3$*, is a set of* n *non-intersecting curves, where each curve connects a point in the set* $\{(1,0,1),(2,0,1),...,(n,0,1)\}$ *with a point in the set* $\{(1,0,0),(2,0,0),...,(n,0,0)\}$ *and each horizontal plane* $z = \alpha$ *with* $\alpha \in [0,1]$ *intersects each curve exactly at one point (examples in fig. 1.8).*

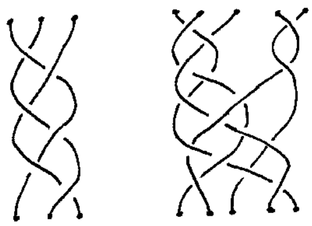


Figure 1.8: Examples of braids

Similarly to what happens with knots, the strands of a braid can be rearranged. If we don't change the initial and final points of each curve and also do not cut and glue back the curves we get a braid that looks different but is equivalent to the first braid. The next figure shows three equivalent braids.

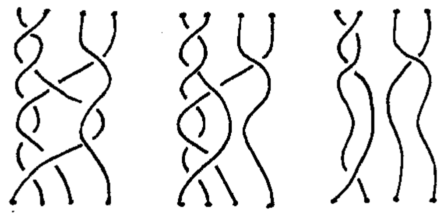


Figure 1.9: Equivalent braids

We can take the **closure** of a braid. Given a braid, B, let I and I' be the smallest intervals such that B is contained in $C_B = I \times I' \times [0, 1]$. Connect with a curve the point $(j, 0, 1)$ with the point $(j, 0, 0)$ for $j = 1, ..., n$; these curves must be parallel in the plane $y = 0$ and lie oustside the region C_B (see fig. 1.10). By this procedure we get the closure of B. This operation will always result in a knot or a link. Furthermore, the inverse is also true, a result due to Alexander.

Theorem 1.1.10 *(Alexander) Each knot or link is the closure of a braid. That is, given a link, L, there is a braid, B, such that L is equivalent to the closure of B.*

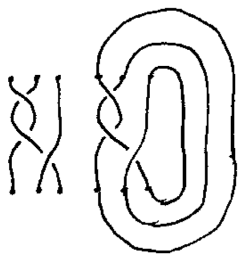


Figure 1.10: The closure of a braid.

1.1.3 Skein-relations and the Conway Polynomial

An important kind of knot invariants are the **polynomial invariants**. These are functions that associate to each class of equivalent knots a polynomial. The first of these invariants was found by Alexander, in the 1920s, and was in use for more than fifty years, until the discovery of the Jones polynomial. However, it was Conway and Kauffman who found an easy way to systematically compute these polynomials.

Theorem 1.1.11 *(Conway) Let $\mathbb{L}$ be the set of all oriented link diagram. There exists a unique map, $\nabla : \mathbb{L} \to \mathbb{Z}[x]$, which satisfies the rules:*

1. *If $L \sim L'$ then $\nabla(L) = \nabla(L')$.*
2. *$\nabla(O) = 1$, where O is the trivial knot.*
3. *$\nabla(L_+) - \nabla(L_-) = x\nabla(L_0)$, where*

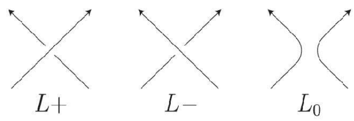


Figure 1.11: Skein-relation diagrams

Remark 1.1.12 *Putting $x = t^{1/2} - t^{-1/2}$ in the Conway polynomial gives the original polynomial found by Alexander, which despite the appearance of the change of variables is in $\mathbb{Z}[t]$.*

Rule 1 says that ∇ is a knot invariant; rule 2 says that the trivial knot has Conway polynomial equal to 1; rule 3 is usually called a **skein-relation**. Skein-relations are often used to define polynomial invariants. They are linear relations between three link diagrams which differ only in one crossing. One diagram has a crossing, another has the opposite crossing, and the third is without a crossing. For the case we are considering of oriented diagrams, we have L_+, L_- and L_0, respectively (figure 1.11).

With the skein-relation we can easily compute the Conway polynomial of the trefoil.

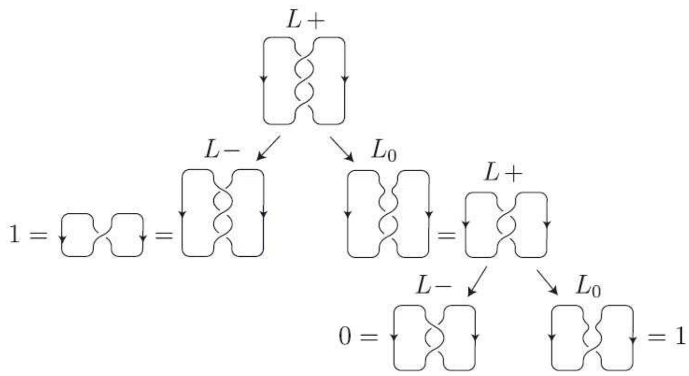


Figure 1.12: Trefoil resolving tree

We get $\nabla(T) = 1 - z(0 - z) = 1 + z^2$ (**Fact:** the unlink has Conway polynomial zero).

Furthermore, we can compute the polynomial for every link diagram. Choosing convenient crossings allows us to turn a diagram into two simpler diagrams: one with less crossings and the other closer to an unlink (see proposition and remark below). In a finite number of steps we find the value. In particular, Conway's theorem states that the values obtained are independent of the order in which we pick the crossings.

Proposition 1.1.13 *Any knot diagram can be turned into a diagram of the unknot by changing some of the crossings from over to under or vice versa.*

Proof: Given a knot diagram, pick one point on the knot which is not a crossing, and a direction to travel along the knot. While travelling along the knot once you will pass each crossing exactly twice. Each time you pass under a crossing, change that crossing. Do this until you return to the initial point. Notice that if you go around the knot again, without changing crossings, the second time you pass at each crossing you will be passing over. Now go around in the opposite direction. You will find a set of successive crossings that you will pass over. The first crossing that you pass under is one that you have already passed over. Then again you will pass over another set of crossings until you pass under one that you already have passed above (it may happen that is a crossing from the first set). Continue until returning to the initial point. As you always pass under crossings that you already have been above, this knot is unknotted.

∎

Remark 1.1.14 *It can also be shown that any n-component link can be turned into the n-component unlink by changing crossings.*

1.2 The Kauffman Bracket

In 1986, Vaughan Jones discovered another polynomial invariant. This invariant is more refined than the Alexander polynomial and was a very important achievement in knot theory. It came up when Jones was working in operator algebras, and established a connection between two previously unrelated areas. Now it is known as the **Jones polynomial**, and it can also be defined and computed using a skein-relation.

Definition 1.2.1 *The Jones polynomial is a map $V : \mathbb{L} \to \mathbb{Z}[q, q^{-1}]$, defined by the following rules:*

1. *If $L \sim L^{'}$ then $V(L) = V(L^{'})$.*
2. *$V(O) = 1$, where O is the trivial knot.*
3. *$q^2 V(L_+) - q^{-2} V(L_-) = (q - q^{-1}) V(L_0)$*

Remark 1.2.2 *If we substitute 2 by the rule $V(O) = q + q^{-1}$, we would get what is called the unnormalized Jones polynomial.*

The same reasoning as was used in the case of the Conway polynomial shows us that there is a value for each link diagram using rule 3. However, there are more aspects which need consideration to make sure that V is well defined, such as:

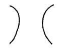
A1. The value must be independent of the order in which we choose the crossings.

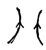
A2. This value must be consistent with rule 1.

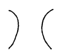
A3. The uniqueness of such a function.

The work of Jones in operator algebras provides a proof for these facts, but here we will follow the approach of Kauffman.

Suppose we want to define a polynomial invariant, $\langle L \rangle$, that can be calculated from the link diagrams. The following rules are a good start:

1. $\langle O \rangle = 1$, where O is the trivial knot,
2. $\langle L \cup O \rangle = C\langle L \rangle$,
3. $\langle$  $\rangle = A\langle$  $\rangle + B\langle$  $\rangle$.

