Quotient 2-groups and the Postnikov decomposition

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1 Introduction

Groups are a mathematical object that has proved to be useful in many areas of mathematics. One of the possible generalizations of the concept of group is that of a *group object* in a category, with groups being the group objects of the category of sets. In this article, we will be concerned with 2-groups, the group objects of the category of categories.

In this article we extend the basics of group theory to 2-groups, such as subgroups and quotient groups; we call the 2-group analogues of these concepts 2-subgroups and quotient 2-groups, respectively. We provide crossed module versions of these concepts; along the way, we provide a full proof of the equivalence between the 2-category of 2-groups and the 2-category of crossed modules, a result known by Brown and Spencer ([2]), but a full proof could not be found in the literature. We also provide some constructions which we think might be useful in the future, such as the 2-subgroup generated by a set of morphisms, and the normal closure of a 2-subgroup.

Most group theory results also hold for 2-groups, such as the isomorphism theorems; we state these for 2-groups and prove them. The work mentioned is quite straightforward and similar to already existing group theory; however, the concept of 2-group quotients allows for a different point of view on the Postnikov decomposition: this provides a decomposition of any 2-group $\mathcal{G}$ by a "2-exact" sequence (as Elgueta calls it in [4])

$$\mathrm{Id}_1 \to \pi_1(\mathcal{G})[1] \xrightarrow{J} \mathcal{G} \xrightarrow{P} \pi_0(\mathcal{G})[0] \to \mathrm{Id}_1;$$

we provide a decomposition of any 2-group $\mathcal{G}$ by a central series

$$\mathrm{Id}_1 \triangleleft \pi_1(\mathcal{G})[1] \triangleleft \mathcal{O}(\mathcal{G}) \triangleleft \mathcal{G};$$

with quotients $\mathcal{G}/\mathcal{O}(\mathcal{G}) \cong \pi_0(\mathcal{G})[0], \mathcal{O}(\mathcal{G})/\pi_1(\mathcal{G})[1] \simeq \mathrm{Id}_1$ and $\pi_1(\mathcal{G})[1]$.

The differences between group theory and 2-group theory start to appear when we start considering equivalence between 2-groups, which is a concept that does not make sense for groups. There are different notions of 2-groups, and also different notions of equivalence between them, so there must be different ways to approach this question; indeed, when studying weak 2-groups, the tendency is to use the weakest notion of equivalence available, as is done in [4]; when studying strict 2-groups, we will use the strictest notion available.

Weak 2-groups can be classified up to equivalence by their homotopy invariants: the homotopy groups and the Postnikov invariant, which is a 3-cocycle (for details, see [4] and its references). We will see that for strict 2-groups these invariants are not actually invariants; we propose an alternative for the Postnikov invariant which is not a 3-cocycle, but rather an equivalence class of exact sequences which seems to be impossible to represent in terms of group cohomology.

We discuss the problem of determining wether two given 2-groups are equivalent in some detail; it turns out that every finite 2-group has a minimal 2-subgroup to which it is equivalent, called its *homotopically minimal* 2-subgroup, and that any two finite 2-groups are equivalent if and only if their homotopically minimal 2-subgroups are isomorphic. We compute the homotopically minimal 2-subgroups of the automorphism 2-group of an arbitrary dihedral group.

This article intends to be quite clear and self-contained, providing proofs of most statements, even some well-known results. However, for the sake of brevity, some of the longer proofs are ommited, such as the proof of the equivalence of the categories of 2-groups and crossed modules, and the proof of any result regarding the Postnikov invariant.

It should also be said that in this article we will not cover generalizations of concepts such as group actions, free groups, group presentations and group representations. We will focus on strict 2-groups, mentioning weak 2-groups, as we have, to discuss their classification up to equivalence, but no more than that. Also, we will assume that all 2-groups are small groupoids, in order to avoid any possible set theoretical issues.

We finish this introduction with a brief description of the structure of the article. Sections 2-4 set the notation for the rest of the article and gather together well known information about 2-groups and crossed modules, including homotopy invariants; the only new thing here seems to be the proof of the equivalence of the 2-categories of 2-groups and crossed modules. Sections 5-6 cover the technical work on 2-subgroups and quotient 2-subgroups in detail; the main definitions, examples and constructions are also done in crossed module language; in particular, section 6 goes over our new interpretation of the Postnikov decomposition. Section 7, the final section, covers the main results of this article regarding 2-group equivalence.

Group theory notation: We denote groups by $G, H, K, \dots$. We write $H \leq G$ if H is a subgroup of G and $H \lhd G$ if it is a normal subgroup. We write $Z(G), \mathrm{Inn}(G), \mathrm{Out}(G), \mathrm{Aut}(G)$ for the center, inner automorphism group, outer automorphism group and automorphism group of G, respectively. Given $g \in G$, *conjugation by g* is the automorphism $\gamma(g) : G \to G$ given by $\gamma(g) : h \mapsto ghg^{-1}$. We write $G \cong H$ to say that the groups G and H are isomorphic. Given $X \subseteq G$, we denote the subgroup generated by X by $\langle X \rangle$. Given $H \leq G$, we denote the normal closure of H by H^G. We denote commutators $ghg^{-1}h^{-1}$ by $[g, h]$, and the commutator subgroup of G by $[G, G]$. We denote the dihedral group with $2n$ elements by D_{2n}.

Categorical notation: We denote categories by $\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots$. The set of objects of a small category $\mathcal{C}$ is denoted by $\mathrm{Obj}(\mathcal{C})$; the set of morphisms between a and b is denoted $\mathrm{Hom}_{\mathcal{C}}(a, b)$ or just $\mathrm{Hom}(a, b)$ when no confusion is possible; the set of all the morphisms of $\mathcal{C}$ is denoted $\mathrm{Mor}(\mathcal{C})$. The identity on an object a is denoted Id_a. We denote the source and target maps $\mathrm{Mor}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{C})$ by s, t, respectively. Composition is denoted by $\cdot$; we take the direction of composition to be as follows: if $f : a \to b$ and $g : b \to c$, then $g \cdot f : a \to c$.

When talking about the 2-category of categories, we denote composition of functors by concatenation, and vertical/horizontal composition of natural transformations by $\cdot/\circ$, respectively. We denote the identity natural transformation on a functor F by Id_F. We denote the vertical/horizontal inverse of a natural transformation τ (when it exists) by τ^{-v}, τ^{-h}, respectively. We take the direction of vertical/horizontal composition to agree with the direction of composition used: if $F, G, H : \mathcal{C} \to \mathcal{D}$ are functors and $\tau : F \to G, \sigma : G \to H$ are natural transformations, then $\sigma \cdot \tau : F \to H$ is the vertical composition of τ and σ; if $F, G : \mathcal{C} \to \mathcal{D}$ and $H, K : \mathcal{D} \to \mathcal{E}$ are functors and $\tau : F \to G, \sigma : H \to K$, then $\sigma \circ \tau : HF \to KG$ is the horizontal composition of τ and σ.

Module language: Given an abelian group E and a group G, we say that E is a *G-module* when we are thinking about a particular action $\triangleright$ of G on E by automorphisms, meaning that

$g \triangleright (e_1 e_2) = (g \triangleright e_1)(g \triangleright e_2)$ for all $g \in G$ and $e_1, e_2 \in E$. More than often we will want to focus on the action, so we call the triplet $(G, E, \triangleright)$ a *module*. A *morphism* between modules is a pair $(f_1, f_2) : (G, E, \triangleright) \to (G', E', \triangleright')$ such that $f_1 : G \to G'$, $f_2 : E \to E'$ are group homomorphisms such that $f_1(g) \triangleright f_2(e) = f_2(g \triangleright e)$ for all $g \in G, e \in E$. The pair (f_1, f_2) is called an *isomorphism* if f_1, f_2 are both group isomorphisms; in this case, we say that $(G, E, \triangleright)$ and $(G', E', \triangleright')$ are *isomorphic modules*; this is denoted $(G, E, \triangleright) \cong (G', E', \triangleright')$. Notice that modules and their morphisms form a category, which we call **GMod**. We call isomorphisms from $(G, E, \triangleright)$ to itself *automorphisms* and denote the group of automorphisms of $(G, E, \triangleright)$ by $\mathrm{Aut}(G, E, \triangleright)$.

Groupoid language: A *groupoid* $\mathcal{C}$ is a category whose morphisms are invertible. The *isotropy group* of an object $c \in \mathrm{Obj}(\mathcal{C})$ is $\mathrm{Hom}(c, c)$; the *orbit*, or *connected component* of an object $c \in \mathrm{Obj}(\mathcal{C})$ is the set of objects which are connected to c by some morphism in $\mathcal{C}$. A groupoid is said to be *connected* if it has only one orbit.

2 2-Groups and crossed modules; basic identities and results

In this article we will be studying strict 2-groups. It is well known that these are equivalent to crossed modules ([1, 2]), and we will want to switch back and forth between these perspectives. In section 2.1 we describe **2Grp**, the category of 2-groups, in section 2.2 we describe **XMod**, the category of crossed modules, and in section 2.3 we describe their equivalence, as proven by Brown and Spencer in [1]. In section 2.4 we provide some classic examples and a couple of less common ones. In section 2.5 we state some basic identities regarding 2-groups that we will need throughout this article.

2.1 The category of 2-groups

As already mentioned, we introduce 2-groups as group objects.

Definition 2.1. *A* (strict) 2-group*, also called* (strict) categorical group *is a (strict) group object in the category of small categories.*

Remark 2.2. *A 2-group can also be defined as a strict 2-category with only one object and all morphisms/2-morphisms invertible.*

Remark 2.3. *In more detail, a categorical group is a small category $\mathcal{G}$, together with a functor $\otimes : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, called* monoidal product*, such that $(\mathrm{Obj}(\mathcal{G}), \otimes)$ is a group with identity denoted 1, and $(\mathrm{Mor}(\mathcal{G}), \otimes)$ is a group with identity Id_1.*

From the functoriality of $\otimes$ follows the interchange law: if $\chi_1 : g_1 \to h_1$, $\eta_1 : h_1 \to k_1$, $\chi_2 : g_2 \to h_2$ and $\eta_2 : h_2 \to k_2$ are morphisms of a 2-group $\mathcal{G}$, then

$$\begin{pmatrix} \eta_1 \\ \cdot \\ \chi_1 \end{pmatrix} \otimes \begin{pmatrix} \eta_2 \\ \cdot \\ \chi_2 \end{pmatrix} = \begin{matrix} (\eta_1 \otimes \eta_2) \\ \cdot \\ (\chi_1 \otimes \chi_2) \end{matrix}$$

It also follows that $\mathrm{Id}_g \otimes \mathrm{Id}_h = \mathrm{Id}_{g \otimes h}$.

From the definition of group object, we also need an inverse functor; however, its functoriality can be deduced from the properties already stated, as we will see in section 2.5.

Notation. *For a 2-group $\mathcal{G}$ we will be using the following notation throughout the article:*

- *We denote the group of objects $(\mathrm{Obj}(\mathcal{G}), \otimes)$ by G_0;*

- *We denote the group of morphisms* $(\mathrm{Mor}(\mathcal{G}), \otimes)$ *by* G_1*;*
- $\mathrm{Id} : G_0 \to G_1$ *denotes the function given by* $g \mapsto \mathrm{Id}_g$*, where* Id_g *is the identity on* g *in the category* $\mathcal{G}$*;*
- *We denote the inverse of an object* $g \in G_0$ *under* $\otimes$ *by* g^{-1}*, and the inverse of a morphism* $\chi \in G_1$ *under* $\otimes$ *by* χ^{-h}*; we call this the* horizontal inverse *of* χ*. We denote the inverse of a morphism* $\chi \in G_1$ *under composition by* χ^{-v}*, when it exists (we will see that this is always the case in section 2.5).*

Also, we will call $\cdot$ *"composition", and* $\otimes$ *the "monoidal operation", "monoidal product" or "monoidal multiplication". If we want to restrict* $\otimes$ *to* G_0 *or* G_1 *we may talk about the "monoidal product on objects" and "monoidal product on morphisms", respectively.*

Since the underlying category of a 2-group $\mathcal{G}$ *is a groupoid, as we will see in section 2.5, we say that objects* g, h *are* isomorphic *if there is a morphism* $\chi : g \to h$*. Being isomorphic is an equivalence relation, and we denote the equivalence class of* g *by* $[g]$*, and will call it the* connected component *of* g*. We say that* $\mathcal{G}$ *is* connected *if its underlying groupoid is connected; that is, if all objects are isomorphic. We say that* $\mathcal{G}$ *is* discrete *if all connected components have exactly one object.*

We call a 2-group $\mathcal{G}$ finite *if* G_1 *is finite.*

Remark 2.4. *Note that* $s, t : G_1 \to G_0$ *and* $\mathrm{Id} : G_0 \to G_1$ *are group homomorphisms; furthermore,* Id *is injective, and so* $G_0 \cong \mathrm{Id}(G_0) \leq G_1$*.*

We now describe 2-group morphisms.

Definition 2.5. *Given 2-groups* $\mathcal{G}, \mathcal{H}$*, a* 2-group homomorphism $F : \mathcal{G} \to \mathcal{H}$ *is a (strict) functor from* $\mathcal{G}$ *to* $\mathcal{H}$ *such that* $F : G_0 \to H_0$ *and* $F : G_1 \to H_1$ *are group homomorphisms. Composition of 2-group homomorphisms is the usual functor composition.*

The category of 2-groups*, denoted* **2Grp***, has objects the 2-groups, and morphisms the 2-group homomorphisms between them; the identity morphisms and composition are obvious.*

A 2-group isomorphism *is a 2-group homomorphism which is bijective on objects and morphisms. The 2-groups* $\mathcal{G}$ *and* $\mathcal{H}$ *are called* isomorphic *if there is a 2-group isomorphism* $\mathcal{G} \to \mathcal{H}$*; we denote this by* $\mathcal{G} \cong \mathcal{H}$*.*

In more detail, saying that $F : \mathcal{G} \to \mathcal{H}$ is a 2-group homomorphism means that F is a pair of group homomorphisms $f_0 : G_0 \to H_0$ and $f_1 : G_1 \to H_1$ which form a functor. As usual, we denote $f_0(g)$ by $F(g)$ and $f_1(\chi)$ by $F(\chi)$.

In order for our notion of isomorphic 2-groups to agree with the categorical notion, 2-group isomorphisms need to be exactly the invertible 2-group homomorphisms. This is done in the following lemma.

Lemma 2.6. *A 2-group homomorphism is invertible if and only if it is a 2-group isomorphism.*

Proof. If $F : \mathcal{G} \to \mathcal{H}$ is invertible, then f_0 and f_1 are invertible, thus bijections; this means that F is a 2-group isomorphism.

If $F : \mathcal{G} \to \mathcal{H}$ is a 2-group isomorphism, then $F^{-1} : \mathcal{H} \to \mathcal{G}$ defined as (f_0^{-1}, f_1^{-1}) is the inverse functor of F^{-1}; also f_0^{-1}, f_1^{-1} are group homomorphisms, since f_0, f_1 are group isomorphisms. Thus F^{-1} is a 2-group isomorphism, which is the inverse of F. $\square$

2.2 The category of crossed modules

We introduce crossed modules as a useful alternative language to describe and study 2-groups. The discussion of the relation between these and 2-groups is postponed to the next section.

Definition 2.7. *A* crossed module *is $\mathcal{G} = (G, E, \partial, \triangleright)$, where G, E are groups, $\partial : E \to G$ is a group homomorphism (sometimes called* boundary morphism*) and $\triangleright$ is an action of G on E by automorphisms (one can think of it as a group homomorphism $\triangleright : G \to \mathrm{Aut}(E)$, and write $g \triangleright e$ to denote $\triangleright(g)(e)$), subject to the* Peiffer laws*:*

1. *$\partial(g \triangleright e) = g\partial(e)g^{-1}$ for all $g \in G, e \in E$;*
2. *$\partial(e) \triangleright f = efe^{-1}$ for all $e, f \in E$.*

In more detail, saying that $\triangleright$ is an action of G on E by automorphisms means that, in addition to the usual action rules, we also have $g \triangleright (e_1 e_2) = (g \triangleright e_1)(g \triangleright e_2)$ for all $g \in G, e_1, e_2 \in E$; in particular, $g \triangleright 1 = 1$ and $g \triangleright e^{-1} = (g \triangleright e)^{-1}$.

Remark 2.8. *Some authors, like Brown and Spencer ([1],[2]), have a slightly different definition of a crossed module, in which the Peiffer laws used are $\partial(g \triangleright e) = g^{-1}\partial(e)g$ and $\partial(e) \triangleright f = e^{-1}fe$. This will make our definitions slightly different; however, everything should be equivalent.*

The next proposition is an easy consequence of the Peiffer laws.

Proposition 2.9. *Given a crossed module $(G, E, \partial, \triangleright)$:*

- *$\mathrm{im}\,(\partial)$ is a normal subgroup of G;*
- *$\ker(\partial) \leq Z(E)$; in particular, $\ker(\partial)$ is abelian.*

We now describe crossed module morphisms.

Definition 2.10. *Let $\mathcal{G} = (G, E, \partial, \triangleright)$ and $\mathcal{G}' = (G', E', \partial', \triangleright')$ be crossed modules. A* crossed module homomorphism *$F : \mathcal{G} \to \mathcal{G}'$ is a pair (f_1, f_2) of group homomorphisms $f_1 : G \to G', f_2 : E \to E'$, such that $\partial' f_2 = f_1 \partial$ and $f_1(g) \triangleright' f_2(e) = f_2(g \triangleright e)$ for all $g \in G, e \in E$.*

Composition *of homomorphisms is defined as follows: if $\mathcal{G}, \mathcal{H}, \mathcal{K}$ are crossed modules, and $F : \mathcal{G} \to \mathcal{H}, F' : \mathcal{H} \to \mathcal{K}$ are crossed module homomorphisms, then $F'F : \mathcal{G} \to \mathcal{K}$ is the pair $(f_1' f_1, f_2' f_2)$.*

The category of crossed modules, denoted **XMod***, has objects the crossed modules, and morphisms the crossed module homomorphisms with composition as just defined; the identity morphism on $\mathcal{G}$ is the pair $(\mathrm{Id}_G, \mathrm{Id}_E)$.*

A crossed module homomorphism $F : \mathcal{G} \to \mathcal{H}$ is an isomorphism *if f_1, f_2 are group isomorphisms. The crossed modules $\mathcal{G}$ and $\mathcal{H}$ are called* isomorphic *if there is a crossed module isomorphism $\mathcal{G} \to \mathcal{H}$; we denoted this by $\mathcal{G} \cong \mathcal{H}$.*

We can also write $F(g)$ for $f_1(g)$ and $F(e)$ for $f_2(e)$.

Like in the 2-group case, we have to check that our notion of isomorphic crossed modules is the same as the categorical notion; this is easy to check, and can be done using the analogous result for 2-groups.

2.3 Equivalence between 2Grp and XMod

By equivalent categories $\mathcal{C}$ and $\mathcal{D}$ we mean that there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $FG \cong \mathrm{Id}_{\mathcal{D}}$ and $GF \cong \mathrm{Id}_{\mathcal{C}}$; that is, there are natural isomorphisms $\tau : \mathrm{Id}_{\mathcal{D}} \to FG$ and $\sigma : \mathrm{Id}_{\mathcal{C}} \to GF$.

We now state the equivalence between the categories **2Grp** and **XMod**; more details will be given in section 4.

Theorem 2.11. *The categories* **2Grp** *and* **XMod** *are equivalent; indeed, functors γ :* **2Grp** $\to$ **XMod** *and ξ :* **XMod** $\to$ **2Grp** *that form an equivalence are given as follows.*

Given a 2-group $\mathcal{G}$, there is an associated crossed module $\gamma(\mathcal{G}) = (G, E, \partial, \triangleright)$ *given by:*

- *$G = G_0$;*
- *$E = \ker s$ as a subgroup of G_1; we could also write $E = \mathrm{Hom}_{\mathcal{G}}(1, -)$;*
- *∂ is the restriction of t on E;*
- *Given $g \in G, \eta \in E$, define $g \triangleright \eta = \mathrm{Id}_g \otimes \eta \otimes \mathrm{Id}_g^{-h}$.*

Given a 2-group homomorphism $F : \mathcal{G} \to \mathcal{H}$, there is an associated crossed module homomorphism *$\gamma(F) : \gamma(\mathcal{G}) \to \gamma(\mathcal{H})$, given by $\gamma(F)(g) = F(g)$ and $\gamma(F)(e) = F(e)$, where $g \in G_0$ and $e \in \mathrm{Hom}_{\mathcal{G}}(1, -)$.*

Given a crossed module $\mathfrak{G} = (G, E, \partial, \triangleright)$, there is an associated 2-group $(\mathcal{G}, \otimes) = \xi(\mathfrak{G})$ *given by:*

- *$G_0 = G$;*
- *For all $g, h \in G$, the set* $\mathrm{Hom}(g, h)$ *is $\{g\} \times \partial^{-1}(hg^{-1})$;*
- *On objects, $\otimes$ is the operation of G;*
- *Composition of morphisms $(\partial(e)g, f) \cdot (g, e) = (g, fe)$ for all $g, h \in G$ and $e, f \in E$;*
- *On morphisms, $\otimes$ is the operation of the semidirect product $G \ltimes_{\triangleright} E$: that is, $(g, e) \otimes (h, f) = (gh, e(g \triangleright f))$ for all $g, h \in G$ and $e, f \in E$.*

Given a crossed module homomorphism $F : \mathfrak{G} \to \mathfrak{H}$ be a crossed module homomorphism. There is an associated 2-group homomorphism *$\xi(F) : \xi(\mathfrak{G}) \to \xi(\mathfrak{H})$, given by $\xi(F)(g) = f_1(g)$ for $g \in \mathrm{Obj}(\xi(\mathfrak{G}))$ and $\xi(F)(g, e) = (f_1(g), f_2(e))$ for every $(g, e) \in \mathrm{Mor}(\xi(\mathfrak{G}))$.*

Proof. See [1]. □

The reason why crossed modules are so closely related to 2-groups is that all structure can be recovered by the groups $G_0, \mathrm{Hom}(1, -)$, the morphism ∂ and the action $\triangleright$; this follows from the following observation: G_1 is isomorphic to the semidirect product $\mathrm{Id}(G_0) \ltimes \ker s$, and $G_0 \cong \mathrm{Id}(G_0)$.

From now on we will use γ and ξ only when we wish to be more rigorous; elsewhere we will write $\mathcal{G} = \gamma(\mathcal{G})$ and $\mathcal{G} = \xi(\mathcal{G})$, and switch freely between both these descriptions. We will denote crossed modules by $\mathcal{G}, \mathcal{H}, \mathcal{K}, \ldots$, unless we want to include 2-groups in the same discussion, in which case we will save $\mathcal{G}, \mathcal{H}, \mathcal{K}, \ldots$ for 2-groups, and denote crossed modules $\mathfrak{G}, \mathfrak{H}, \mathfrak{K}, \ldots$.

2.4 Examples

Let us give some examples. The first two of these are essential.

Example 2.12. *The category of modules and their morphisms, as described in the introduction, can be embedded in the category of 2-groups/the category of crossed modules. We define a functor* $\iota : \mathbf{GMod} \to \mathbf{2Grp}$ *as follows. Given a module* $(G, E, \triangleright)$*, let* $\iota(G, E, \triangleright)$ *be the 2-group which we denote* $G[0] \ltimes_{\triangleright} E[1]$ *with object group* G *and morphism group* $G \ltimes_{\triangleright} E$*, where* $s(g, e) = g = t(g, e)$ *and composition is* $(g, e) \cdot (g, f) = (g, ef)$*. Given a module morphism* $(f_1, f_2) : (G, E, \triangleright) \to (G', E', \triangleright)$*, define* $\iota(f_1, f_2) : G[0] \ltimes_{\triangleright} E[1] \to G'[0] \ltimes_{\triangleright'} E'[1]$ *by* $\iota(f_1, f_2)(g) = f_1(g)$ *and* $\iota(f_1, f_2)(g, e) = (f_1(g), f_2(e))$*. It is easy to check that* ι *is a well-defined full and faithful functor.*

When $\triangleright$ *is the trivial action, we denote this 2-group simply by* $G[0] \times E[1]$*; when* $E = 1$*, we denote it by* $G[0]$*, and when* $G = 1$ *we denote it by* $E[1]$*. Notice that* $G[0]$ *is a 2-group whose morphisms are only the identities, and that* $E[1]$ *is a 2-group with only one object.*

It is easy to check that the associated crossed module to $G[0] \ltimes_{\triangleright} E[1]$ *is isomorphic to* $(G, E, 1, \triangleright)$*, where* 1 *is the trivial homomorphism.*

Notice that it follows from the interchange law/second Peiffer law that E *must be abelian for* $G[0] \ltimes_{\triangleright} E[1]$ *to be a 2-group.*

From the fullness of ι *follows that* $G[0] \ltimes_{\triangleright} E[1]$ *and* $G'[0] \ltimes_{\triangleright'} E'[1]$ *are isomorphic 2-groups if and only if* $(G, E, \triangleright)$ *and* $(G', E', \triangleright')$ *are isomorphic modules; in particular,* $G[0] \times E[1] \cong G'[0] \times E'[1]$ *if and only if* $G \cong G'$ *and* $E \cong E'$*.*

Example 2.13. *Given a group* G*, let* $G[\mathrm{Ad}]$ *denote the 2-group with morphism group* G *and each set* $\mathrm{Hom}(g, h)$ *singular with only element* $g \to h$ *for* $g, h \in G$*; composition and* $\otimes$ *on morphisms are defined in the only possible way that is coherent with objects: we have* $(g \to h) \cdot (h \to k) = g \to k$ *and* $(g \to h) \otimes (k \to l) = (gk \to hl)$*. It is easy to check that this does indeed define a 2-group. It is easy to prove that any two 2-groups with object group* G *and each hom-set singular are isomorphic to* $G[\mathrm{Ad}]$*.*

Let us determine the associated crossed module to $G[\mathrm{Ad}]$ *in detail. Let* $H = \{1 \to g : g \in G\}$*. The boundary morphism* $\partial : H \to G$ *is given by* $\partial(1 \to g) = g$*. Finally, the action* $\triangleright$ *is given by* $g \triangleright (1 \to h) = \mathrm{Id}_g \otimes (1 \to h) \otimes \mathrm{Id}_{g^{-1}} = (g \to g) \otimes (1 \to h) \otimes (g^{-1} \to g^{-1}) = 1 \to ghg^{-1}$*. The associated crossed module is thus* $(G, H, \partial, \triangleright)$*.*

Since ∂ *is a isomorphism* $G \cong H$ *and the action is defined by* $g \triangleright h = \partial^{-1}(g) h \partial^{-1}(g)^{-1}$*, it's easy to see that* $(G, H, \partial, \triangleright) \cong (G, G, \mathrm{Id}, \mathrm{Ad})$*, where* Id *is the identity homomorphism and* Ad *is the adjoint action:* $g \mathrm{Ad} h = ghg^{-1}$*; an isomorphism is* (Id, ∂)*, for example. Thus* $\gamma({}^{\mathrm{Ad}}G) \cong (G, G, \mathrm{Id}, \mathrm{Ad})$*.*

This is called the adjoint 2-group/adjoint crossed module on G*.*

The next example is reminiscent of 2-group actions, and is the 2-group analogue to permutation groups of sets.

Example 2.14. *Given a small category* $\mathcal{C}$*, let* $S_{\mathcal{C}}$ *denote the 2-group with* $\mathrm{Obj}(S_{\mathcal{C}})$ *the invertible endofunctors on* $\mathcal{C}$ *and* $\mathrm{Mor}(S_{\mathcal{C}})$ *the natural isomorphisms between said functors; composition of morphisms is the usual composition of functors, and horizontal/vertical composition of 2-morphisms is the usual horizontal/vertical composition of natural transformations. We call* $S_{\mathcal{C}}$ *the* permutation 2-group of $\mathcal{C}$*.*

The associated crossed module to $S_{\mathcal{C}}$ *is* $(F_{\mathcal{C}}, I_{\mathcal{C}}, T, \triangleright)$*, where* $F_{\mathcal{C}}$ *is the group of invertible endofunctors on* $\mathcal{C}$ *under functor composition,* $I_{\mathcal{C}}$ *is the group of natural isomorphisms between* *the identity functor on $\mathcal{C}$ and some other invertible endofunctor on $\mathcal{C}$, T is the target morphism, and $\triangleright$ is the action defined by $(F \triangleright \tau)c = F(\tau(F^{-1}(c)))$.*

Let us analyze a particular case: let $\mathcal{C}$ be the category associated to a group G; that is, the category with one object $*$ and $\mathrm{Hom}(*, *) = G$, *with composition given by the operation of G. In this case we denote $S_{\mathcal{C}}$ by* $\mathrm{Aut}(G)$. *The associated crossed module to* $\mathrm{Aut}(G)$ *is* $(\mathrm{Aut}(G), G, \gamma, |)$, *where $\gamma(g)$ is the conjugation by g; i.e. the map $\gamma(g) : h \mapsto ghg^{-1}$; and $|$ is the action given by evaluation: $\sigma|g = \sigma(g)$ (the details are left to the reader). This is called the* Automorphism 2-group/Automorphism crossed module *of G.*

Remark 2.15. *Given a group G, when we write* $\mathrm{Aut}(G)$ *we might mean the group or the 2-group; whenever* $\mathrm{Aut}(G)$ *is mentioned in this article, it will be clear from context which of the two we are referring to.*

Our final example is topological in nature. There are other well known examples of crossed modules arising from algebraic topology, where crossed modules were first used.