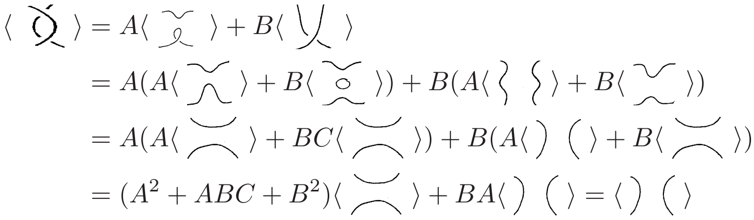
Remark 1.2.3 *Note that we are considering unoriented diagrams. In the case of oriented diagrams rule 3 would look like* $\langle$  $\rangle = A\langle$  $\rangle + B\langle$  $\rangle$ *and there is an incompatibility with the orientations in the last term.*

Remark 1.2.4 *Rule 1 is natural; rule 2 allows us to get rid of unknotted trivial components by multiplication by a constant factor; rule 3 defines our polynomial in terms of two polynomials of simpler diagrams. In the presence of the opposite crossing we can still use rule 3 from a perpendicular point of view, which gives* $\langle$  $\rangle = A\langle$  $\rangle + B\langle$  $\rangle$.

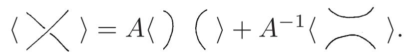
Remark 1.2.5 *To compute the bracket: first apply rule 3 at each crossing to obtain* 2^n *diagrams, where* n *is the number of crossings. Each of these diagrams are made of a finite union of disjoint 'circles'. Then apply rule 1 and 2 until you find the final value. In fact, there is an explicit formula for the bracket and so there is no ambiguity in the final value.*

Now, we want to find values for A, B, C for which $\langle L \rangle$ becomes invariant under the Reidemeister moves.

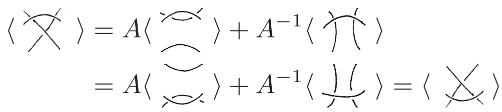
We will start with move 2:



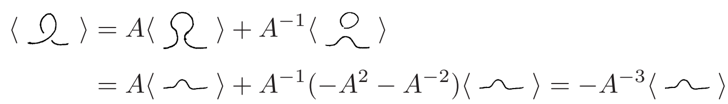
In order to get $\langle \quad \rangle = \langle \;) \; (\; \rangle$ we must have $B = A^{-1}$ and $C = -A^2 - A^{-2}$. So now our rules for the **Kauffman bracket** become

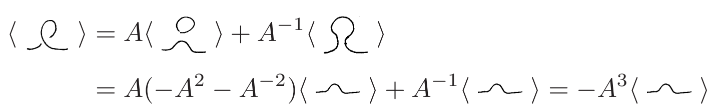
1. $\langle O \rangle = 1$
2. $\langle L \cup O \rangle = (-A^2 - A^{-2})\langle L \rangle$,
3. 

Now, move 3:



The diagrams on either side of the second equality are related by move 2 and planar isotopy, hence we have $\langle \quad \rangle = \langle \quad \rangle$. Finally, move 1:





Although the bracket is not invariant under move 1, there is a clever trick by Kauffman to solve this problem.

Definition 1.2.6 *We define the **X-polynomial** for $L \in \mathbb{L}$ to be*

$$X(L) = (-A^3)^{-\omega(L)} \langle |L| \rangle$$

where ω is the writhe and $|.|$ ignores the orientation.

Let $L, L' \in \mathbb{L}$, where L' is obtained from L by performing move 1 once, in order to get an extra positive crossing. Let us check what happens to $X(L)$:

$$\begin{aligned} X(L') &= (-A^3)^{-\omega(L')}\langle |L| \rangle \\ &= (-A^3)^{-(\omega(L)+1)}\langle |L'| \rangle \\ &= (-A^3)^{-(\omega(L)+1)}((-A^3)\langle |L| \rangle) \\ &= (-A^3)^{-\omega(L)}\langle |L| \rangle = X(L) \end{aligned}$$

Similarly, we could check that $X(L)$ is also unchanged by the other type of move 1. As $\langle L \rangle$ and $\omega(L)$ are unaffected by moves 2 and 3, therefore $X(L)$ is also unaffected by these moves, hence is an oriented link invariant.

At this point we know that $X(L)$ satisfies rules 1 and 2 of the definition of the Jones polynomial. We also have:

$$\begin{aligned} & A^4X(L_+) - A^{-4}X(L_-) = \\ &= A^4(-A^3)^{-\omega(L_0)-1}\langle \quad \rangle - A^{-4}(-A^3)^{-\omega(L_0)+1}\langle \quad \rangle \\ &= (-A^3)^{-\omega(L_0)}(A^{-1}\langle \quad \rangle - A\langle \quad \rangle) \\ &= (-A^3)^{-\omega(L_0)}(A^{-2} - A^2)\langle \ ) \ (\ \rangle \\ &= (A^{-2} - A^2)X(L_0). \end{aligned}$$

that is,

$$A^4X(L_+) - A^{-4}X(L_-) = (A^2 - A^{-2})X(L_0)$$

If we put $A^2 = -q$ we get rule 3 of definition 1.2.1

$$q^2X(L_+) - q^{-2}X(L_-) = (q - q^{-1})X(L_0)$$

Now we know that $X(L)$ satisfies all the rules in the Jones polynomial definition. Furthermore, according to remark 1.2.5 there is no ambiguity in the final value, and by construction this value is consistent with the invariance condition (this solves A1 e A2). It only remains to check uniqueness (A3), and for that the following proposition is of good use:

Proposition 1.2.7 *Let $L \in \mathbb{L}$, then $V(L \cup O) = (q + q^{-1})V(L)$, where O is a trivial component with any orientation.*

Proof: This follows easily from invariance and the skein-relation:

$$\begin{aligned} & q^2V(\quad) - q^{-2}V(\quad) = (q - q^{-1})V(\ \rangle \bigcirc \ ) \Leftrightarrow \\ & q^2V(L) - q^{-2}V(L) = (q - q^{-1})V(L) \\ & \Rightarrow V(L \cup O) = (q + q^{-1})V(L) \end{aligned}$$

■

Corollary 1.2.8 *Let O^n be the n-component unlink. Then $V(O^n) = (q + q^{-1})^{n-1}$*

Now suppose there are two different functions satisfying the Jones polynomial definition. They have to be equal for the links with 0 crossings by the corollary above. Suppose that they are equal for links with up to n crossings and consider $L \in \mathbb{L}$ with $n + 1$ crossings. We know that there is a number, n', of crossings which turn L into the unlink when they are replaced by the opposite crossing. By applying the skein-relation at one of these crossings we get two diagrams: one with n crossings and another in which we need to change $n' - 1$ crossings to get the unlink. We pick one of these crossings in the second diagram and we again apply the skein-relation. By repeating this at the n' crossings, we will get n' diagrams with n crossings and one diagram of the unlink. By the induction hypothesis all these diagrams have the same value for both polynomials, hence we have uniqueness. We have thus concluded that $X(L)$ is the Jones polynomial.

Chapter 2

The HOMFLY n-Specializations and a Bracket for Planar Graphs

In section 2.1, we introduce and discuss briefly the definitions of the HOMFLY polynomial and its n-specializations. We also mention examples for small values of n. In section 2.2, based on the ideas from chapter 1 we construct a bracket using planar graphs and then we obtain an invariant for the HOMFLY n-specializations.

2.1 The HOMFLY Polynomial and its n-Specialization

After the discovery of the Jones polynomial, mathematicians started searching for two variable polynomials that were a generalization of both Alexander and Jones polynomials. Six mathematicians discovered, independently, the same polynomial which is now called the **HOMFLY polynomial.** It is a two variable Laurent polynomial and can also be defined by a skein-relation.

Definition 2.1.1 *The HOMFLY polynomial is a map* $J : \mathbb{L} \to \mathbb{Z}[a, a^{-1}, q, q^{-1}]$, *defined by the following rules:*

1. *If* $L \sim L^{'}$ *then* $J(L) = J(L^{'})$.
2. $J(O) = 1$, *where* O *is the trivial knot.*
3. $aJ(L_+) - a^{-1}J(L_-) = (q - q^{-1})J(L_0)$

Remark 2.1.2 *The original definition was given by four different group of authors from widely different perspectives. Three groups published a joint paper on this result [2], see also the paper from the fourth group [9].*

This polynomial is in fact a generalization of both the Alexander and Jones polynomial and it is more powerful than both of them. By putting $a = q^n$ we get a polynomial invariant for each $n \in \mathbb{N}$ and considering the unnormalized version of it we have

Definition 2.1.3 *The **HOMFLY n-specialization** is a map $J_n : \mathbb{L} \to \mathbb{Z}[q, q^{-1}]$, defined by the following rules:*

1. *If $L \sim L^{'}$ then $J_n(L) = J_n(L^{'})$.*
2. *$J_n(O) = [n]$, where O is the trivial knot.*
3. *$q^n J_n(L_+) - q^{-n} J_n(L_-) = (q - q^{-1}) J_n(L_0)$*

In rule 2 we used the notation of the quantum integers, $[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + ... + q^{-n+3} + q^{-n+1}$.

Remark 2.1.4 *We can also define the 0-specialization, but in this case we have to substitute rule 2 by $J_0(O) = 1$.*

We could have substituted rule 2 by its normalized version, $J_n(O) = 1$. However, independently of the value of the trivial knot, we have $J_n(L \cup O) = [n] J_n(L)$ (in a similar way to prop. 1.2.7) and because of this relation it is natural to use the unnormalized version. The capacity of J_n to distingush knots does not change with the value of the trivial knot (so long as it is non zero) and we can get the normalized version by dividing J_n by $[n]$.

Some examples: for $n = 0$, putting $x = q - q^{-1}$ gives the skein-relation of the Conway polynomial, and from remark 2.1.4 the 0-specialization is in fact the Conway polynomial; for $n = 1$ it can be shown by using an induction argument that J_1 is constant and equal to 1; for $n = 2$ it is obvious that we have the unnormalized Jones polynomial.

Our interest in the n-specialization is mainly related to the fact that there are elementary methods for defining it. Since all approaches to the HOMFLY polynomial are theoretically sophisticated, the elementary approach that we will take to the n-specialization is interesting in itself but also very useful as regards the computation of the link invariants. These methods generalize the ideas of the Kauffman bracket and also provide a systematic way to compute the polynomials. It is true that we lose information when we pass from the two variables (a, q) of the HOMFLY to the one variable (q) of the specialization, but we can compensate this loss by considering the whole family of specializations, i.e. the information contained in the HOMFLY polynomial is the same as that contained in $(J_n)_{n \in \mathbb{N}_0}$. Here we mean that if two links are distinguished by the HOMFLY polynomial then there is an n for which they are also distinguished by J_n.

2.2 A Bracket for Planar Trivalent Graphs with Thick Edges

We could state that if the HOMFLY polynomial is well defined, and since the n-specialization is a particular case of it, it also must be well defined. However, we are interested in an approach in which we follow Kauffman's ideas. We start by finding a bracket for oriented link diagrams and then we show that with a little trick it satisfies the skein-relation.

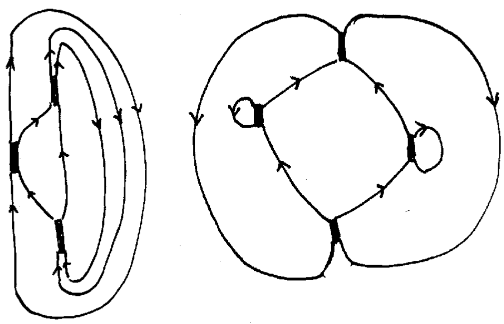
First we define a certain kind of planar graphs that we will call **classic**. Figure 2.2 illustrates two examples.

Definition 2.2.1 *Let Γ be an oriented trivalent planar graph. We say that Γ is classic if:*

1. *At each vertex two edges are "in" and one edge is "out", or two are "out" and one is "in".*
2. *An edge that at one vertex is the single "outgoing" edge must be the single "ingoing" edge at its other vertex and vice-versa. We call it a thick edge and we represent it as in figure 2.1.*

Figure 2.1: A thick edge. It is supposed to be oriented in the obvious way.

The graph with no vertices and one connected component (we will denote it by O), and the empty graph are also classic graphs. We will call O the trivial graph.

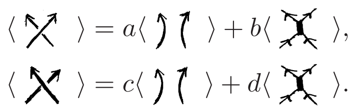


(a) Example 1 (b) Example 2

Figure 2.2: Examples of two classic graphs

As in the case of the Jones polynomial, we will define a bracket and then search for conditions that will make it a generalized invariant under the Reidemeister moves. We will call it the **n-bracket**. A first difference compared to the Kauffman bracket is that in this case we want to consider oriented diagrams. From remark 1.2.3 we know there is a problem with orientations. In order to solve it we will define the n-bracket through an auxiliary bracket for classic graphs, the **Γ-bracket**.

To simplify a link diagram, we want the n-bracket to satisfy the following recursive relations:

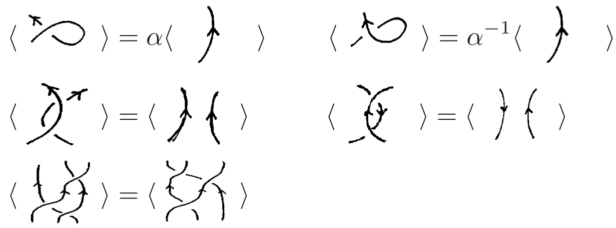


(2.1)

(2.2)

Although there is an abuse of notation in these rules, because we are using the same notation for the n-bracket and for the Γ-bracket, this should not be a problem. What we mean is that to compute the value of the n-bracket for a link diagram (m crossings) we first have to apply these rules at each crossing to get 2^m classic graphs. These graphs might have from 0 up to m thick edges (see figure 2.3 for an example). Then use the Γ-bracket, to determine the corresponding value of each graph and take the appropriate linear combination of these values.

As with the Kauffman bracket we want the n-bracket to be a generalized invariant, but this time for oriented moves, i.e. it should be invariant under the oriented versions of the Reidemeister moves 2 and 3 and invariant up to a factor under move 1. Precisely, it has to satisfy the following relations:



(2.3)

(2.4)

(2.5)

Remark 2.2.2 *Since the other oriented versions of Reidemeister move 3 can be obtained from 2.4 and 2.5, we only have to consider the case of move 3 where all the crossings are positive, i.e. 2.5.*

It is clear that for the relations to become true we have to find appropriate values for a, b, c, d and impose conditions on the Γ-bracket. A natural rule for the Γ-bracket is

$$\langle G \cup O \rangle = [n]\langle G \rangle, \tag{2.6}$$

where $\cup$ denotes the juxtaposition of two graphs. If we set $\langle \emptyset \rangle = 1$ the previous rule gives