Example 2.16. *Given a topological group G, let $G[\mathrm{Top}]$ denote the 2-group with object group G and monoidal product on objects the operation of G. For $g, h \in G$, the set* $\mathrm{Hom}(g, h)$ *consists of the paths from g to h, up to homotopy. Composition is the usual concatenation of classes of paths; the monoidal product on morphisms is given as follows: if $\chi, \eta : [0, 1] \to G$ are paths, then $[\chi] \otimes [\eta]$ is defined as the homotopy class of $\chi \otimes \eta$, which is defined by $(\chi \otimes \eta)(t) = \chi(t)\eta(t)$ for $t \in [0, 1]$.*

The associated crossed module to $G[\mathrm{Top}]$ is $(G, P(1, -), \partial, \triangleright)$, where $P(1, -)$ denotes the group of classes of paths starting at the identity, with horizontal composition as the group operation, ∂ denotes the target morphism, and $\triangleright$ is the action given by $g \triangleright [\chi]$ the class of $g \triangleright \chi$, defined by $(g \triangleright \chi)(t) = g\chi(t)g^{-1}$ for $t \in [0, 1]$.

Let us analyze the case $G = O(2)$ in more detail. It is known that $O(2)$ is homeomorphic to $S^1 \times \{\pm 1\}$; thus every homotopy class of paths P in $O(2)$ starting from I can be identified with a real number r; if $r = 2\pi k + s$, where $k \in \mathbb{Z}$ and $s \in [0, 2\pi[$, then we can think of P as the path which goes around the circle $|k|$ times in a positive or negative direction, and then goes directly to the angle s. It is easy to check that if P is identified with r and Q is identified with s, then $P \otimes Q$ is identified with $r + s$; thus $P(1, -)$ is isomorphic to $\mathbb{R}$. The boundary morphism is given by $\partial(r) = \begin{bmatrix} \cos(r) & \sin(r) \\ -\sin(r) & \cos(r) \end{bmatrix}$. Finally, the action is given by $g \triangleright r = \det(g)r$

We can thus denote the associated crossed module to ${}^{\mathrm{Top}}O(2)$ by

$$(O(2), \mathbb{R}, \partial, \triangleright).$$

2.5 Basic results

We now state and prove some basic results about 2-groups.

We start with a very useful identity which writes compositions in terms of the monoidal product; this will be especially useful when working with the 2-subgroup generated by a set of morphisms, later on.

Proposition 2.17. *Let $\chi : h \to k, \eta : g \to h$ be morphisms. Then*

$$\chi \cdot \eta = \chi \otimes \mathrm{Id}_{h^{-1}} \otimes \eta = \eta \otimes \mathrm{Id}_{h^{-1}} \otimes \chi.$$

Proof. We have

$$\begin{aligned}\chi\cdot\eta &= (\chi\otimes \mathrm{Id}_1)\cdot[\mathrm{Id}_h\otimes(\mathrm{Id}_{h^{-1}}\otimes\eta)]\\ &= (\chi\cdot\mathrm{Id}_h)\otimes[\mathrm{Id}_1\cdot(\mathrm{Id}_{h^{-1}}\otimes\eta)] = \chi\otimes\mathrm{Id}_{h^{-1}}\otimes\eta,\end{aligned}$$

on the one hand, and

$$\begin{aligned}\chi\cdot\eta &= (\mathrm{Id}_1\otimes\chi)\cdot[(\eta\otimes\mathrm{Id}_{h^{-1}})\otimes\mathrm{Id}_h]\\ &= [\mathrm{Id}_1\cdot(\eta\otimes\mathrm{Id}_{h^{-1}})]\otimes(\chi\cdot\mathrm{Id}_h) = \eta\otimes\mathrm{Id}_{h^{-1}}\otimes\chi,\end{aligned}$$

on the other. □

Corollary 2.18. *The category $\mathcal{G}$ is a groupoid. In particular, if $\chi: g\to h$ is a morphism of $\mathcal{G}$, then $\chi^{-v} = \mathrm{Id}_g\otimes\chi^{-h}\otimes\mathrm{Id}_h$.*

Proof. Let $\chi: g\to h$. Writing $\chi^{-v} = \mathrm{Id}_g\otimes\chi^{-h}\otimes\mathrm{Id}_h$, we have $\chi^{-v}: h\to g$, so that both $\chi\cdot\chi^{-v}$ and $\chi^{-v}\cdot\chi$ both make sense. We have

$$\chi\cdot(\mathrm{Id}_g\otimes\chi^{-h}\otimes\mathrm{Id}_h) = \chi\otimes\mathrm{Id}_{g^{-1}}\otimes(\mathrm{Id}_g\otimes\chi^{-h}\otimes\mathrm{Id}_h) = \chi\otimes\mathrm{Id}_1\otimes\chi^{-h}\otimes\mathrm{Id}_h = \mathrm{Id}_h,$$

and

$$(\mathrm{Id}_g\otimes\chi^{-h}\otimes\mathrm{Id}_h)\cdot\chi = (\mathrm{Id}_g\otimes\chi^{-h}\otimes\mathrm{Id}_h)\otimes\mathrm{Id}_{h^{-1}}\otimes\chi) = \mathrm{Id}_g\otimes\chi^{-h}\otimes\mathrm{Id}_1\otimes\chi = \mathrm{Id}_g,$$

as desired. □

There is an identity which partially rewrites the monoidal product in terms of compositions; however, this is not as efficient as the previous ones, since the monoidal product is still necessary in the rewritten version.

Proposition 2.19. *Let $\chi_1: g_1\to h_1, \chi_2: g_2\to h_2$ be two 2-morphisms. Then*

$$\chi_1\otimes\chi_2 = (\mathrm{Id}_{h_1}\otimes\chi_2)\cdot(\chi_1\otimes\mathrm{Id}_{g_2}) = (\chi_1\otimes\mathrm{Id}_{h_2})\cdot(\mathrm{Id}_{g_1}\otimes\chi_2).$$

Proof. All that is needed is to apply the interchange law to the expressions on the right hand side. □

We now state some typical group/categorical properties for 2-groups; these follow imediately from $\mathcal{G}$ being a groupoid and G_1 a group under $\otimes$.

Proposition 2.20. *The usual group/categorical properties hold for 2-groups:*

- *Vertical inverses are unique;*
- *There are cancellation laws for the monoidal product and the composition;*
- $(\chi\cdot\eta)^{-v} = \eta^{-v}\cdot\chi^{-v}$ *for all $\chi,\eta\in\mathcal{G}_1$ such that $\chi\cdot\eta$ makes sense;*
- $(\chi\otimes\eta)^{-h} = \eta^{-h}\otimes\chi^{-h}$ *for all $\chi,\eta\in G_1$.*

There are another two properties relating the monoidal product and the composition, which follow immediately from the interchange law.

Proposition 2.21. *Let $\chi, \eta \in G_1$. Then:*

- $(\chi \otimes \eta)^{-v} = \chi^{-v} \otimes \eta^{-v}$*;*
- *If $\chi \cdot \eta$ makes sense, then $(\chi \cdot \eta)^{-h} = \chi^{-h} \cdot \eta^{-h}$.*

There is a final result on 2-groups worth mentioning; it is also an easy corollary to Proposition 2.17.

Corollary 2.22. $\mathrm{Hom}(1,1)$ *is an abelian group under $\otimes$; furthermore, $\otimes$ and $\cdot$ are the same in* $\mathrm{Hom}(1,1)$.

Proof. Let $\chi, \eta \in \mathrm{Hom}(1,1)$. From Proposition 2.17 follows $\chi \otimes \eta = \chi \otimes \mathrm{Id}_{1^{-1}} \otimes \eta = \chi \cdot \eta = \eta \otimes \mathrm{Id}_{1^{-1}} \otimes \chi = \eta \otimes \chi$, as desired. ☐

3 Homotopy of 2-groups and crossed modules and homotopy groups

In this section we describe the 2-morphisms of **2Grp** and **XMod**. This gives us a definition of equivalence of 2-groups and of crossed modules, akin to equivalence of categories. We discuss the so called "homotopy groups" of 2-groups.

3.1 2-Group 2-homomorphisms

We begin with the definition of a 2-group 2-homomorphism, which is the same as in [4, 2].

Definition 3.1. *Given 2-groups $\mathcal{G}, \mathcal{H}$ and homomorphisms $F, G : \mathcal{G} \to \mathcal{H}$, a* 2-group 2-homomorphism *(sometimes called* 2-group homotopy*) $\tau : F \to G$ is a natural transformation between the functors F, G, such that the correspondence $c \mapsto \tau(c)$ is a group homomorphism $G_0 \to H_1$.*

The next proposition shows how to get 2-group 2-homomorphisms from group homomorphisms $G_0 \to H_1$.

Proposition 3.2. *Let F be a 2-group homomorphism $\mathcal{G} \to \mathcal{H}$ and $\tau : G_0 \to H_1$ a group homomorphism such that $s\tau = F$ on objects. Then there is a 2-group homomorphism $G : \mathcal{G} \to \mathcal{H}$ such that $\tau : F \to G$ is a 2-group 2-homomorphism, given by:*

- *For $g \in G_0$, we have $G(g) = t(\tau(g))$;*
- *For $\chi : g \to h$ in G_1, we have $G(\chi) = \tau(h) \cdot F(\chi) \cdot \tau(g)^{-v}$.*

Proof. Since the naturality of τ is obvious, and since it is a group homomorphism, by hypothesis, we need only prove that G is indeed a 2-group homomorphism. Let us prove that G is a functor. If $\chi : g \to h$ is a morphism, then $G(\chi) = \tau(h) \cdot F(\chi) \cdot \tau(g)^{-v} : G(g) \to G(h)$. If $g \in G_0$, then $G(\mathrm{Id}_g) = \tau(g) \cdot F(\mathrm{Id}_g) \cdot \tau(g)^{-v} = \tau(g) \cdot \mathrm{Id}_{F(g)} \cdot \tau(g)^{-v} = \mathrm{Id}_{G(g)}$. Finally, if $\chi : g \to h$ and $\eta : h \to k$, then

$$\begin{aligned} G(\eta \cdot \chi) &= \tau(k) \cdot F(\eta \cdot \chi) \cdot \tau(g)^{-v} = \tau(k) \cdot F(\eta) \cdot F(\chi) \cdot \tau(g)^{-v} \\ &= [\tau(k) \cdot F(\eta) \cdot \tau(h)^{-v}] \cdot [\tau(h) \cdot F(\chi) \cdot \tau(g^{-v}] = G(\eta) \cdot G(\chi), \end{aligned}$$

as desired.

Now let us prove that G on objects/morphisms is a group homomorphism. For $g, h \in G_0$ we have

$$G(g \otimes h) = t(\tau(g \otimes h)) = t(\tau(g) \otimes \tau(h)) = t(\tau(g)) \otimes t(\tau(h)) = G(g) \otimes G(h);$$

for $\chi : g \to h, \chi' : g' \to h'$ we have

$$\begin{aligned} G(\chi \otimes \chi') &= \tau(h \otimes h') \cdot F(\chi \otimes \chi') \cdot \tau(g \otimes g')^{-v} \\ &= [\tau(h) \otimes \tau(h')] \cdot [F(\chi) \otimes F(\chi')] \cdot [\tau(g) \otimes \tau(g')]^{-v} \\ &= [\tau(h) \otimes \tau(h')] \cdot [F(\chi) \otimes F(\chi')] \cdot [\tau(g)^{-v} \otimes \tau(g')^{-v}] \\ &= [\tau(h) \cdot F(\chi) \cdot \tau(g)^{-v}] \otimes [\tau(h') \cdot F(\chi') \cdot \tau(g')^{-v}] = G(\chi) \otimes G(\chi'), \end{aligned}$$

as desired. □

Proposition 3.3. *The class of 2-groups can be made into a 2-category* **2Grp** *with objects the 2-groups, morphisms the homomorphisms, 2-morphisms the 2-group 2-homomorphisms; morphism composition the usual composition of functors, and vertical/horizontal composition of 2-morphisms the usual vertical/horizontal composition of natural transformations.*

Proof. Let $\mathcal{G}, \mathcal{H}$ be 2-groups, $F, G, H : \mathcal{G} \to \mathcal{H}$ be 2-group homomorphisms and $\tau : F \to G, \sigma : G \to H$ be 2-group 2-homomorphisms. Let us prove that $\sigma \cdot \tau$ is a 2-group 2-homomorphism: since it is a natural transformation, we need only check that the correspondence $c \mapsto (\sigma \cdot \tau)(c)$ is a group homomorphism $G_0 \to H_1$. We have $(\sigma \cdot \tau)(g \otimes h) = \sigma(gh) \cdot \tau(gh) = (\sigma(g) \otimes \sigma(h)) \cdot (\tau(g) \otimes \tau(h)) = (\sigma(g) \cdot \tau(g)) \otimes (\sigma(h) \cdot \tau(h)) = (\sigma \cdot \tau)(g) \otimes (\sigma \cdot \tau)(h)$ for every $g, h \in G_0$, as desired.

Now, let $F, G : \mathcal{H} \to \mathcal{K}$ and $H, K : \mathcal{G} \to \mathcal{H}$ be 2-group homomorphisms and $\tau : F \to G, \sigma : H \to K$ be 2-group 2-homomorphisms. Let us prove that $\tau \circ \sigma$ is a 2-group 2-homomorphism: since it is a natural transformation, we need only check that the correspondence $c \mapsto (\tau \circ \sigma)(c)$ is a group homomorphism $G_0 \to K_1$. We have

$$\begin{aligned} (\tau \circ \sigma)(g \otimes h) &= \tau(K(g \otimes h)) \cdot F(\sigma(g \otimes h)) = \tau(K(g) \otimes K(h)) \cdot F(\sigma(g) \otimes \sigma(h)) \\ &= [\tau(K(g)) \otimes \tau(K(h))] \cdot [F(\sigma(g)) \otimes F(\sigma(h))] = \\ &= [\tau(K(g) \cdot F(\sigma(g))] \otimes [\tau(K(h)) \cdot F(\sigma(h))] \\ &= (\tau \circ \sigma)(g) \otimes (\tau \circ \sigma)(h), \end{aligned}$$

as desired.