$$\langle O \rangle = [n] \tag{2.7}$$

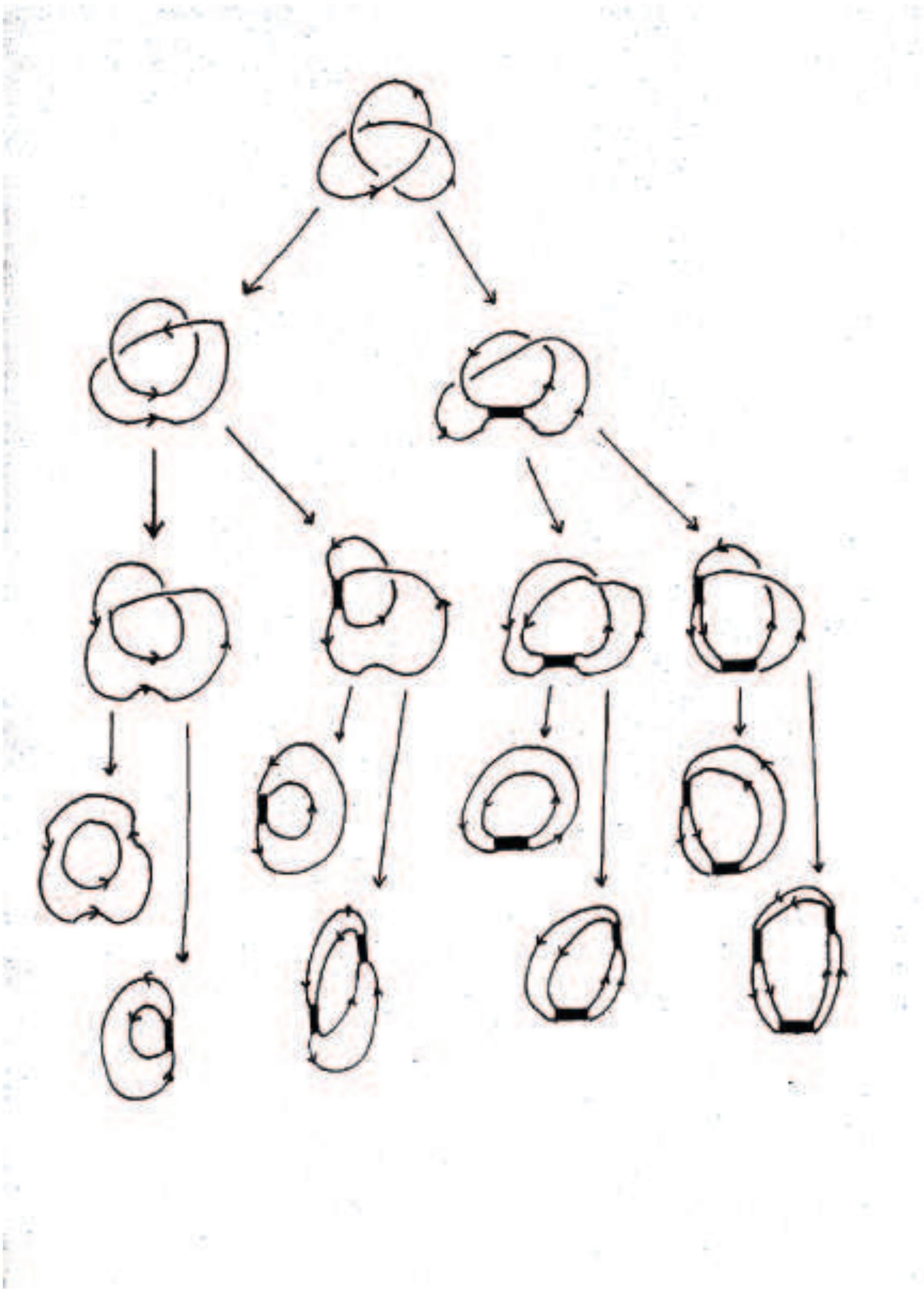


Figure 2.3: Turning the Trefoil into eight classic graphs

To find more conditions and values for a, b, c, d we will compute the n-bracket for the Reidemeister moves.

Starting with move 1:

$$\begin{aligned} \langle \quad \rangle &= c\langle \quad \rangle + d\langle \quad \rangle \\ &= c[n]\langle \quad \rangle + d\langle \quad \rangle \end{aligned}$$

hence we have $\langle \quad \rangle = \alpha^{-1}\langle \quad \rangle$ if and only if

$$\langle \quad \rangle = m\langle \quad \rangle \tag{2.8}$$

where $m \in \mathbb{Z}[q, q^{-1}]$ and we have

$$\alpha^{-1} = c[n] + dm. \tag{2.9}$$

From the other type of orientation we conclude that

$$\alpha = a[n] + bm. \tag{2.10}$$

From $\alpha\alpha^{-1} = 1$ we conclude that

$$m = \frac{-(ad+bc)[n] \pm \sqrt{(ad-bc)^2[n]^2 + 4bd}}{2bd}$$

Now, move 2:

$$\begin{aligned} \langle \quad \rangle &= a\langle \quad \rangle + b\langle \quad \rangle \\ &= ac\langle \quad \rangle + ad\langle \quad \rangle + bc\langle \quad \rangle + bd\langle \quad \rangle \\ &= ac\langle \quad \rangle + (ad+bc)\langle \quad \rangle + bd\langle \quad \rangle \end{aligned}$$

which gives $\langle \quad \rangle = \langle \quad \rangle$ if $ac = 1$ and

$$\langle \quad \rangle = -(ad+bc)(db)^{-1}\langle \quad \rangle. \tag{2.11}$$

Before checking the rest of the moves, we recall that our objective is to use this bracket to get an invariant which satisfies the skein-relation of the HOMFLY n-specialization. This way, suppose for a moment that the n-bracket satisfies equations (2.3), (2.4) and (2.5), and using Kauffman's trick define

$$I(L) = \alpha^{-\omega(L)}\langle L\rangle \tag{2.12}$$

From the same arguments as used in the case of the X-polynomial (see section 1.2) it follows that $I(L)$ is a link invariant. We observe that the n-bracket satisfies

$$d\langle \quad \rangle - b\langle \quad \rangle = (da - bc)\langle \quad \rangle$$

thus we also have

$$-d\alpha I(\quad) + b\alpha^{-1} I(\quad) = -(da - bc) I(\quad) \tag{2.13}$$

and we will get the desired skein-relation if we impose

$$-d\alpha = q^n \qquad b\alpha^{-1} = -q^{-n} \qquad -(da - bc) = q - q^{-1}.$$

As a consequence,

$$\begin{aligned} -db &= (-d\alpha)(b\alpha^{-1}) = (q^n)(-q^{-n}) = -1 \\ &\Rightarrow db = 1. \end{aligned}$$

therefore, the formula for m simplifies to:

$$m = \frac{-(ad + bc)[n] \pm \sqrt{(ad - bc)^2 [n]^2 + 4}}{2} \tag{2.14}$$

Rewriting (2.13) using (2.10) and (2.9) gives

$$(-da[n] - m) I(\quad) + (bc[n] + m) I(\quad) = -(da - bc) I(\quad) \tag{2.15}$$

Since $[n]$ is fixed and $m = m(ad, bc)$ it is clear that (2.15) only depends on the products da and bc. In practice, regarding the complete invariant $I(L)$, this means that we only have two variables which allows us to put $a = c = 1$ in (2.1) and (2.2) (since despite the simpler form of the n-bracket we still obtain a sufficiently general (2.15), for $I(L)$). Indeed by taking $a = c = 1$ and replacing b by bc in (2.1), d by dc in (2.2) we would also obtain (2.15) and it is clearly superfluous to have the product of two variables as coefficients in the relations (2.1) and (2.2).

For the conditions $db = 1$ and $-d + b = q - q^{-1}$ there is the natural solution $d = -q$ and $b = -q^{-1}$. Now, substituting in (2.14), and choosing the minus sign gives $m = [n-1]$. Thus we have found values for a, b, c, d and m and the n-bracket can be written as:

$$\langle \quad \rangle = \langle \quad \rangle - q^{-1}\langle \quad \rangle \tag{2.16}$$

$$\langle \quad \rangle = \langle \quad \rangle - q\langle \quad \rangle \tag{2.17}$$

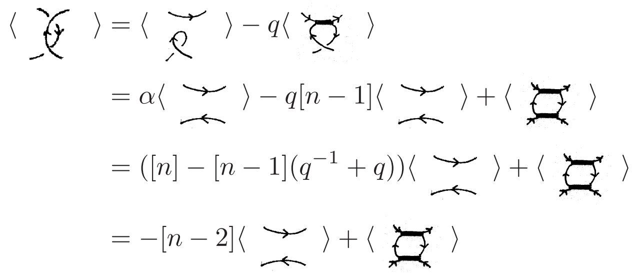
We can also rewrite (2.8) and (2.11), respectively as

$$\langle \quad \rangle = [n-1]\langle \quad \rangle, \tag{2.18}$$

$$\langle \quad \rangle = (q + q^{-1})\langle \quad \rangle = [2]\langle \quad \rangle \tag{2.19}$$

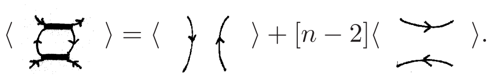
Now we return to the Reidemeister moves and we continue searching for conditions for the Γ-bracket.