It is easy to check that the identity natural transformation on a 2-group homomorphism is a 2-group 2-homomorphism. The remaining properties of 2-categories hold for 2-groups/2-group homomorphisms/2-group 2-homomorphisms in particular since they hold for the 2-category of categories/functors/natural transformations in general. This concludes our proof. □

Proposition 3.4. *Every 2-morphism of* **2Grp** *is vertically invertible.*

Proof. Let $\tau : F \to G$ be a 2-group 2-homomorphism, where $F, G : \mathcal{G} \to \mathcal{H}$ are 2-group homomorphisms. Since every morphism of $\mathcal{H}$ is invertible, τ is a natural isomorphism and thus has a vertical inverse τ^{-v} given by $\tau^{-v}(g) = \tau(g)^{-v}$ for all $g \in G_0$. We need only check that τ^{-v} is in fact a 2-homomorphism: indeed,

$$\tau^{-v}(g \otimes h) = \tau(g \otimes h)^{-v} = (\tau(g) \otimes \tau(h))^{-v} = \tau(g)^{-v} \otimes \tau(h)^{-v} = \tau^{-v}(g) \otimes \tau^{-v}(h)$$

for all $g, h \in G_0$, as desired. □

Definition 3.5. *Given 2-groups $\mathcal{G}, \mathcal{H}$, two 2-group homomorphisms $F, G : \mathcal{G} \to \mathcal{H}$ are said to be* isomorphic *(or* homotopic*) if they are isomorphic morphisms in* **2Grp***; we denote this by $F \cong G$.*

The 2-groups $\mathcal{G}$ and $\mathcal{H}$ are said to be equivalent *(or* homotopically equivalent*) if they are equivalent objects in* **2Grp***; that is, if there are 2-group homomorphisms $F : \mathcal{G} \to \mathcal{H}$ and $G : \mathcal{H} \to \mathcal{G}$ such that $FG \cong \mathrm{Id}_{\mathcal{H}}$ and $GF \cong \mathrm{Id}_{\mathcal{G}}$. We denote this by $\mathcal{G} \simeq \mathcal{H}$.*

Note that since every 2-morphism of **2Grp** is invertible, morphisms F and G are isomorphic if and only if there is a 2-morphism $\tau : F \to G$.

3.2 Crossed module derivations

In this section we will define the 2-category of crossed modules, **XMod**. The proof that **XMod** is indeed a 2-category is postponed to section 4.

We start by defining crossed module derivations and their compositions; these are very similar to [2] (the differences arise from using different versions of the Peiffer laws, as mentioned in section 2).

Definition 3.6. *Given crossed module homomorphisms $F, F' : \mathcal{G} \to \mathcal{G}'$, where $\mathcal{G} = (G, E, \partial, \triangleright)$ and $\mathcal{G}' = (G', E', \partial', \triangleright')$, a* derivation *$s : F \to F'$ is a function $s : G \to E'$ such that $s(gh) = s(g)(f_1(g) \triangleright s(h))$ for all $g, h \in G$, and additionally $f_1'(g) = \partial'(s(g))f_1(g)$ and $f_2'(e) = s(\partial(e))f_2(e)$ for all $g \in G, e \in E$*

Vertical composition *of derivations is defined as follows: if $\mathcal{G}, \mathcal{G}'$ are crossed modules, $F_1, F_2, F_3 : \mathcal{G} \to \mathcal{G}'$ are crossed module homomorphisms, and $s : F_1 \to F_2, s' : F_2 \to F_3$ are derivations, then $s' \cdot s : F_1 \to F_3$ is defined by $(s' \cdot s)(g) = s'(g)s(g)$ for every $g \in G$.*

Horizontal composition *of derivations is defined as follows: if $\mathcal{G}, \mathcal{H}, \mathcal{K}$ are crossed modules, $F_1, F_1' : \mathcal{G} \to \mathcal{H}, F_2, F_2' : \mathcal{H} \to \mathcal{K}$ are crossed module homomorphisms, and $s_1 : F_1 \to F_1', s_2 : F_2 \to F_2'$ are derivations, then $s_2 \circ s_1$ is defined by*

$$(s_2 \circ s_1)(g) = s_2(f_{1,1}(g))f_{2,2}'(s_1(g))$$

for every $g \in G$.

XMod *has 2-morphisms the derivations between crossed module homomorphisms, and vertical/horizontal composition as just defined.*

We call crossed modules isomorphic/equivalent *and crossed module homomorphisms* isomorphic *just as we do in any 2-category, and just like we did in the case of 2-groups and 2-group homomorphisms.*

We now to verify that all the compositions are well defined. We start with a lemma in which we define whiskering of morphisms on 2-morphisms in **XMod**, which we will use for proving that the horizontal composition of derivations is a derivation.

Lemma 3.7. *Let $\mathcal{G} = (G, E, \partial, \triangleright), \mathcal{G}' = (G', E', \partial', \triangleright'), \mathcal{G}'' = (G'', E'', \partial'', \triangleright'')$ be crossed modules. If $F : \mathcal{G} \to \mathcal{G}', F_1, F_2 : \mathcal{G}' \to \mathcal{G}''$ are homomorphisms and $s : F_1 \to F_2$ is a derivation, then $s \circ F : F_1F \to F_2F$ defined by $(s \circ F)(g) = s(f_1(g))$ is a derivation.*

Similarly, if $F_1', F_2' : \mathcal{G} \to \mathcal{G}', F' : \mathcal{G}' \to \mathcal{G}''$ are homomorphisms and $s' : F_1' \to F_2'$ is a derivation, then $F' \circ s' : F'F_1' \to F'F_2'$ defined by $(F' \circ s')(g) = f_2'(s'(g))$ is also a derivation.

Proof. We have

$$\begin{aligned}(s\circ F)(gh) &= s(F(gh)) = s(F(g)F(h)) = s(F(g))(F_1(F(g)) \triangleright s(F(h)))\\ &= (s\circ F)(g)((F_1F)(g) \triangleright (s\circ F)(h))\end{aligned}$$

for all $g, h \in G$. Also

$$(F_2F)(g) = F_2(F(g)) = \partial''(s(F(g)))F_1(F(g)) = \partial''((s\circ F)(g))(F_1F)(g)$$

and

$$(F_2F)(e) = F_2(F(e)) = s(\partial(F(e)))F_1(F(e)) = s(F(\partial(e)))(F_1F)(e) = (s\circ F)(\partial(e))(F_1F)(e)$$

for all $g \in G, e \in E$. Thus $s\circ F : F_1F \to F_2F$ is a derivation.

We have

$$\begin{aligned}(F'\circ s')(gh) &= F'(s'(gh)) = F'(s'(g)(F_1(g) \triangleright s'(h))) = F'(s'(g))(F'(F_1(g)) \triangleright F'(s'(h)))\\ &= (F'\circ s')(g)((F'F_1)(g) \triangleright ((F'\circ s')(h)))\end{aligned}$$

for all $g, h \in G$. Also

$$(F'F_2')(g) = F'(F_2'(g)) = F'(\partial'(s(g))F_1'(g)) = \partial''(F'(s'(g)))F'(F_1'(g)) = \partial''((F'\circ s')(g))(F'F_1')(g)$$

and

$$(F'F_2')(e) = F'(F_2'(e)) = F'(s'(\partial(e))F_1'(e)) = F'(s'(\partial(e))F'(F_1')(e) = (F'\circ s')(\partial(e))(F'F_1')(e)$$

for all $g \in G, e \in E$. Thus $F'\circ s' : F'F_1' \to F'F_2'$ is a derivation. □

Lemma 3.8. *The vertical/horizontal composition of derivations yields another derivation.*

Proof. Now let $F, F', F'' : \mathcal{G} \to \mathcal{G}'$ be crossed module homomorphisms, and $s : F \to F', s' : F' \to F''$ be derivations. We prove that $s' \cdot s$ is a derivation $F \to F''$. Firstly,

$$\begin{aligned}(s'\cdot s)(gh) &= s'(gh)s(gh) = [s'(g)(f_1'(g) \triangleright s'(h))][s(g)(f_1(g) \triangleright s(h))]\\ &= s'(g)\{[\partial'(s(g))f_1(g)] \triangleright s'(h)\}s(g)[f_1(g) \triangleright s(h)]\\ &= s'(g)\{\partial'(s(g)) \triangleright [f_1(g) \triangleright s'(h)]\}s(g)[f_1(g) \triangleright s(h)]\\ &= s'(g)s(g)[f_1(g) \triangleright s'(h)]s(g)^{-1}s(g)[f_1(g) \triangleright s(h)]\\ &= (s'\cdot s)(g)[f_1(g) \triangleright s'(h)][f_1(g) \triangleright s(h)]\\ &= (s'\cdot s)(g)[f_1(g) \triangleright (s'\cdot s)(h)],\end{aligned}$$

as desired.

Also $f_1''(g) = \partial'(s'(g))f_1'(g) = \partial'(s'(g))(\partial'(s(g))f_1(g)) = \partial'((s'\cdot s)(g))f_1(g)$ and $f_2''(e) = s'(\partial(e))f_2'(e) = s'(\partial(e))(s(\partial(e)f_2(e)) = (s'\cdot s)(\partial(e))f_2(e)$, as needed.

Finally, we prove that the horizontal composition of derivations is a derivation. Let $F_1, F_1' : \mathcal{G} \to \mathcal{G}', F_2, F_2' : \mathcal{G}' \to \mathcal{G}''$ be crossed module homomorphisms and $s_1 : F_1 \to F_1', s_2 : F_2 \to F_2'$ be derivations. Note that $s_2 \circ s_1 = (F_2' \circ s_1) \cdot (s_2 \circ F_1)$; from the previous lemma it follows that $s_2 \circ s_1$ is the vertical composition of derivations, therefore it's a derivation; furthermore, since $s_2 \circ F_1 : F_2F_1 \to F_2'F_1$ and $F_2' \circ s_1 : F_2'F_1 \to F_2'F_1'$, $s_2 \circ s_1$ is a derivation $F_2F_1 \to F_2'F_1'$. □

3.3 Homotopy invariants

There are *homotopy invariants* for 2-groups: given a 2-group $\mathcal{G}$, there is a *homotopy module* $\pi(\mathcal{G}) = (\pi_0(\mathcal{G}), \pi_1(\mathcal{G}), \triangleright)$ and the Postnikov invariant $\alpha(\mathcal{G}) \in H^3(\pi_0(\mathcal{G}), \pi_1(\mathcal{G}))$, which is a 3-cocycle. These invariants also exist for weak 2-groups, and they classify them up to equivalence: given two weak 2-groups $\mathcal{G}$ and $\mathcal{H}$, they are equivalent if and only if there is a module isomorphism $(f_0, f_1) : \pi(\mathcal{G}) \to \pi(\mathcal{H})$ and the isomorphism $H^3(\pi_0(\mathcal{G}), \pi_1(\mathcal{G})) \to H^3(\pi_0(\mathcal{H}), \pi_1(\mathcal{H}))$ induced by f_0 and f_1 maps $\alpha(\mathcal{G})$ to $\alpha(\mathcal{H})$ (see [4]). However, for strict 2-groups this is not the case, as we will see in more detail in section 7.

Let us describe the homotopy module in more detail. The groups $\pi_0(\mathcal{G})$ and $\pi_1(\mathcal{G})$ are respectively called the *first homotopy group* and the *second homotopy group*. The first homotopy group consists of the isomorphism classes of the objects in $\mathcal{G}$ with the operation induced by $\otimes$: i.e. $[g][h] = [g \otimes h]$; we can also say that $\pi_0(\mathcal{G})$ is the group $G_0/t(\ker s)$. The second homotopy group consists of the morphisms $1 \to 1$, under the monoidal product; that is, $\pi_1(\mathcal{G}) = \mathrm{Hom}(1, 1)$; note that this group is abelian. The action $\triangleright$ of $\pi_0(\mathcal{G})$ on $\pi_1(\mathcal{G})$ is given by $[g] \triangleright \eta = \mathrm{Id}_g \otimes \eta \otimes \mathrm{Id}_{g^{-1}}$; it is easy to check that $\triangleright$ is well-defined and indeed an action by automorphisms.

If $\gamma(\mathcal{G}) = (G, E, \partial, \triangleright)$, then $\pi(\mathcal{G}) = (G/\mathrm{im}\,\partial, \ker \partial, \triangleright)$ (we use the same symbol for the actions since the action of $\pi_0(\mathcal{G})$ on $\pi_1(\mathcal{G})$ is induced from the one of G on E).

We now describe the Postnikov invariant. Given a module $(Q, A, \triangleright)$, there is an equivalent description of $H^3(Q, A)$ by equivalence classes of exact sequences, which we will now give; details about the equivalence can be seen in [3] and [5]. Given a module $(Q, A, \triangleright)$, consider all possible exact sequences of the form

$$1 \to A \xrightarrow{i} E \xrightarrow{\partial} G \xrightarrow{\pi} Q \to 1,$$

where $(G, E, \partial, \triangleright)$ is a crossed module such that the action of G on E induces the given action of Q on A. These exact sequences can be given an equivalence relation, which is the smallest equivalence relation such that $1 \to A \xrightarrow{i} E \xrightarrow{\partial} G \xrightarrow{\pi} Q \to 1$ is equivalent to $1 \to A \xrightarrow{i'} E' \xrightarrow{\partial'} G' \xrightarrow{\pi'} Q \to 1$ whenever there is a crossed module morphism $(f_1, f_2) : (G, E) \to (G', E')$ such that the following diagram commutes:

$$\begin{tikzcd}
 & & E \arrow[r, "\partial"] \arrow[dd, "f_2"] & G \arrow[dd, "f_1"] \arrow[dr, "\pi"] & & \\
1 \arrow[r] & A \arrow[ur, "i"] \arrow[dr, "i'"'] & & & Q \arrow[r] & 1 \\
 & & E' \arrow[r, "\partial'"] & G' \arrow[ur, "\pi'"'] & &
\end{tikzcd}$$

The equivalence classes of sequences can be given a product, as we now describe: the product of the classes of the sequences $1 \to A \xrightarrow{i} E \xrightarrow{\partial} G \xrightarrow{\pi} Q \to 1$ and $1 \to A \xrightarrow{i'} E' \xrightarrow{\partial'} G' \xrightarrow{\pi'} Q \to 1$, is the class of the sequence $1 \to A \xrightarrow{I} (E \times E')/K \xrightarrow{\partial \times \partial'} G \times_Q G' \xrightarrow{\Pi} Q \to 1$, where $K = \{(a, a^{-1}) : a \in A\}$, and $I(a) = (I(a), 1)K$ for all $a \in A$, and $G \times_Q G' = \{(g, g') \in G \times G' : \pi(g) = \pi'(g')\}$, and $\Pi(g, g') = \pi(g) = \pi'(g')$ for all $(g, g') \in G \times_Q G'$.

We write $\mathcal{E}^3(Q, A)$ to denote the group just described; it is well known that $\mathcal{E}^3(Q, A) \cong H^3(Q, A)$ (see [5]). The *Postnikov invariant* of a 2-group $\mathcal{G}$ is the cohomology class $\alpha(\mathcal{G}) \in H^3(\pi_0(\mathcal{G}), \pi_1(\mathcal{G}))$ (where $\pi_1(\mathcal{G})$ is a $\pi_0(\mathcal{G})$-module as seen above) of the image of the extension given by

$$1 \to \pi_1(\mathcal{G}) \xrightarrow{i} \ker s \xrightarrow{t} G_0 \xrightarrow{\pi} \pi_0(\mathcal{G}) \to 1,$$

under isomorphism, where π is the projection $G_0 \to \pi_0(\mathcal{G})$. We will also use $\alpha(\mathcal{G})$ to denote the equivalence class of this extension in $\mathcal{E}^3(\pi_0(\mathcal{G}), \pi_1(\mathcal{G}))$. If $\gamma(\mathcal{G}) = (G, E, \partial, \triangleright)$, then the extension is

$$1 \to \ker\partial \xrightarrow{i} E \xrightarrow{\partial} G \xrightarrow{\pi} G/\mathrm{im}\ \partial \to 1.$$

We now provide a proof that the homotopy invariants are indeed invariants.