Still working with move 2, from the other orientation we have



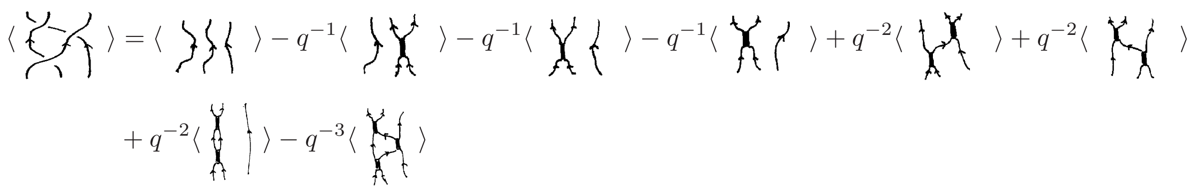
$$\begin{aligned}\langle\ \rangle &= \langle\ \rangle - q\langle\ \rangle\\ &= \alpha\langle\ \rangle - q[n-1]\langle\ \rangle + \langle\ \rangle\\ &= ([n]-[n-1](q^{-1}+q))\langle\ \rangle + \langle\ \rangle\\ &= -[n-2]\langle\ \rangle + \langle\ \rangle\end{aligned}$$

that is $\langle\ \rangle = \langle\ \rangle$ if and only if



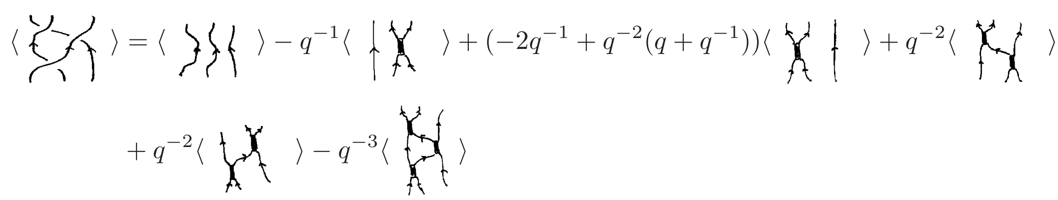
$$\langle\ \rangle = \langle\ \rangle + [n-2]\langle\ \rangle. \tag{2.20}$$

Finally, move 3



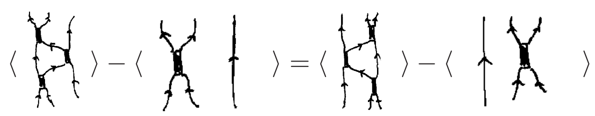
$$\langle\ \rangle = \langle\ \rangle - q^{-1}\langle\ \rangle - q^{-1}\langle\ \rangle - q^{-1}\langle\ \rangle + q^{-2}\langle\ \rangle + q^{-2}\langle\ \rangle + q^{-2}\langle\ \rangle - q^{-3}\langle\ \rangle$$

and using equation 2.19 in the penultimate term we can rewrite this as



$$\langle\ \rangle = \langle\ \rangle - q^{-1}\langle\ \rangle + (-2q^{-1}+q^{-2}(q+q^{-1}))\langle\ \rangle + q^{-2}\langle\ \rangle + q^{-2}\langle\ \rangle - q^{-3}\langle\ \rangle$$

Similarly, we can expand $\langle\ \rangle$ and easily check that $\langle\ \rangle = \langle\ \rangle$ iff



$$\langle\ \rangle - \langle\ \rangle = \langle\ \rangle - \langle\ \rangle \tag{2.21}$$

and this is our final condition.

Summarizing what we have achieved in this section: we defined the n-bracket by (2.16) and (2.17); we have shown that it is a generalized invariant (i.e. satisfies (2.3), (2.4) and

(2.5)) if the Γ-bracket satisfies (2.6), (2.18), (2.19), (2.20), (2.21); we have also used the Kauffman trick to define the link invariant $I(L)$ which satisfies the skein-relation in the definition of the HOMFLY n-specialization.

Although $I(L)$ satisfies the three rules of the definition of J_n , we still have to show that we can pick the crossings in any order to compute J_n using the skein-relation. In the case of the Jones polynomial we accomplished this because there is no ambiguity in the computation of the Kauffman bracket (see remark 1.2.5). With J_n we intend to use the n-bracket in the same way. Unfortunately, this is not as simple as in the Kauffman bracket case and for a good understanding of the computation of the n-bracket we need to look further into the Γ-bracket. In fact, at this point, we know which properties the Γ-bracket must satisfy, but we still do not know if there is a function which satisfies these properties. These questions will be treated in the next chapter.

Chapter 3

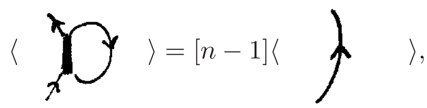
The Γ-bracket and the MOY Formula

In section 3.1, we discuss the existence of a function satisfying the properties obtained in the previous chapter and how these properties can be used to compute the Γ-bracket. In section 3.2 we show that a particular case of a formula due to H. Murakami, T. Ohtsuki and S. Yamada (see [8]), which we will refer to as the MOY formula, corresponds to the Γ-bracket.

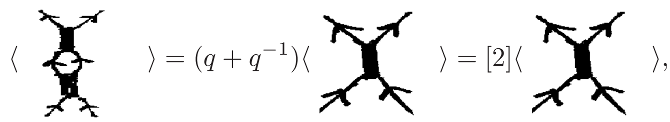
3.1 Computing the Γ-bracket

In chapter 2 we have seen that in order for the n-bracket be a link invariant we need the Γ-bracket to satisfy the following properties:

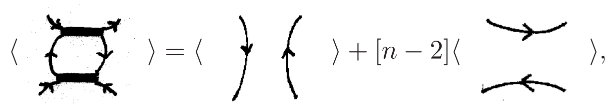
$$\langle G \cup O \rangle = [n]\langle G \rangle, \tag{3.1}$$



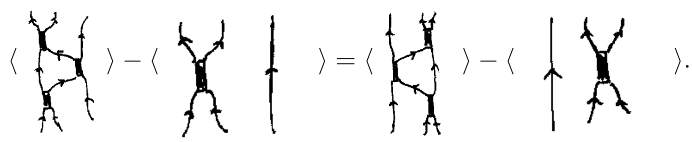
(3.2)



(3.3)



(3.4)



(3.5)

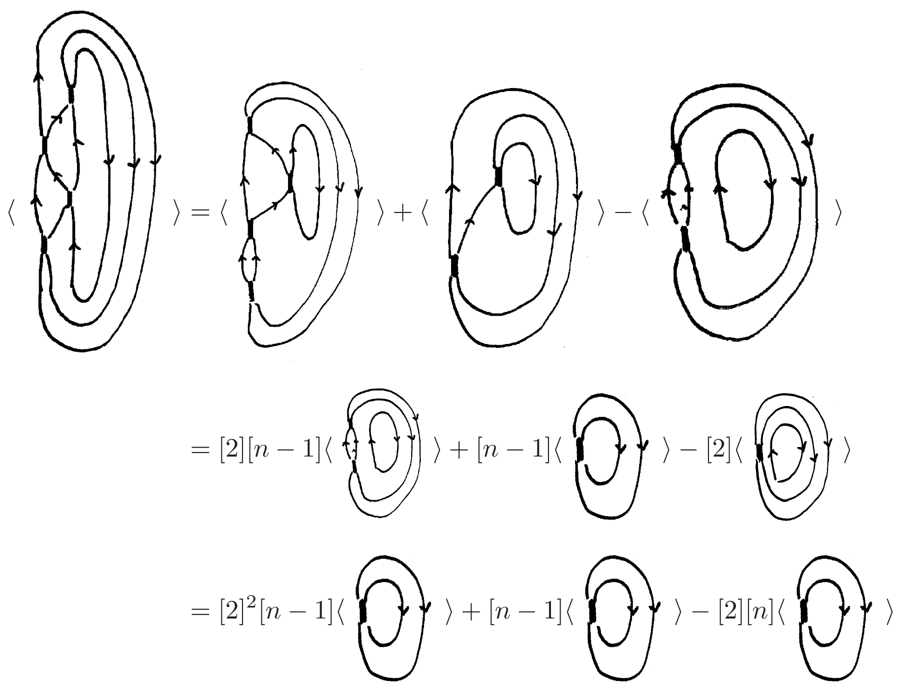
A natural question that arises at this point is if there is any function which actually satisfies this list of properties. The answer is yes, and our purpose in this chapter is to justify this.

First remember that (3.1) implies $\langle O\rangle = [n]$ (we have set $\langle \emptyset \rangle = 1$), which means that we have a value for the trivial graph. Given a classic graph, if we are able to turn it into a linear combination of trivial graphs then we are able to get a value for the initial graph; we just have to take the sum of the coefficients times $[n]$. Notice that this is completely analogous to what happens when computing the Kauffman bracket.

We hope that to achieve the desired linear combination we only have to use (3.1), (3.2), (3.3), (3.4), (3.5), this way we do not have to impose more conditions. In other words, we want to simplify a given classic graph using only the last four of these five properties. Here simplify means decrease the number of thick edges.

According to theorem 1.1.10, we can assume that every classic graph comes from an oriented link that is the closure of a braid. Since we are interested in obtaining a link invariant, despite the fact that we are reducing the set of classic graphs for which we will study how to compute the Γ-bracket, this assumption does not cause a loss of generality. The value that the link invariant gives to a link and to its braid form must be equal. The next example illustrates how we can apply the rules to simplify a graph and obtain its value.

Example 3.1.1



$$= [n-1]([2]^2[n-1]\langle O\rangle + [n-1]\langle O\rangle - [2][n]\langle O\rangle)$$
$$= [n][n-1]^2[2]^2 + [n-1]^2[n] - [n]^2[n-1][2]$$

Remark 3.1.2 *Since we are interested in graphs coming from braids we never have to use (3.4), because its edges are oriented in a way which can not occur coming from a braid.*

In the rest of this section, every time we write graph we want to mean classic graph coming from a braid. Now we will clarify two important points of this process of simplification:

1. It must be possible to use (3.2) or (3.3) or (3.5) for any graph with more than zero thick edges. By use we mean substitute one side of a rule by the other side of the same rule, as was done in the example.
2. It must be possible to simplify any graph by applying a finite sequence of these rules.

Let G be a graph and consider it as living in S^2. In order to answer point 1 we are interested in thinking of G as a finite number of polygons (faces); these polygons may share an edge; some edges might be curved and both thick and normal edges count as edges. We have the following proposition and lemma:

Proposition 3.1.3 *Every face of G, including the outer face, has an even number of edges.*

Proof: Let P be a face of G with w edges. If P has no thick edges then the orientation of the edges must change at each vertex and thus we must have an even number of edges. Now suppose P has at least one thick edge. Then P must also have two normal edges, one before and other after the thick edge, and these three edges have compatible orientations. Now we can substitute these by a single normal edge which respects the orientation, and we obtain P'. The number of edges of P' is $w - 2$. If P' still has a thick edge we perform this operation again and get P''. We repeat this for each thick edge of P. In the end we get a polygon with only normal edges (see figure below).

(a) Removing an edge　　(b) Removing another edge　　(c) Only normal edges

Figure 3.1: Simplifying a face

As we already know this final polygon must have an even number of edges. Moreover, since at each step the number of edges decreases by 2, we also have removed an even number of edges. Therefore, P has an even number of edges.

■

Lemma 3.1.4 *Every classic graph ($\neq \emptyset, O$) has a face with two or four edges.*

Proof: Let V, A, F be the number of vertices, edges and faces, respectively. Since the graph is trivalent we have $2A = 3V$. From the previous proposition we can write

$$F = F_2 + F_4 + F_6 + ... + F_{2k} + ...$$

where F_{2k} is the number of faces with $2k$ edges.

Each edge belongs to at most two faces and thus we have

$$2A \geq \Sigma_{n=1}^{\infty} 2nF_{2n} \geq 2F_2 + 4F_4 + 6\Sigma_{n=3}^{\infty} F_{2n} = 2F_2 + 4F_4 + 6F'$$

Supposing that $F_2 = F_4 = 0$ then $F = F' \Rightarrow \frac{A}{3} \geq F$.

From the Euler formula we must have $V - A + F = 2$, but this gives a contradiction, because

$$V - A + F = \frac{2}{3}A - A + F = -\frac{A}{3} + F \leq 0 < 2$$

We conclude that we must have $F_2 \neq 0$ or $F_4 \neq 0$.

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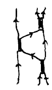
A simple inspection of a short list of cases shows that the possibilities for a face with two edges in a graph are  and  ; moreover, for a face with four edges the possibilities are  and  . Notice, as an example, that  can not occur because of the orientation.

From lemma 3.1.4 we know that given any graph there is always a face with two or four edges, hence we can always apply (3.2) or (3.3) or (3.5). This concludes the first point.

It is clear from the expressions of (3.2), (3.3) (and also (3.4)) that these rules simplify a diagram. However, (3.5) allows us to substitute a diagram with three thick edges in a specific position by three diagrams, two of which are simpler diagrams (two less thick edges) and one diagram with three thick edges in another specific position. It is not immediate that this new diagram with three thick edges can be simplified. Therefore, we need to be careful with graphs where only (3.5) can used (like the graph in example 3.1.1) before we take point 2 to be true. To end this section we will present the idea of an algorithm to simplify this kind of graph.

Remark 3.1.5 *In the following algorithm we will often say 'apply (3.5)'. We want this to mean that: we use (3.5) after using a planar isotopy in order to get the specific position for the thick edges, if needed; also, we ignore the simpler diagrams resulting from (3.5). This* *way, in practice we are only substituting locally a specific distribution of three thick edges by another one.*

Suppose that only (3.5) can be performed in our graph. The algorithm consists in applying (3.5) repeatedly in a way which will lead us to a configuration that can be simplified using (3.2) (depending on the graph it may happen that we also apply (3.3) a number of times, which speeds up the simplification). This will be achieved by successively reducing the number of thick edges between the last two strands (i.e. furthest to the right). When this number is one we are able to perform (3.2). In example 3.1.1, we only needed to apply (3.5) once before we were able to simplify the graph.

Let n be the number of strands. It should be observed that there must be at least two thick edges between the $(n-1)$-th and n-th strands. Otherwise there are only one or zero, but if there are 0 then in the initial link there were no crossings involving the last strand and thus it is an unlinked component and can be ignored. If there is one we can use (3.2) which contradicts the hypothesis.

We can suppose that all the thick edges are at different heights. We execute the following procedure: (*) focus on the horizontal band between the first two (counting from top to bottom) thick edges between the $(n-1)$-th and n-th strands. In this band there must be at least one thick edge between the $(n-2)$-th and $(n-1)$-th strands, because otherwise we could apply (3.3).

(A) If there is only one (as in ex. 3.1.1) we can apply (3.5) and we have reduced by one the total number of thick edges between the $(n-1)$-th and n-th strands. Now if it is possible to apply (3.2) we are done. If not apply (3.3) as many times as possible (it may be zero) and then repeat the procedure (*) with the new initial pair of thick edges between the last two strands.

(B) If there are more than one thick edges between the $(n-2)$-th and $(n-1)$-th strands we focus on the band between the first two thick edges between these strands. Note that this band is included in the previous one. Once again there must be at least one thick edge between the $(n-3)$-th and $(n-2)$-th strands: (1) if there is only one thick edge we can apply (3.5) and we have reduced the number of thick edges between the $(n-2)$-th and $(n-1)$-th strands. Now apply (3.3) as many times as possible (it may be zero) and go back to (*). (2) If there are more than one thick edges we go one step left and we keep repeating the procedure.

Regarding point B above, since the number of strands and thick edges is finite and we are considering a sequence of bands each of which is included in the previous one there always must be a step where we find only one thick edge. When this happen we use (3.5) and return to (*). This concludes the algorithm and establishes point 2.

At this point we know that given any link diagram there is a value for it using the n-bracket. We start by picking one of its braid representations and we apply (2.16) or (2.17) at each crossing to get a finite number of graphs. Finally we use the rules of the Γ-bracket to simplify them and we take the sum of the values.

However, this is not everything, since we still do not know if the values obtained with the Γ-bracket are independent of the order in which we apply the rules during the simplification. We answer this question in the next section.

3.2 The MOY Formula for Classic Graphs.

The formula that we will use here is a special case of the MOY formula which is defined in [8] for a more general type of graphs. We will show that this formula satisfies all the required properties of the Γ-bracket. This way we guarantee that the values obtained using the simplifying rules are independent of the order in which we use them and thus we establish the uniqueness of the Γ-bracket defined in terms of (3.1-3.5).

3.2.1 The Definition

Let $n \in \mathbb{N}$ and $n \geq 2$ and put $X = \{-(n-1), -(n-1)+2, ..., n-1\}$.

Remark 3.2.1 *X is the set of exponents appearing in the sum formula for $[n]$.*

A **state**, σ, of a classic graph is an assignment of an element $e \in X$ to each normal edge such that if two normal edges end in the same thick edge they must have different elements assigned to them and, moreover, if two elements go into a thick edge the same two elements must also go out of the same edge at the end. Figure 3.2 illustrates a state.