Proposition 3.9. *There is a functor $\pi : \mathbf{2Grp} \to \mathbf{GMod}$, defined as already seen for objects, and as follows for morphisms: given a 2-group homomorphism $F : \mathcal{G} \to \mathcal{H}$, define $\pi(F) : \pi(\mathcal{G}) \to \pi(\mathcal{H})$ by $\pi(F) = (\pi_0(F), \pi_1(F))$, where $\pi_0(F)([g]) = [F(g)]$ for all $g \in G_0$ and $\pi_1(F)(\eta) = F(\eta)$ for all $\eta \in \pi_1(\mathcal{G})$.*

Proof. Notice that $\pi_0(F) : \pi_0(\mathcal{G}) \to \pi_0(\mathcal{H})$ is well defined: if $[g] = [h]$, then $[F(g)] = [F(h)]$. Furthermore, it is a group homomorphism: $\pi_0(F)([g][h]) = \pi_0(F)([g \otimes h] = [F(g \otimes h)] = [F(g) \otimes F(h)] = [F(g)][F(h)] = \pi_0(F)([g])\pi_0(F)([h])$ for all $[g], [h] \in \pi_0(\mathcal{G})$.

Our $\pi_1(F) : \pi_1(\mathcal{G}) \to \pi_1(\mathcal{H})$ is also well defined: if $\eta : 1 \to 1$, then $F(\eta) : 1 \to 1$, thus $F(\eta) \in \pi_1(\mathcal{H})$. Furthermore, it is clearly a group homomorphism.

Now we prove that $\pi_0(F), \pi_1(F)$ are coherent with the group actions: if $[g] \in \pi_0(\mathcal{G}), \eta \in \pi_1(\mathcal{G})$, then

$$\begin{aligned}\pi_0(F)([g]) \triangleright \pi_1(F)(\eta) &= [F(g)] \triangleright F(\eta) = \mathrm{Id}_{F(g)} \otimes F(\eta) \otimes \mathrm{Id}_{F(g)^{-1}} \\ &= F(\mathrm{Id}_g \otimes \eta \otimes \mathrm{Id}_{g^{-1}}) = F([g] \triangleright \eta) = \pi_1(F)(g \triangleright \eta).\end{aligned}$$

Finally, it is easy to see that $\pi(\mathrm{Id}_{\mathcal{G}}) = \mathrm{Id}_{\pi(\mathcal{G})}$ and $\pi(GF) = \pi(G)\pi(F)$, given any 2-group homomorphisms $F : \mathcal{G} \to \mathcal{H}$ and $G : \mathcal{H} \to \mathcal{G}$. □

Notice that $\pi_0 : \mathbf{2Grp} \to \mathbf{Grp}$ and $\pi_1 : \mathbf{2Grp} \to \mathbf{AbGrp}$ are functors as well.

The next lemma follows straightforwardly from the definitions.

Lemma 3.10. *Let $F, G : \mathcal{G} \to \mathcal{H}$ be 2-group homomorphisms and $\tau : F \to G$ be a 2-group 2-homomorphism. Then $\pi(F) = \pi(G)$.*

The next proposition is an easy corollary of the previous two lemmas.

Proposition 3.11. *If $\mathcal{G} \simeq \mathcal{H}$ then $\pi(\mathcal{G}) \cong \pi(\mathcal{H})$ are isomorphic modules.*

Proof. Let $F : \mathcal{G} \to \mathcal{H}$ and $G : \mathcal{H} \to \mathcal{G}$ be 2-group homomorphisms such that $\mathrm{Id}_{\mathcal{G}} \cong GF$ and $\mathrm{Id}_{\mathcal{H}} \cong FG$. From the previous lemma and functoriality of π follows that $\mathrm{Id}_{\pi(\mathcal{G})} = \pi(\mathrm{Id}_{\mathcal{G}}) = \pi(GF) = \pi(G)\pi(F)$ and, similarly, $\mathrm{Id}_{\pi(\mathcal{H})} = \pi(F)\pi(G)$. Thus $\pi(\mathcal{G})$ and $\pi(\mathcal{H})$ are isomorphic modules. □

The proof of the next proposition is straightforward.

Proposition 3.12. *Let $\mathcal{G} \simeq \mathcal{H}$ in such a way that $F : \mathcal{G} \to \mathcal{H}, G : \mathcal{H} \to \mathcal{G}$ are 2-group homomorphisms and $\tau : \mathrm{Id}_{\mathcal{G}} \to GF$ and $\sigma : \mathrm{Id}_{\mathcal{H}} \to FG$ are 2-group 2-homomorphisms. Then there is an isomorphism $\alpha(F, G) : \mathcal{E}^3(\pi_0(\mathcal{G}), \pi_1(\mathcal{G})) \to \mathcal{E}^3(\pi_0(\mathcal{H}), \pi_1(\mathcal{H}))$, defined as follows: given an exact sequence*

$$1 \to \pi_0(\mathcal{G}) \xrightarrow{i} E \xrightarrow{\partial} G \xrightarrow{\pi} \pi_1(\mathcal{G}) \to 1,$$

the image of its equivalence class is the equivalence class of the exact sequence

$$1 \to \pi_0(\mathcal{H}) \xrightarrow{i\pi_1(G)} E \xrightarrow{\partial} G \xrightarrow{\pi_0(F)\pi} \pi_1(\mathcal{H}) \to 1.$$

Furthermore, the image of $\alpha(\mathcal{G})$ is $\alpha(\mathcal{H})$.

In section 7 we will provide a different point of view on homotopy invariants.

4 The equivalence of 2Grp and XMod; examples and applications

In this section we state and prove the equivalence between **2Grp** and **XMod**. We give some examples of 2-group equivalence, and compute some automorphism 2-groups of 2-groups.

4.1 The equivalence

We start by stating clearly what we mean by 2-category equivalence. Given two 2-categories $\mathcal{C}, \mathcal{D}$, we say that they are *equivalent* if there are 2-functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $FG \cong \mathrm{Id}_{\mathcal{D}}$ and $GF \cong \mathrm{Id}_{\mathcal{C}}$; that is, there are natural isomorphisms $\tau : FG \to \mathrm{Id}_{\mathcal{D}}$ and $\sigma : GF \to \mathrm{Id}_{\mathcal{C}}$. We are purposely vague in this definition, because different meanings of the words "2-functor" and "natural transformation" give different kinds of equivalence; in the equivalence we want to prove, we will be talking about strict 2-functors and strict natural transformations, and we will call it *strict equivalence*. Notice that the statement we prove, that **2Grp** and **XMod** are strictly equivalent, is stronger than just the statement that they are equivalent.

We now give the definitions of 2-functor and natural transformation in this context; the original ones can be seen in [7]; in [6] the definitions are given in their most general forms, for bicategories; the reader can check that for (strict) 2-categories they reduce to our definition.

Definition 4.1. *Let $\mathcal{C}, \mathcal{D}$ be two 2-categories. A* (strict) 2-functor *$F : \mathcal{C} \to \mathcal{D}$ maps $\mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D}), \mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$ for all objects A, B and $2\mathrm{Hom}(f, g) \to 2\mathrm{Hom}(F(f), F(g))$ for all morphisms f, g, in such a way that preserves all 2-categorical structure: composition of morphisms, vertical and horizontal composition of 2-morphisms, identity morphisms and identity 2-morphisms.*

Given strict 2-functors $F, G : \mathcal{C} \to \mathcal{D}$, a natural transformation *$\tau : F \to G$ is a map $\mathrm{Obj}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$ such that $\tau(A) : F(A) \to G(A)$, in such a way that for every morphism $f : A \to B$ we have*

$$G(f)\tau(A) = \tau(B)F(f),$$

and for every 2-morphism $\chi : f \to g$, where $f, g : A \to B$, we have

$$G(\chi)\mathrm{Id}_{\tau(A)} = \mathrm{Id}_{\tau(B)} \circ F(\chi).$$

A natural transformation such that $\tau(A)$ is invertible for all objects A is called a natural isomorphism*.*

We now define ξ and γ on morphisms and 2-morphisms, extending Proposition 2.11.

Proposition 4.2. *Let $F, F' : \mathcal{G} \to \mathcal{H}$ be 2-group homomorphisms and $\tau : F \to F'$ a 2-group 2-homomorphism. There is an* associated derivation *$\gamma(\tau) : \gamma(F) \to \gamma(F')$, given by $\gamma(\tau)(g) = \tau(g) \otimes \mathrm{Id}_{F(g)}^{-h}$ for every $g \in G_0$.*

Let $F, F' : \mathfrak{G} \to \mathfrak{H}$ be crossed module homomorphisms and $s : F \to F'$ a derivation. There is an associated 2-group 2-homomorphism *$\xi(s) : \xi(F) \to \xi(F')$ given by $\xi(s)(g) = (f_1(g), s(g))$ for every $g \in G$.*

Furthermore, $\xi(s \cdot s') = \xi(s) \cdot \xi(s'), \xi(s \circ s') = \xi(s) \circ \xi(s')$ for all derivations s, s' such that $s \cdot s', s \circ s'$ make sense, respectively; also $\gamma(\tau \cdot \sigma) = \gamma(\tau) \cdot \gamma(\sigma), \gamma(\tau \circ \sigma) = \gamma(\tau) \circ \gamma(\sigma)$ for all 2-group 2-homomorphisms τ, σ such that $\tau \cdot \sigma, \tau \circ \sigma$ make sense, respectively.

Proof. Let $\gamma(\mathcal{G}) = (G, E, \partial, \triangleright)$ and $\gamma(\mathcal{H}) = (G', E', \partial', \triangleright')$. Let us prove that $s = \gamma(\tau)$ is indeed a derivation. First notice that s maps G into E': if $g \in G$, then $s(g) = \tau(g) \otimes \mathrm{Id}^{-h}_{F(g)}$ is a morphism $1 \to F'(g)F(g)^{-1}$; that is, $s(g) \in \mathrm{Hom}_{\mathcal{H}}(1, -) = E'$.

If $g, h \in G$, then

$$\begin{aligned} s(g)\left(\gamma(F)(g) \triangleright s(h)\right) &= \left(\tau(g) \otimes \mathrm{Id}^{-h}_{F(g)}\right) \otimes \left[F(g) \triangleright \left(\tau(h) \otimes \mathrm{Id}^{-h}_{F(h)}\right)\right] \\ &= \left(\tau(g) \otimes \mathrm{Id}^{-h}_{F(g)}\right) \otimes \left[\mathrm{Id}_{F(g)} \otimes \left(\tau(h) \otimes \mathrm{Id}^{-h}_{F(h)}\right) \otimes \mathrm{Id}^{-h}_{F(g)}\right] \\ &= \tau(g) \otimes \tau(h) \otimes \mathrm{Id}^{-h}_{F(h)} \otimes \mathrm{Id}^{-h}_{F(g)} = \tau(g \otimes h) \otimes \mathrm{Id}^{-h}_{F(g \otimes h)} = s(gh); \end{aligned}$$

furthermore, $\gamma_1(F')(g) = F'(g) = t(\tau(g)) = t(s(g) \otimes \mathrm{Id}_{F(g)}) = \partial(s(g)) \otimes F(g) = \partial(s(g)) \otimes \gamma_1(F)(g)$ for every $g \in G$. From the naturality of τ comes $F'(e) \cdot \tau(g) = \tau(h) \cdot F(e)$ whenever $e : g \to h$; in particular, when $e \in E$, we have $e : 1 \to \partial(e)$, and

$$\begin{aligned} \gamma_2(F')(e) = F'(e) = \tau(\partial(e)) \cdot F(e) &= \left(s(\partial(e)) \otimes \mathrm{Id}_{F(\partial(e))}\right) \cdot (\mathrm{Id}_1 \otimes F(e)) \\ &= \left(s(\partial(e) \cdot \mathrm{Id}_1) \otimes (\mathrm{Id}_{F(\partial(e))} \cdot F(e)\right) = s(\partial(e))F(e) = s(\partial(e))\gamma_2(F)(e), \end{aligned}$$

as needed.

We now prove that $\xi(s)$ is a 2-homomorphism. We have $\xi(s)(g) = (f_1(g), s(g)) : f_1(g) \to \partial'(s(g)) \otimes f_1(g)$; since $\partial'(s(g)) \otimes f_1(g) = f_1'(g)$ we have $\xi(s)(g) : \xi(F)(g) \to \xi(F')(g)$. The naturality square commutes: given a morphism $(g, e) : g \to \partial(e)g$ in $\xi(\mathfrak{G})$, we have

$$\begin{aligned} \xi(F')(g, e) \cdot \xi(s)(g) &= (f_1'(g), f_2'(e)) \cdot (f_1(g), s(g)) = (f_1(g), f_2'(e)s(g)) = (f_1(g), s(\partial(e))f_2(e)s(g)) \\ &= (f_1(g), s(\partial(e))(f_2(e)s(g)f_2(e)^{-1})f_2(e)) = (f_1(g), s(\partial(e))(\partial'(f_2(e)) \triangleright' s(g))f_2(e)) \\ &= (f_1(g), s(\partial(e))(f_1(\partial(e)) \triangleright' s(g))f_2(e)) \\ &= (f_1(g), s(\partial(e)g)f_2(e)) = (f_1(\partial(e)g), s(\partial(e)g)) \cdot (f_1(g), f_2(e)) \\ &= \xi(s)(\partial(e)g) \cdot \xi(F)(g, e). \end{aligned}$$

Finally,

$$\begin{aligned} \xi(s)(g \otimes h) &= (f_1(g \otimes h), s(g \otimes h)) = (f_1(g)f_1(h), s(g)(f_1(g) \triangleright' s(h))) \\ &= (f_1(g), s(g)) \otimes (f_1(h), s(h)) = \xi(s)(g) \otimes \xi(s)(h) \end{aligned}$$

for all $g, h \in G$.

The remainder of the proposition is easy to show. □

We will need the following short lemma, which follows straightforwardly from the definitions.

Lemma 4.3. *The map ξ is faithful on 2-morphisms: if $s, s' : F \to F'$ are derivations such that $\xi(s) = \xi(s')$, then $s = s'$.*

Proposition 4.4. XMod *is a 2-category.*

Proof. We have already proved that **XMod** is a category; thus what remains to be proved is the existence of identity 2-morphisms, that horizontal/vertical compositions are associative, that identity 2-morphisms on identity morphisms are horizontal identities, that the horizontal composition of identity 2-morphisms is an identity 2-morphism, and that the interchange law holds.