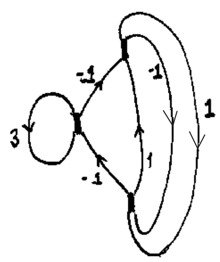


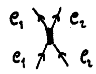
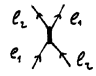
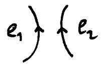
Figure 3.2: Example of a state for $n = 4$

Given a state, σ, we define the **weight** of a vertex v to be

$$w(v, \sigma) = q^{\delta},$$

where $\delta = 1/2$ if $e_1 < e_2$ or $\delta = -1/2$ if $e_1 > e_2$ and the outgoing/incoming edges are labelled as follows 

.

After fixing a state, at each thick edge we must have  or . Now replace every thick edge by  or , respectively. This gives rise to a finite set of simple closed curves (which may intersect each other) such that each curve is associated with an element $e(C) \in X$ (see fig. 3.3). Then we define the **rotation number**, $rot(\sigma)$ to be

$$\sum_C e(C) rot(C)$$

where the sum is over all the curves and $rot(C)$ is -1 or 1 if C is oriented clockwise or counter-clockwise, respectively.

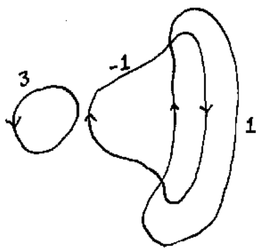


Figure 3.3: Curves corresponding to the state in fig. 3.2

As an example, the state represented in figure 3.2 with curves in fig. 3.3 has rotation number $(3)(1) + (-1)(-1) + (1)(-1) = 3$. Finally, we define the Γ-bracket to be

$$\langle G \rangle_\Gamma = \sum_\sigma \left\{ \prod_v w(v, \sigma) \right\} q^{rot(\sigma)}, \tag{3.6}$$

where σ runs over all states of G and v runs over all vertices of G. We also put $\langle \emptyset \rangle_\Gamma = 1$. It is clear that $\langle \quad \rangle_\Gamma$ is invariant under planar isotopy.

3.2.2 The Properties

Now we will prove the properties. We will denote the Γ-bracket simply by $\langle \quad \rangle$.

Proposition 3.2.2 *Let G be a classic graph. The Γ-bracket satisfies*

$$\langle G \cup O \rangle = [n] \langle G \rangle$$

where O is oriented counter-clockwise.

Proof: Observe that a state, σ, of $G \cup O$ is given by a state of G, σ_G, and an element $k \in X$ associated to O; the vertices of $G \cup O$ are those of G and $rot(\sigma) = rot(\sigma_G) + rot(O)$. Then we have

$$\begin{aligned}
\langle G \cup O \rangle &= \sum_{\sigma} \left\{ \prod_{v} w(v, \sigma) \right\} q^{rot(\sigma)} = \sum_{(\sigma_G, k)} \left\{ \prod_{v_G} w(v_G, \sigma_G) \right\} q^{rot(\sigma_G) + rot(O)} \\
&= \sum_{k \in X} \left(\sum_{\sigma_G} \left\{ \prod_{v_G} w(v_G, \sigma_G) \right\} q^{rot(\sigma_G)} \right) q^k \\
&= \sum_{k \in X} \langle G \rangle q^k = [n] \langle G \rangle
\end{aligned}$$

■

Remark 3.2.3 *If O were oriented clockwise the same proof would hold beacause of the symmetry in the elements of X.*

Proposition 3.2.4 *The Γ-bracket satisfies*

$\langle$  $\rangle = [2] \langle$  $\rangle$

Proof: Consider a state of the left hand side; each edge of the 2-face in the middle must have an element assigned to it. They have to be two different elements, say α and β, and suppose $\alpha < \beta$. Now it should be observed that by permuting α with β we get another state of the left hand side, and by removing the assignments to the 2-face we get a state of the right hand side; also it is easy to see that the rotation number of a state on the left is independent of the assignments to the 2-face and equal to the rotation number of the state of the right hand side obtained as above.

Denote the vertices at the ends of the edges bounding the 2-face by v_1 and v_2. Suppose that the left edge has α assigned to it, this way the weight of both v_1 and v_2 is $q^{1/2}$ and in the permutated state it is $q^{-1/2}$. Thus the contribution of v_1 and v_2 is a factor $q + q^{-1} = [2]$ and we can write

$\langle$  $\rangle = \sum_{\sigma} \left\{ w(v_1, \sigma) w(v_2, \sigma) \prod_{v \neq v_1, v_2} w(v, \sigma) \right\} q^{rot(\sigma)}$

$$= [2] \sum_{\sigma_{right}} \left\{ \prod_{v_{right}} w(v_{right}, \sigma_{right}) \right\} q^{rot(\sigma_{right})}$$

$= [2] \langle$  $\rangle$

where σ_{right} runs over all the states of the right hand side and v_{right} runs over all the vertices of the right hand side.

■

Proposition 3.2.5 *The* Γ*-bracket satisfies*

$\langle$  $\rangle = [n-1]\langle$  $\rangle$

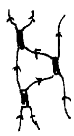
Proof: Consider a state of the left hand side. Observe that the left hand ingoing/outgoing edges must have the same element, α, assigned to them. Notice that a state of the left hand side can be obtained from a state of the right hand side by assigning an extra element to the new normal edge. If the new normal edge is labeled with $\beta \neq \alpha$ the weight of the vertices in the thick edge is $q^{sign(\beta-\alpha)/2}$. Since the contribution to the rotation number on the left hand side is $-\beta$, for each α (determined by a state of the right side) the additional factor on the left hand side is

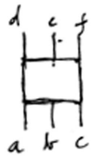
$$\begin{aligned}\sum_{\substack{\beta\neq\alpha\\ \beta\in X}} q^{sign(\beta-\alpha)-\beta} &= \sum_{\substack{\beta=-(n-1)\\ \beta\in X}}^{\alpha-2} q^{-1-\beta} + \sum_{\substack{\beta=\alpha+2\\ \beta\in X}}^{n-1} q^{1-\beta}\\ &= q^{n-2} + q^{n-4} + ... + q^{-\alpha+1} + q^{-\alpha-1} + ... + q^{-n+2}\\ &= [n-1]\end{aligned}$$

which completes the proof.

■

Proposition 3.2.6 *The* Γ*-bracket satisfies*

$\langle$  $\rangle - \langle$  $\rangle = \langle$  $\rangle - \langle$  $\rangle$ (3.7)

Proof: We will denote by D_i $(i = 1, 2, 3, 4)$ the diagram corresponding to the i-th term in the equation counting from left to right and by G_i the whole graph. Given a state of G_i, D_i has the form of a permutation box, that is,  , where $(d, e, f) = \pi(a, b, c)$ is a permutation of $a, b, c \in X$. We should specify the permutation before possibly choosing values for a, b, c which are not distinct, so that when in this case we know which upper strand is associated to which lower strand.

Let Z denote the complement of D_i in G_i (note that Z is the same for each D_i); for a given state σ, let $w_Z(\sigma)$ and $w_{D_i}(\sigma)$ be the product of the weights of the vertices in Z and D_i, respectively. Fix three different values for a, b, c and a permutation, $\pi \in S_3$, and define $S_i = S_i(a, b, c, \pi(a, b, c))$ to be the set of the states of G_i which are compatible with the fixed choice; notice that for $i = 2, 4$ it may happen that for some choices there are zero compatible states. Moreover, let M_i be the set of the states of G_i where a, b, c do not have three different values and observe that the following equation holds

$$\begin{aligned}\langle D_i \rangle &= \sum_{\sigma} \left\{ \prod_{v} w(v, \sigma) \right\} q^{rot(\sigma)} \\ &= \sum_{\substack{a,b,c \in X \\ \pi(a,b,c)}} \sum_{\sigma \in S_i} w_Z(\sigma) w_{D_i}(\sigma) q^{rot(\sigma)} + \sum_{\sigma \in M_i} \left\{ \prod_{v} w(v, \sigma) \right\} q^{rot(\sigma)}\end{aligned}$$

The first equality is just the definition; in the second, besides the comments just before the equation we are using the fact that any state of G_i for which a, b, c are distinct must be in S_i for some choice.