We define the identity derivation on a crossed module homomorphism $F : \mathcal{G} \to \mathcal{H}$ as $\mathrm{Id}_F : g \mapsto 1$ for all $g \in G$. It is easy to check that this is a derivation, and that $\xi(\mathrm{Id}_F) = \mathrm{Id}_{\xi}(F)$. The proof of all these properties is very similar to the proof of Proposition **??**; we prove the interchange law as another example: let $\mathcal{G}, \mathcal{H}, \mathcal{K}$ be crossed modules, $F_1, F_2, F_3 : \mathcal{G} \to \mathcal{H}, F_1', F_2', F_3' : \mathcal{H} \to \mathcal{K}$ be crossed module homomorphisms, and $s_1 : F_1 \to F_2, s_2 : F_2 \to F_3, s_1' : F_1' \to F_2', s_2' : F_2' \to F_3'$ be derivations. Then

$$\begin{aligned}\xi((s_2' \cdot s_1') \circ (s_2 \cdot s_1)) &= (\xi(s_2') \cdot \xi(s_1')) \circ (\xi(s_2) \cdot \xi(s_1)) \\ &= (\xi(s_2') \circ \xi(s_2)) \cdot (\xi(s_1') \circ \xi(s_1)) = \xi((s_2' \circ s_2) \cdot (s_1' \circ s_1));\end{aligned}$$

since both $(s_2' \cdot s_1') \circ (s_2 \cdot s_1)$ and $(s_2' \circ s_2) \cdot (s_1' \circ s_1)$ are 2-morphisms $F_1'F_1 \to F_3'F_3$ and ξ is injective on 2-morphisms with the same source and target, $(s_2' \cdot s_1') \circ (s_2 \cdot s_1) = (s_2' \circ s_2) \cdot (s_1' \circ s_1)$, as desired. $\square$

Theorem 4.5. **2Grp** *and* **XMod** *are equivalent 2-categories.*

Proof. Recall Proposition 2.11. Notice also that $\gamma : \mathbf{2Grp} \to \mathbf{XMod}$ and $\xi : \mathbf{XMod} \to \mathbf{2Grp}$ are functors.

We define a strict natural transformation $\tau : \mathrm{Id}_{\mathbf{XMod}} \to \gamma\xi$ as follows. Given a crossed module $\mathcal{G} = (G, E, \partial, \triangleright)$, we have $\gamma(\xi(\mathfrak{G})) = (G, \{1\} \times E, \partial', \triangleright')$, where $\partial'(1, e) = \partial(e)$ and $g \triangleright' (1, e) = (1, g \triangleright e)$. The isomorphism $\tau(\mathfrak{G}) : \mathfrak{G} \to \gamma\xi(\mathfrak{G})$ is given by $\tau(\mathfrak{G}) = (t_1, t_2)$, where $t_1 : G \to G$ is the identity and $t_2 : E \to \{1\} \times E$, where $t_2(e) = (1, e)$. It is easy to see that $\tau(\mathfrak{G})$ is indeed a isomorphism. Furthermore, if $F : \mathcal{G} \to \mathcal{G}'$ is a homomorphism, then it is easy to see that the corresponding naturality square commutes; the same holds for derivations.

We now define a strict natural transformation $\sigma : \mathrm{Id}_{\mathbf{2Grp}} \to \xi\gamma$ as follows. Given a 2-group $\mathcal{G}$, the isomorphism $\sigma(\mathcal{G}) : \mathcal{G} \to \xi\gamma(\mathcal{G})$ is given by $\sigma(\mathcal{G})(g) = g$ and $\sigma(\mathcal{G})(\chi) = (s(\chi), \chi \otimes \mathrm{Id}_{s(\chi)}^{-h})$ for all $g \in G_0$ and $\chi \in G_1$. Let us check that $\sigma(\mathcal{G})$ is indeed an isomorphism. It is a functor: $\sigma(\mathcal{G})(\mathrm{Id}_g) = (g, \mathrm{Id}_1)$, which is the identity on g on $\xi\gamma(\mathcal{G})$); also if $\chi : g \to h$ and $\eta : h \to k$ are morphisms in $\mathcal{G}$, then

$$\sigma(\mathcal{G})(\eta \cdot \chi) = (g, (\eta \cdot \chi) \otimes \mathrm{Id}_g^{-h}) = (g, (\eta \otimes \mathrm{Id}_h^{-h}) \otimes (\chi \otimes \mathrm{Id}_g^{-h}) = (h, \eta \otimes \mathrm{Id}_h^{-h}) \cdot (g, \chi \otimes \mathrm{Id}_g^{-h}) = \sigma(\mathcal{G})(\eta) \cdot \sigma(\mathcal{G})(\chi),$$

where $(\eta \cdot \chi) \otimes \mathrm{Id}_g^{-h} = (\eta \otimes \mathrm{Id}_h^{-h}) \otimes (\chi \otimes \mathrm{Id}_g^{-h})$ follows from Proposition 2.17. Obviously $\sigma(\mathcal{G})$ behaves well under $\otimes$ on objects. It remains to be proved that it behaves well under $\otimes$ on objects. Let $\chi : g \to h, \eta : k \to l$ be morphisms on $\mathcal{G}$. Then

$$\begin{aligned}\sigma(\mathcal{G})(\chi \otimes \eta) &= (gk, (\chi \otimes \eta) \otimes \mathrm{Id}_{g \otimes k}^{-h}) = (gk, (\chi \otimes \mathrm{Id}_g^{-h}) \otimes [\mathrm{Id}_g \otimes (\eta \otimes \mathrm{Id}_k^{-h}) \otimes \mathrm{Id}_g^{-h}]) \\ &= (gk, (\chi \otimes \mathrm{Id}_g^{-h}) \otimes [g \triangleright (\eta \otimes \mathrm{Id}_k^{-h})]) = (g, \chi \otimes \mathrm{Id}_g^{-h}) \otimes (k, \eta \otimes \mathrm{Id}_k^{-h}) = \sigma(\mathcal{G})(\chi) \otimes \sigma(\mathcal{G})(\eta).\end{aligned}$$

Finally, it is easy to see that $\sigma(\mathcal{G})$ is bijective on objects and morphisms, and thus concluding the proof that $\sigma(\mathcal{G})$ is an isomorphism.

Now we prove that the naturality squares for σ commute: given a 2-group homomorphism $F : \mathcal{G} \to \mathcal{H}$, we have $(\xi\gamma)(F)\sigma(\mathcal{G}) = \sigma(\mathcal{H})F$: if $\chi : g \to h$ is a morphism on $\mathcal{G}$, then

$$\begin{aligned}(\xi\gamma)(F)(\sigma(\mathcal{G})(\chi)) &= (\xi\gamma)(F)(g, \chi \otimes \mathrm{Id}_g^{-h}) = (F(g), F(\chi \otimes \mathrm{Id}_g^{-h})) \\ &= (F(g), F(\chi) \otimes \mathrm{Id}_{F(g)}^{-h}) = \sigma(\mathcal{H})(F(\chi)).\end{aligned}$$

Given a 2-homomorphism $\tau : F \to G$, we have $\mathrm{Id}_{\sigma(\mathcal{H})} \circ \mathrm{Id}_{\mathbf{2Grp}}(\tau) = \xi\gamma(\tau) \circ \mathrm{Id}_{\sigma(\mathcal{G})}$: if $g \in G_0$, then

$$(\mathrm{Id}_{\sigma(\mathcal{H})} \circ \tau)(g) = \sigma(\mathcal{H})(\tau(g)) = (F(g), \tau(g) \otimes \mathrm{Id}^{-1}_{F(g)}) = \xi(\gamma(\tau))(g) = (\xi(\gamma(\tau)) \circ \mathrm{Id}_{\sigma(\mathcal{G})})(g).$$

Thus we have $\xi\gamma \cong \mathrm{Id}_{\mathbf{2Grp}}$ and $\gamma\xi \cong \mathrm{Id}_{\mathbf{XMod}}$, and so it is proven that $\mathbf{2Grp} \simeq \mathbf{XMod}$. $\square$

4.2 Properties, examples and computations of automorphism 2-groups

Using a similar strategy to the one employed in the proof of Proposition 4.4, crossed module analogues to Propositions 3.2 and 3.4 can be proved. We now state these without a proof.

Proposition 4.6. *Let $F : \mathcal{G} \to \mathcal{G}'$ be a crossed module homomorphism and $s : G \to E'$ a map such that $s(gh) = s(g)(f_1(g)) \triangleright s(h))$ for all $g, h \in G$. Then there is a crossed module homomorphism $F' : \mathcal{G} \to \mathcal{G}'$ such that $s : F \to F'$ is a derivation, given by $f_1'(g) = \partial(s(g))f_1(g)$ and $f_2'(e) = s(\partial(e))f_2(e)$ for all $g \in G, e \in E$.*

Proposition 4.7. *Every derivation is vertically invertible.*

We now determine the 2-groups which are equivalent to Id_1.

Proposition 4.8. *A 2-group $\mathcal{G}$ is equivalent to the trivial 2-group if and only if it is (isomorphic to) an adjoint 2-group.*

Proof. Let us prove that every group isomorphic to an adjoint 2-group is equivalent to the trivial 2-group; we only need to prove this for adjoint 2-groups.

Let $\mathcal{G} = G[\mathrm{Ad}]$ be an adjoint 2-group. Define $F : G[\mathrm{Ad}] \to \mathrm{Id}_1$ and $G : \mathrm{Id}_1 \to G[\mathrm{Ad}]$ in the only way possible: F maps everything to Id_1 and G maps Id_1 to the identity. We have $FG = \mathrm{Id}_{\mathrm{Id}_1}$, so we need only prove $GF \cong \mathrm{Id}_{^{\mathrm{Ad}}G}$. In order to do so, consider $\tau : GF \to \mathrm{Id}_{^{\mathrm{Ad}}G}$ defined as follows: given $g \in G$, define $\tau(g)$ as the only morphism $1 \to g$. We have $\tau(g) : GF(g) \to \mathrm{Id}_{G[\mathrm{Ad}]}(g)$, and

$$\tau(g \otimes h) = 1 \to (g \otimes h) = (1 \otimes 1) \to (g \otimes h) = (1 \to g) \otimes (1 \to h) = \tau(g) \otimes \tau(h),$$

and, finally, if $\chi = g \to h$ is a morphism in $^{\mathrm{Ad}}G$, then

$$\mathrm{Id}_{G[\mathrm{Ad}]}(\chi) \cdot \tau(g) = (g \to h) \cdot (1 \to g) = 1 \to h = (1 \to h) \cdot (1 \to 1) = \tau(h) \cdot GF(\chi).$$

Now let us prove that a 2-group equivalent to the trivial 2-group is isomorphic to an adjoint 2-group. If $\mathcal{G} \simeq \mathrm{Id}_1$, then the underlying categories $\mathcal{G}$ and Id_1 are equivalent; this means that $\mathrm{Hom}_{\mathcal{G}}(g, h)$ is singular for all $g, h \in G_0$. As seen in the last example, this means that $\mathcal{G} \cong G[\mathrm{Ad}]$ for $G = G_0$. $\square$

We will now introduce the automorphism 2-group; but first, a short lemma is needed.

Lemma 4.9. *Let $F, G : \mathcal{G} \to \mathcal{H}$ be 2-group isomorphisms and $\tau : F \to G$ be a 2-group homomorphism. Then τ is horizontally invertible.*

Proof. Since τ is an invertible natural transformation between invertible functors, there is a natural transformation that is its horizontal inverse, τ^{-h}, given by $\tau^{-h}(g) = F^{-1}(\tau(F(g)))$; this is a group homomorphism since it is the composition of the group homomorphisms F^{-1}, τ and F. $\square$

Definition 4.10. *Let $\mathcal{G}$ be a 2-group. We define the* automorphism 2-group *of $\mathcal{G}$, denoted* $\mathrm{Aut}(\mathcal{G})$*, as the 2-group with:*

- *Object group the isomorphisms $\mathcal{G} \to \mathcal{G}$, with monoidal product on objects the composition of functors;*
- *Morphism group the 2-homomorphisms between those isomorphisms; the composition is the vertical composition of 2-group 2-homomorphisms, and monoidal product on morphisms is the horizontal composition.*

Remark 4.11. *Just like groups have a 2-group of automorphisms, 2-groups should have a 3-group of automorphisms; this is indeed the case, but this discussion is postponed to section 8.*

Note that the automorphism 2-group is indeed a 2-group, since from Lemmas 2.6 and 4.9 both objects and morphisms are horizontally invertible; the remaining properties follow from identity natural transformations on homomorphisms being 2-homomorphisms and from the properties of natural transformations.

We finish this section by computing the automorphism 2-group of some 2-groups.

Example 4.12. *Let $(G, E, \triangleright)$ be a module. We claim that*

$$\mathrm{Aut}(G[0] \ltimes_{\triangleright} E[1]) \cong \mathrm{Aut}(G, E, \triangleright)[0] \ltimes_{*} Z(G, E)[1],$$

where $Z(G, E)$ is the group of 1-cocycles and $*$ is the action of $\mathrm{Aut}(G, E, \triangleright)$ on $Z(G, E)$ given by $(\sigma_1, \sigma_2) * f = \sigma_2 f \sigma_1^{-1}$. In particular, if $\triangleright$ is the trivial action, then

$$\mathrm{Aut}(G[0] \times E[1]) \cong (\mathrm{Aut}(G) \times \mathrm{Aut}(E))[0] \ltimes_{*} \mathrm{Hom}(G, E).$$

Let $\tau : F \to F'$ be a 2-morphism; from the naturality squares of τ follows that $F' = F$. Thus all morphisms in $\mathrm{Aut}(G[0] \ltimes_{\triangleright} E[1])$ have the same source and target, and so $\mathrm{Aut}(G[0] \ltimes_{\triangleright} E[1]) = H[0] \ltimes_{*'} K[1]$ for the groups H of isomorphisms $G[0] \times E[1] \to G[0] \times E[1]$ and K of 2-homomorphisms $\mathrm{Id}_{\mathrm{Aut}(G[0] \times E[1])} \to F$, under horizontal composition, and the action $*'$ given by $F * \tau = F \tau F^{-1}$. We need only prove that $(H, K, *') \cong (\mathrm{Aut}(G, E, \triangleright), Z(G, E), *)$ as modules.

Define $\phi : H \to \mathrm{Aut}(G, E, \triangleright)$ and $\phi' : \mathrm{Aut}(G, E, \triangleright) \to H$ as follows: if $F \in H$, then $\phi(F) = (f_1, f_2)$, where $f_1(g) = F(g)$ for $g \in G$ and $f_2(e)$ is such that $F(1, e) = (1, f_2(e))$ for $e \in E$; if $(f_1, f_2) \in \mathrm{Aut}(G, E, \triangleright)$, then $\phi'(f_1, f_2) = F$ is defined by $F(g) = f_1(g)$ for $g \in G$ and $F(g, e) = (f_1(g), f_2(e))$ for $(g, e) \in G \ltimes_{\triangleright} E$. It is easy to check that ϕ and ϕ' are well defined group homomorphisms; furthermore, they are inverse functions.

Define $\sigma : K \to Z(G, E)$ and $\sigma' : Z(G, E) \to K$ as follows: if $\tau \in K$ and $g \in G$ then $\sigma(\tau)(g)$ is such that $\tau(g) = (g, \sigma(\tau)(g))$ (the cocycle condition arises from τ being multiplicative. If $f \in Z(G, E)$ then $\sigma'(f) : \mathrm{Id}_{G[0] \ltimes_{\triangleright} E[1]} \to \mathrm{Id}_{G[0] \ltimes_{\triangleright} E[1]}$ is defined by $\sigma'(f)(g) = (g, f(g))$. It is easy to check that σ and σ' are well defined group homomorphisms; furthermore, they are inverse functions. It is also easy to check that (ϕ, σ) and (ϕ', σ') agree with the given actions, thus are module isomorphisms. This finishes the proof of our initial claim.

Example 4.13. *Let G be a group. We claim that* $\mathrm{Aut}(G[\mathrm{Ad}]) \cong \mathrm{Aut}(G)[\mathrm{Ad}]$.

Let $\mathcal{K} = \mathrm{Aut}(G[\mathrm{Ad}])$. We only need to prove that $K_0 \cong \mathrm{Aut}(G)$ and that every Hom set in $\mathcal{K}$ is a singular set.

We define $\phi : K_0 \to \mathrm{Aut}(G)$ and $\psi : \mathrm{Aut}(G) \to K_0$ as follows: given $F \in K_0$, define $\phi(F) = f_0$; given $f \in \mathrm{Aut}(G)$, define $\psi(f)$ by $\psi(f)(g) = f(g)$ for $g \in G$ and $\psi(f)(g \to$

$h) = f(g) \to f(h)$ for all $g, h \in G$. It is easy to check that ϕ and ψ are well defined group homomorphisms; furthermore, they are inverse; thus $K_0 \cong \mathrm{Aut}(G)$.

Let $F, F' \in K_0$. Defining $\tau : F \to F'$ by $\tau(g) = (F(g) \to F'(g))$ for $g \in G$; its easy to see that τ is a 2-homomorphism. Furthermore, if $\sigma : F \to F'$ is any 2-homomorphism, then by definition $\sigma(g) \in \mathrm{Hom}(F(g), F'(g))$, therefore $\sigma(g) = (F(g) \to F'(g))$; that is, $\sigma = \tau$. It is thus proven that every Hom set in $\mathcal{K}$ is singular, finishing the proof to our claim.

Example 4.14. *Let G be a group. We claim that $\mathrm{Aut}(G[\mathrm{Aut}]) \cong \xi(\mathrm{Aut}(G), \mathrm{Hom}(\mathrm{Aut}(G), Z(G)), \partial, \triangleright)$, where $\mathrm{Hom}(G, E)$ for groups G, E with E abelian is the same as in example 4.12; the boundary morphism ∂ is given by $\partial(\tau) : g \mapsto \tau(\gamma_g)g$, and the action $\triangleright$ is given by $f \triangleright \tau : g \mapsto f(\tau(f^{-1}(g)))$.*

In order to prove our claim, we will prove that $\gamma(\mathrm{Aut}(\mathcal{G})) \cong (\mathrm{Aut}(G), \mathrm{Hom}(\mathrm{Aut}(G), Z(G)), \partial, \triangleright)$, where $\mathcal{G} = \xi\gamma(G[\mathrm{Aut}]) \cong G[\mathrm{Aut}]$. Write $(K, E, \partial', \triangleright') = \gamma(\mathrm{Aut}(\mathcal{G}))$. Define $\phi : K \to \mathrm{Aut}(G)$ and $\phi' : \mathrm{Aut}(G) \to K_1$ as follows: given $F \in K$ and $g \in G$, define $\phi(F)(g)$ by $F(1, g) = (1, \phi(F)(g))$; given $\sigma \in \mathrm{Aut}(G)$, define $\phi'(\sigma)$ by $\phi'(\sigma)(\tau) = \sigma\tau\sigma^{-1}$ for $\tau \in G_0$ and $\phi'(\sigma)(\tau, g) = (\sigma\tau\sigma^{-1}, \sigma(g))$. It is easy to check that ϕ and ϕ' are well defined group homomorphisms and that $\phi\phi' = \mathrm{Id}_{\mathrm{Aut}(G)}$; if we show that $\phi'\phi = \mathrm{Id}_K$, we conclude that ϕ is an isomorphism. Indeed, if $F \in K$, then $\gamma(F) = (f_1, f_2) : \gamma(\mathcal{G}) \to \gamma(\mathcal{G})$ is a crossed module homomorphism; we have $f_1(\sigma)(f_2(g)) = f_1(\sigma)|f_2(g) = f_2(\sigma|g) = f_2(\sigma(g))$, therefore $f_1(\sigma) = f_2\sigma f_2^{-1}$; thus, since $f_2 = \phi(F)$, we have $F(\sigma) = \phi(F)\sigma\phi(F)^{-1}$ for $\sigma \in \mathrm{Aut}(G)$ and $F(\sigma, g) = (\phi(F)\sigma\phi(F)^{-1}, \phi(F)(g))$; in summary, $\phi'\phi(F) = F$, as desired.

Define $\psi : E \to \mathrm{Hom}(\mathrm{Aut}(G), Z(G))$ and $\psi' : \mathrm{Hom}(\mathrm{Aut}(G), Z(G)) \to E$ as follows. Given a 2-homomorphism $\tau : \mathrm{Id}_{\mathrm{Aut}(\mathcal{G})} \to F$ in E, define $\psi(\tau)(\sigma)$ by g, where $\tau(\sigma) = (\sigma, g)$; given a homomorphism $f \in \mathrm{Hom}(\mathrm{Aut}(G), Z(G))$, define $\psi'(f) : \mathrm{Id}_{\mathrm{Aut}(\mathcal{G})} \to \phi'(\sigma_f)$, where $\sigma_f : g \mapsto f(\gamma_g)g$, by $\psi'(f)(\rho) = (\rho, f(\rho))$.

Let us prove that ψ is well-defined: if $\tau : \mathrm{Id}_{\mathrm{Aut}(\mathcal{G})} \to \phi(\sigma)$ is a 2-homomorphism, and $\psi(\tau) = f$ then $f : \mathrm{Aut}(G) \to G$ is a group homomorphism; we need only check that $\mathrm{im}\ f \leq Z(G)$. Indeed, consider a morphism $(\rho, g) \in G_1$; the corresponding naturality square yields $\sigma(g)f(\rho) = f(\gamma_g\sigma)g$. Setting $\rho = \mathrm{Id}_G$, we get $\sigma(g) = f(\gamma_g)g$; thus in general

$$f(\gamma_g)f(\rho)g = f(\gamma_g\rho)g = \sigma(g)f(\rho) = f(\gamma_g)gf(\rho),$$

and therefore $f(\rho)g = gf(\rho)$ for all $g \in G, \rho \in \mathrm{Aut}(G)$; that is, $f(\rho) \in Z(G)$ for all $\rho \in \mathrm{Aut}(G)$, as needed.

It is now easy to check that ψ, ψ' are well-defined group homomorphisms; furthermore, they are inverse. It is now easy to finish the proof of our initial claim, by seeing that $(\phi, \psi) : \gamma(\mathrm{Aut}(\mathcal{G})) \to (\mathrm{Aut}(G), \mathrm{Hom}(\mathrm{Aut}(G), Z(G)), \partial, \triangleright)$ is an isomorphism.

5 2-Subgroups

We start our generalizations with the notion of 2-subgroup, the idea of having a structure of some type inside a structure of the same type, in the case of 2-groups. In section 5.1, we define 2-subgroups and give criteria to determine 2-subgroups; in section 5.2 we define crossed submodules, making use of the equivalence described in section 2.3. In section 5.3 we will see examples of 2-subgroups, and in section 5.4 we will see the 2-subgroup generated by a subset of 2-morphisms.

5.1 Definition

The notion of subgroup in group theory is very well known: a subgroup H of a given group G is a subset of G which is a group under the operation of G. Of course this is a painful definition: if this were the working definition, anytime one proved that a subset of a given group is a subgroup, one would have to check all the group axioms. Instead, in practice, one only checks three axioms: that H contains the identity of G, and is closed under its operation and inverses; the axioms for a group hold in particular for H since they hold in general for G.

With this in mind, we give a practical definition of a 2-subgroup $\mathcal{H}$ of a given 2-group $\mathcal{G}$, with a relatively short list of axioms, and then prove that it is equivalent to $\mathcal{H}$ being a 2-group under the operations of $\mathcal{G}$.

However, there is an issue of rigour for these notions that we feel the need to discuss beforehand. This issue shows up in a subtle way in group theory, but it is mostly ignored; however, it is harder to ignore in the context of 2-groups, since there is a larger amount of structure involved. The usual definition of a subgroup H of a given group G states that H is a *subset* of G; strictly speaking, this means that we can't call H a group. However, it is well known that H gives rise to a group, simply by giving it the restriction of the operation of G to H. We could define a subgroup in such a way that this issue disappears: for example, we might say that $(H, \circ)$ is a subgroup of $(G, *)$ if $H \subseteq G$ and the inclusion $i : H \to G$ is a group homomorphism. However, if we did that, then it wouldn't make sense to ask questions such as "does this particular *subset* S of a given group G form a subgroup?". Additionally, it is useful to do these things in an informal fashion, such that one doesn't have to mention the operation being used all the time. We will study these concepts with this mindset, providing clarification where needed.

We now proceed to the actual definition.

Definition 5.1. *A* 2-subgroup *$\mathcal{H}$ of a given 2-group $\mathcal{G}$ is a pair (H_0, H_1) of subsets $H_0 \subseteq G_0, H_1 \subseteq G_1$ such that:*

- *H_1 is a subgroup of G_1;*
- *$s(H_1), t(H_1) \subseteq H_0$;*
- *$\mathrm{Id}(H_0) \subseteq H_1$.*

We adopt the notation from group theory, writing $\mathcal{H} \leq \mathcal{G}$ for "$\mathcal{H}$ is a 2-subgroup of $\mathcal{G}$".

Proposition 5.2. *A 2-subgroup $\mathcal{H} = (H_0, H_1)$ of a given 2-group $\mathcal{G}$ is a 2-group under the restrictions of the operations of $\mathcal{G}$ to $\mathcal{H}$; taking $\mathcal{H}$ as the subcategory of $\mathcal{G}$ with objects H_0 and morphisms H_1, and restricting the functor $\otimes$ to $\mathcal{H}$, we get a 2-group.*

Proof. Since $H_1 \leq G_1$, the set H_1 is closed under $\otimes$ and horizontal inverses.

To prove that $H_0 \leq G_0$, we simply notice that $\mathrm{Id}(H_0)$ must be a subgroup of H_1, and thus $H_0 = s(\mathrm{Id}(H_0))$ must be a subgroup of G_0, since s is a group homomorphism.

Let us prove that $\mathcal{H}$ is closed under composition. This follows from Proposition 2.17, as follows. If $\chi, \eta \in H_1$ and $\chi : h_2 \to h_3, \eta : h_1 \to h_2$, then $h_1, h_2, h_3 \in H_0$; in particular, $h_2^{-1} \in H_0$, and thus $\mathrm{Id}_{h_2^{-1}} \in H_1$. Since H_1 is closed under $\otimes$, we have $\chi \cdot \eta = \chi \otimes \mathrm{Id}_{h_2^{-1}} \otimes \eta \in H_1$.

Let us now prove that $\mathcal{H}$ is closed under vertical inverses. This follows from Corollary 2.18, as follows. If $\chi \in H_1$ and $\chi : g \to h$, then $g, h \in H_0$, and thus $\mathrm{Id}_g, \mathrm{Id}_h \in H_1$. Furthermore, since $H_1 \leq G_1$, we have $\chi^{-h} \in H_1$. Now $\chi^{-v} = \mathrm{Id}_g \otimes \chi^{-h} \otimes \mathrm{Id}_h \in H_1$, since H_1 is closed under $\otimes$.

These and the remaining conditions in the definition of subgroup prove that the restrictions of composition, horizontal composition and vertical composition to $\mathcal{H}$ are well defined. The remaining 2-group axioms (associativities, interchange law) hold for $\mathcal{H}$ in particular, since they hold for $\mathcal{G}$ in general. $\square$

Remark 5.3. *From now on, when we talk about 2-subgroup, we can mean either the pair of subsets or the 2-group we can get from a 2-subgroup. In any discussion in this article, this ambiguity will not cause any issues.*

Remark 5.4. *It is easy to check that if a pair (H_0, H_1) of subsets $H_0 \subseteq G_0, H_1 \subseteq G_1$ gives rise to a 2-group in the same way as described in the previous proposition (i.e. restricting operations), then (H_0, H_1) is a 2-subgroup of $\mathcal{G}$. This means that (H_0, H_1) is a 2-subgroup of $\mathcal{G}$ if and only if it gives rise to a 2-group under the operations of $\mathcal{G}$.*

Definition 5.5. *Given a collection of pairs of sets $\{P_i\}_{i\in I}$, where each $P_i = (P_{i,1}, P_{i,2})$, we define its* intersection *as $\bigcap_{i\in I} P_i = (\bigcap_{i\in I} P_{i,1}, \bigcap_{i\in I} P_{i,2})$.*

Proposition 5.6. *Let $\{\mathcal{H}_i\}_{i\in I}$ be a collection of 2-subgroups of a given 2-group $\mathcal{G}$. Then $\bigcap_{i\in I} \mathcal{H}_i$ is a 2-subgroup of $\mathcal{G}$.*

Proof. It is well known that the intersection of groups is a group, thus $\bigcap_{i\in I} \mathcal{H}_{i,1}$ is a group. We have

$$s\left(\bigcap_{i\in I} \mathcal{H}_{i,1}\right) \subseteq \bigcap_{i\in I} s(\mathcal{H}_{i,1}) \subseteq \bigcap_{i\in I} \mathcal{H}_{i,0};$$

analogous arguments work for t and Id. $\square$

5.2 Crossed submodules

When using crossed modules to study 2-groups, we want a simple and direct way to look at the 2-subgroup relation involved; that is, given crossed modules $\mathcal{G}, \mathcal{H}$, we want to know when it is the case that $\xi(\mathcal{H}) \leq \xi(\mathcal{G})$. This is equivalent to the following definition, as we will prove.

Definition 5.7. *Given a crossed module $\mathcal{G} = (G, E, \partial, \triangleright)$, we say a pair (H, F) of subsets $H \subseteq G, F \subseteq E$ is a* crossed submodule *of $\mathcal{G}$ if $H \leq G, F \leq E$, $\partial(F) \leq H$ and $h \triangleright f \in F$ for every $h \in H, f \in F$. We write $(H, F) \leq \mathcal{G}$ to denote this.*

Remark 5.8. *It is easy to check that a crossed submodule (H, F) gives rise to a crossed module $(H, F, \partial', \triangleright')$, where $\partial', \triangleright'$ are the appropriate restrictions of ∂ and $\triangleright$; since these are obvious from the context, we shorten this notation to (H, F). When we talk about a crossed submodule we can either mean the pair (H, F) or the crossed module itself, just as in the case of 2-subgroups, as discussed in the last section. We also write $(H, F, \partial', \triangleright') \leq \mathcal{G}$ to denote that (H, F) is a crossed submodule of $\mathcal{G}$.*

Proposition 5.9. *Let $\mathfrak{G} = (G, E, \partial, \triangleright), \mathfrak{H} = (H, F, \partial', \triangleright')$ be crossed modules and $\mathcal{G} = \xi(\mathfrak{G}), \mathcal{H} = \xi(\mathfrak{H})$. Then $\mathfrak{H} \leq \mathfrak{G}$ if and only if $\mathcal{H} \leq \mathcal{G}$.*

Proof. Suppose $\mathfrak{H} \leq \mathfrak{G}$. Then $H_0 = H \leq G = G_0$ and $H_1 = H \ltimes_{\triangleright'} F \leq G \ltimes_{\triangleright} E = G_1$, since $H \leq G, F \leq E$ and $\triangleright'$ is a restriction of $\triangleright$ (to H on the "acting on" group and to F on the "acted on" group). If $(h, f) \in H_1$, then $s(h, f) = h \in H = H_0$ and $t(h, f) = \partial(f)h \in H = H_0$, since $\partial(f) \in H, h \in H$. Finally, if $h \in H_0$, then $\mathrm{Id}_h = (h, 1) \in H_1$, since $1 \in F$. Thus $\mathcal{H} \leq \mathcal{G}$, as desired.