Any state of G_i also determines a labeling of D_i and Z; we will call these labelings local states of D_i and Z, and write $S_{|D_i}$ and $S_{|Z}$ for the sets of all the possible local states (compatible with the choice previously made) of D_i and Z, respectively; notice that for $a, b, c, \pi(a, b, c)$ fixed $S_{|Z}$ is the same for all i. Now observe that the contribution of each vertex is local and regrouping the states by these labelings we can write

$$\sum_{\sigma \in S_i} w_Z(\sigma) w_{D_i}(\sigma) = \sum_{\sigma' \in S_{|D_i}} w_{D_i}(\sigma') \sum_{\sigma' \in S_{|Z}} w_Z(\sigma').$$

Furthermore, the rotation number is not affected by the local state of D_i, the labelings in Z are enough to determine it. Here we mean that when constructing the curves for computing $rot(\sigma)$ (as in fig. 3.3) we can simply connect the ingoing and outgoing strands of D_i which have the same labeling. Therefore, we have the following expression:

$$\langle D_i \rangle = \sum_{\substack{a,b,c \in X \\ \pi(a,b,c)}} \left(\sum_{\sigma' \in S_{|Z}} w_Z(\sigma') q^{rot(\sigma')} \right) \sum_{\sigma' \in S_{|D_i}} w_{D_i}(\sigma') + \sum_{\sigma \in M_i} \left\{ \prod_{v} w(v, \sigma) \right\} q^{rot(\sigma)}.$$

Now observe that (3.7) will be true if we have the following three equations:

$$\sum_{\sigma' \in S_{|D_1}} w_{D_1}(\sigma') - \sum_{\sigma' \in S_{|D_2}} w_{D_2}(\sigma') - \sum_{\sigma' \in S_{|D_3}} w_{D_3}(\sigma') + \sum_{\sigma' \in S_{|D_4}} w_{D_4}(\sigma') = 0, \tag{3.8}$$

for each choice of distinct a, b, c and $\pi \in S_3$, and

$$\sum_{\sigma \in M_1} \left\{ \prod_v w(v,\sigma) \right\} q^{rot(\sigma)} - \sum_{\sigma \in M_2} \left\{ \prod_v w(v,\sigma) \right\} q^{rot(\sigma)} = 0, \tag{3.9}$$

$$\sum_{\sigma \in M_3} \left\{ \prod_v w(v,\sigma) \right\} q^{rot(\sigma)} - \sum_{\sigma \in M_4} \left\{ \prod_v w(v,\sigma) \right\} q^{rot(\sigma)} = 0. \tag{3.10}$$

Checking that (3.8) holds for each choice is a repetitive task and we will present here only the simplest and hardest cases; all the others follow exactly in the same way.

First notice that the number of compatible labelings for D_i is between 0 and 2 depending on i and on the permutation; it should also be observed that there are only two possibilities for the labeling of a thick edge:  and . In the first case the product of the weight of the two vertices is always 1; in the second case it is q if $a < b$ and q^{-1} if $b < a$.

Now consider the permutation: $\mathbf{d} = \mathbf{c}$, $\mathbf{e} = \mathbf{b}$, $\mathbf{f} = \mathbf{a}$.

There are no local states for D_2, D_4 and only one for D_1 and D_3, which are represented in the following figures.

Figure 3.4: Single possibility for D_1

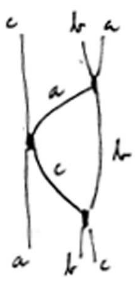
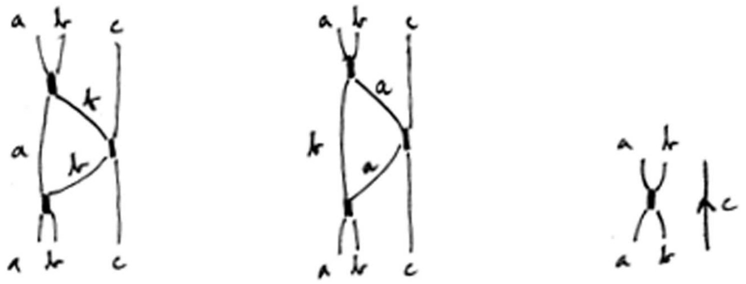


Figure 3.5: Single possibility for D_3

Looking at the product of weights of the vertices coming from each thick edge, it is clear that in this case $w_{D_1} = w_{D_3} = 1$ and thus (3.8) holds.

Now consider the permutation: $\mathbf{d} = \mathbf{a}$, $\mathbf{e} = \mathbf{b}$, $\mathbf{f} = \mathbf{c}$.

For the terms on the left hand side there are two local states for D_1 and one local state for D_2, which are represented in the next figures.

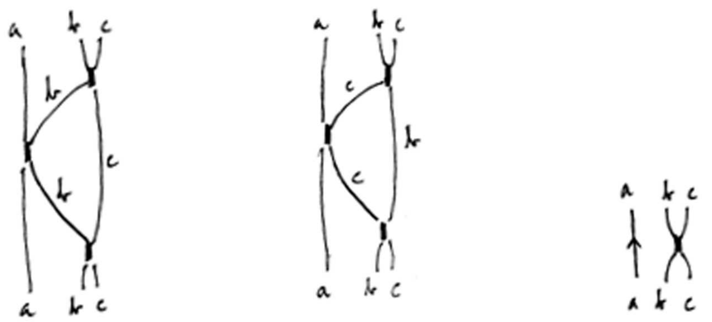


(a) D_1 (b) D_1 (c) D_2

Figure 3.6: Terms on the left hand side.

From figure (a) we have (1) if $a < b$ then $w_{D_1} = q^3$ if $b < c$ or $w_{D_1} = q$ if $c < b$; (2) if $b < a$ then $w_{D_1} = q^{-3}$ if $c < b$ or $w_{D_1} = q^{-1}$ if $b < c$. From figure (b) we have $w_{D_1} = q$ if $a < c$ or $w_{D_1} = q^{-1}$ if $c < a$; from figure (c) $w_{D_2} = q$ if $a < b$ or $w_{D_2} = q^{-1}$ if $b < a$.

Now for the right hand side there are two local states for D_3 and one for D_4, which are represented in the next figures.



(a) D_3 (b) D_3 (c) D_4

Figure 3.7: Terms on the right hand side.

From figure (a) we have (1) if $b < c$ then $w_{D_3} = q^3$ if $a < b$ or $w_{D_3} = q$ if $b < a$; (2) if $c < b$ then $w_{D_3} = q^{-3}$ if $b < a$ or $w_{D_3} = q^{-1}$ if $a < b$. From figure (b) we have $w_{D_3} = q$ if $a < c$ or $w_{D_1} = q^{-1}$ if $c < a$; from figure (c) $w_{D_4} = q$ if $b < c$ or $w_{D_4} = q^{-1}$ if $c < b$.

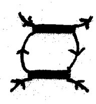
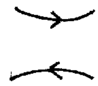
Now it remains to check that for any value of a, b, c we have (3.8) for this permutation. As an example, suppose $a < b < c$. In this case we have $q^3 + q - q - q^3 - q + q = 0$ and (3.8) holds. A completely analogous analysis can be performed for all the other choices of a, b, c and $\pi(a, b, c)$.

The same line of argument as was used to obtain (3.8) can also be used to turn equations (3.9) and (3.10) into a form such that these equations only depend on the local states for D_1,D_2 and D_3,D_4, respectively. Then we only have to perform a similar case by case inspection to the one just described.

■

At last, we reach our goal with the next proposition.

Proposition 3.2.7 *The* Γ*-bracket satisfies*

$\langle$  $\rangle = \langle$  $\rangle + [n-2]\langle$  $\rangle$

Since we are considering links in braid form, rule (3.4) takes on a background role. Also, its proof does not involve any new idea so we omit the proof. Finally, we summarize everything with

Theorem 3.2.8 *The* Γ*-bracket satisfies the rules (3.1), (3.2), (3.3), (3.4), (3.5) and is explicitly given by (3.6). These rules can be applied without ambiguity in order to compute the* Γ*-bracket instead of using the formula.*

Proof: The first sentence follows immediately from the propositions proved in this section and the second is a direct consequence of the existence of a well-defined formula.

■

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