Suppose now that $\mathcal{H} \leq \mathcal{G}$. Then ∂' and $\triangleright'$ are restrictions of ∂ and $\triangleright$, respectively. We have $H = H_0 \leq G_0 = G$ and $F = \ker s \cap H_1 \leq \ker s = E$. Furthermore, $\partial(F) = t(F) \leq H_0 = H$. Finally, if $h \in H$ and $f \in F$, then $(h, h \triangleright f) = (h, 1) \otimes (1, f) \in H_1$, therefore $h \triangleright f \in F$. Thus $\mathfrak{H} \leq \mathfrak{G}$, as desired. $\square$

5.3 Examples

The details of the proof of the next proposition are left to the reader.

Proposition 5.10. *Given a 2-group homomorphism $F : \mathcal{G} \to \mathcal{H}$, and a 2-subgroup $\mathcal{K} \leq \mathcal{H}$, the pair $(f_0^{-1}(K_0), f_1^{-1}(K_1))$ is a 2-subgroup of $\mathcal{G}$, denoted $F^{-1}(\mathcal{K})$. Given a 2-subgroup $\mathcal{K}' \leq \mathcal{G}$ the pair $(f_0(K_0'), f_1(K_1'))$ is a 2-subgroup of $\mathcal{H}$, denoted $F(\mathcal{K}')$. In particular, $F^{-1}(\mathcal{H}) = \mathcal{G}$, and $F(\mathrm{Id}_1) = \mathrm{Id}_1$.*

Definition 5.11. *Given a 2-group homomorphism $F : \mathcal{G} \to \mathcal{H}$, we define its* kernel *as the 2-subgroup* $\ker F = F^{-1}(\mathrm{Id}_1)$ *of $\mathcal{G}$, and its* image *as the 2-subgroup* $\mathrm{im}\, F = F(\mathcal{G})$.

Example 5.12. *Let $\mathcal{G}$ be a 2-group and $H \leq G_0$. There is a 2-subgroup $\mathcal{G}|H$ of $\mathcal{G}$, called the* 2-subgroup of $\mathcal{G}$ induced by H*, with object subgroup H and morphism subgroup $s^{-1}(H) \cap t^{-1}(H)$.*

Example 5.13. *Any 2-group $\mathcal{G}$ has five 2-subgroups worth mentioning:*

- *The* trivial 2-subgroup*, denoted* Id_1*, which is the pair* $(\{1\}, \{\mathrm{Id}_1\})$*;*
- *The* identities 2-subgroup*, given by the pair $(G_0, \mathrm{Id}(G_0))$; it is isomorphic to $G_0[0]$;*
- *The* base isotropy 2-subgroup *of $\mathcal{G}$, which is $\mathcal{G}|1$; another possible notation is $\pi_1(\mathcal{G})[1]$;*
- *The* base orbit*, denoted $\mathcal{O}(\mathcal{G})$, which is $\mathcal{G}|t(\ker s)$; it can also be denoted $\mathcal{G}|[1]$.*
- *$\mathcal{G}$ itself.*

Note that $\mathrm{Id}_1 \leq \pi_1(\mathcal{G})[1] \leq \mathcal{O}(\mathcal{G}) \leq \mathcal{G}$.

In the language of groupoid theory, the base isotropy 2-subgroup of $\mathcal{G}$ is the isotropy group of 1. The base orbit of $\mathcal{G}$ is the 2-subgroup of $\mathcal{G}$ induced by the orbit of 1, also in the language of groupoid theory.

If $\gamma(\mathcal{G}) = (G, E, \partial, \triangleright)$, then $\gamma(\mathrm{Id}_1) = (1, 1), \gamma(\pi_1(\mathcal{G})) = (1, \ker \partial)$ and $\gamma(\mathcal{O}(\mathcal{G})) = (\mathrm{im}\, \partial, E)$.

Example 5.14. *Continuing example 2.16, consider $O(2)[\mathrm{Top}]$. Given a positive integer n, the dihedral group D_{2n} is a subgroup of $O(2)$, therefore we can think of the 2-subgroup $O(2)[\mathrm{Top}]|D_{2n}$; in fact, this 2-subgroup gives rise to the crossed submodule $\left(D_{2n}, \frac{2\pi}{n}\mathbb{Z}\right)$. Notice that $O(2)[\mathrm{Top}]|D_{2n}$ is not itself a topological 2-group, since D_{2n} is a discrete set lying inside $O(2)$. In fact, if we give D_{2n} the discrete topology, we get $D_{2n}[\mathrm{Top}] \cong D_{2n}[0]$.*

5.4 The 2-subgroup generated by a set of 2-morphisms

Definition 5.15. *Given a 2-group $\mathcal{G}$ and a subset $X \subseteq G_1$, we say a 2-subgroup $\mathcal{H} \leq \mathcal{G}$ is* generated by X *if $X \subseteq H_1$ and for every 2-subgroup $\mathcal{K} \leq \mathcal{G}$ with $X \subseteq K_1$ we have $\mathcal{H} \leq \mathcal{K}$. We denote this by $\mathcal{H} = \langle\langle X \rangle\rangle$.*

We provide a proof of the existence of $\langle\langle X\rangle\rangle$, and a construction of this 2-subgroup.

Proposition 5.16. *Given a 2-group $\mathcal{G}$ and a subset $X \subseteq G_1$, the 2-subgroup $\langle\langle X\rangle\rangle$ exists.*

Proof. Let S be the set of 2-subgroups $\mathcal{K} \leq \mathcal{G}$ such that $X \subseteq H_1$. We claim that

$$\bigcap_{\mathcal{K}\in S} \mathcal{K} = \langle\langle X\rangle\rangle.$$

Indeed, $X \subseteq K_1$ for every $\mathcal{K} \in S$, therefore $X \subseteq \cap_{\mathcal{K}\in S} K_1 = \mathrm{Mor}(\cap_{\mathcal{K}\in S}\mathcal{K})$. Finally, if $\mathcal{H} \leq \mathcal{G}$ is such that $X \subseteq H_1$, then $\mathcal{H} \in S$, therefore $\cap_{\mathcal{K}\in S}\mathcal{K} \leq \mathcal{H}$. □

Proposition 5.17. *Given a 2-group $\mathcal{G}$ and a subset $X \subseteq G_1$, we have*

$$\langle\langle X\rangle\rangle = (\langle s(X) \cup t(X)\rangle, \langle X \cup \mathrm{Id}(s(X)) \cup \mathrm{Id}(t(X))\rangle).$$

Proof. We start by proving that the pair $\mathcal{H} = (\langle s(X) \cup t(X)\rangle, \langle X \cup \mathrm{Id}(s(X)) \cup \mathrm{Id}(t(X))\rangle)$ is a 2-subgroup of $\mathcal{G}$.

Obviously $\langle X \cup \mathrm{Id}(s(X)) \cup \mathrm{Id}(t(X))\rangle \leq G_1$. We have

$$s(\langle X \cup \mathrm{Id}(s(X)) \cup \mathrm{Id}(t(X))\rangle) = \langle s(X) \cup s(\mathrm{Id}(s(X))) \cup s(\mathrm{Id}(t(X))\rangle = \langle s(X) \cup t(X)\rangle;$$

similarly, $t(\langle X \cup \mathrm{Id}(s(X)) \cup \mathrm{Id}(t(X))\rangle) = \langle s(X) \cup t(X)\rangle$.

Finally, $\mathrm{Id}(\langle s(X) \cup t(X)\rangle) = \langle \mathrm{Id}(s(X) \cup t(X))\rangle \subseteq \langle X \cup \mathrm{Id}(s(X)) \cup \mathrm{Id}(t(X))\rangle$.

Now we prove that if $\mathcal{K} \leq \mathcal{G}$ and $X \subseteq K_1$, then $\mathcal{H} \leq \mathcal{K}$. Since $K_1 \leq G_1$, we have $\langle X\rangle \leq K_1$. Since $s(K_1) \subseteq K_0$, we have $s(\langle X\rangle) \subseteq K_0$; similarly, $t(\langle X\rangle) \subseteq K_0$. Since $K_0 \leq G_0$, we have $\langle s(X) \cup t(X)\rangle \subseteq K_0$. Now $\mathrm{Id}(K_0) \subseteq K_1$, therefore $\mathrm{Id}(\langle s(X) \cup t(X)\rangle) \subseteq K_1$; since $\mathrm{Id}(s(X)), \mathrm{Id}(t(X)), X \subseteq K_1$ and $K_1 \leq G_1$, we have $\langle X \cup \mathrm{Id}(s(X)) \cup \mathrm{Id}(t(X))\rangle \subseteq K_1$. This concludes our proof. □

Let us do this construction in terms of crossed modules.

Proposition 5.18. *Let $\mathcal{G} = (G, E, \partial, \triangleright)$ be a crossed module. Let S be a set and $X = \{(g_s, e_s) : s \in S\} \subseteq \mathrm{Mor}(\xi(\mathcal{G}))$ a set of morphisms, write $X_G = \{g_s : s \in s\}$ and $X_E = \{e_s : s \in S\}$. We have*

$$\langle\langle X\rangle\rangle = \xi(\langle X_G \cup \partial(X_E)\rangle, \langle(\langle X_G\rangle \triangleright X_E)\rangle).$$

Proof. All we have to prove is that

$$\mathrm{Obj}(\langle\langle X\rangle\rangle) = \langle X_G \cup \partial(X_E)\rangle$$

and

$$E' := \{e \in E : \exists(g, e) \in \mathrm{Mor}(\langle\langle X\rangle\rangle)\} = \langle(\langle X_G\rangle \triangleright X_E)\rangle.$$

We have $\mathrm{Obj}(\langle\langle X\rangle\rangle) = \langle s(X) \cup t(X)\rangle = \langle X_G \cup \{\partial(e_s)g_s : s \in S\}\rangle = \langle X_G \cup \partial(X_E)\rangle$.

We now prove that $\langle(\langle X_G\rangle \triangleright X_E)\rangle \leq E'$. We need only prove that for $e \in \langle X_G\rangle \triangleright X_E$ we have $e \in E'$. Indeed, if $(g_{s_1}^{a_{s_1}} \cdots g_{s_n}^{a_{s_n}} \triangleright e_t) \in (\langle X_G\rangle \triangleright X_E)$, where every $a_i \in \{-1, 1\}$, then

$$(gg_t, g \triangleright a_t) = (g_{s_1}, 1)^{a_1} \otimes \cdots \otimes (g_{s_n}, 1)^{a_n} \otimes (g_t, e_t) \in \mathrm{Mor}(\langle\langle X\rangle\rangle).$$

Finally, we prove that if $E' \leq \langle(\langle X_G\rangle \triangleright X_E)\rangle$. In order to do so, we will use a short lemma.

Lemma 5.19. *If $g \in X_G \cup \partial(X_E)$ and $e \in \langle(\langle X_G\rangle \triangleright X_E)\rangle$, then $g^{\pm 1} \triangleright e \in \langle(\langle X_G\rangle \triangleright X_E)\rangle$.*

Proof. It suffices to prove what is asked for $e \in (\langle X_G \rangle \triangleright X_E)$, since $g \triangleright (e_1^{a_1} \cdots e_n^{a_n}) = (g \triangleright e_1)^{a_1} \cdots (g \triangleright e_n)^{a_n}$. Either $g \in X_G$ or $g \in \partial(X_E)$, cases we now analyse.

Take $g = g_s \in X_G$. If $e = g' \triangleright e_t$, where $g' \in \langle X_G \rangle$ and $e_t \in X_E$, then $g^{\pm 1} \triangleright e = (g_s^{\pm 1}) \triangleright (g' \triangleright e_t) = (g_s^{\pm 1} g') \triangleright e_t \in \langle X_G \rangle X_E$, since $g_s^{\pm 1} g' \in \langle X_G \rangle X_E$.

Now take $g = \partial(e_s) \in \partial(X_E)$. We have $g^{\pm 1} \triangleright e = \partial(e_s^{\pm 1}) \triangleright e = e_s^{\pm 1} e e_s^{\mp 1} \in \langle (\langle X_G \rangle \triangleright X_E) \rangle$, since $e_s \in X_E \subseteq (\langle X_G \rangle \triangleright X_E)$. ☐

If $e \in E'$, then there is $(g, e) \in \mathrm{Mor}(\langle\langle X \rangle\rangle)$. We proceed by induction on the length of the word writing (g, e) in terms of elements $X \cup s(X) \cup t(X)$. If this length is 0, then $(g, e) = (1, 1)$, and this case is trivial. Suppose now that $(g, e) = (g_0, e_0)^{\pm 1} \otimes (g', e')$, where (g_0, e_0) is either $(g_s, 1), (\partial(e_s), 1)$ or (g_s, e_s), and (g', e') can be written as a shorter word; by hypothesis, $e' \in \langle (\langle X_G \rangle \triangleright X_E) \rangle$. If we take ± 1 to be 1, then $(g_0, e_0) \otimes (g', e') = (g_0 g', e_0 (g_0 \triangleright e'))$, thus $e = e_0 (g_0 \triangleright e') \in \langle (\langle X_G \rangle \triangleright X_E) \rangle$, using the above lemma. If we take ± 1 to be -1, then

$$(g_0, e_0)^{-1} \otimes (g', e') = (g_0^{-1}, g_0^{-1} \triangleright e_0^{-1}) \otimes (g', e') = (g_0^{-1} g', (g_0^{-1} \triangleright e_0^{-1})(g_0^{-1} \triangleright e')),$$

and thus $e = (g_0^{-1} \triangleright e_0)^{-1} (g_0^{-1} \triangleright e') \in \langle (\langle X_G \rangle \triangleright X_E) \rangle$, again, using the above lemma. This concludes our proof. ☐

6 Normal 2-subgroups, quotient 2-groups and the Postnikov series

In universal algebra, congruences are the way of taking quotients of algebraic structures; this is necessary for example with monoids, which can have congruences that don't arise from submonoids. However, we can define quotients on certain algebraic structures without using congruences, as is the case of groups and rings; this is also the case for 2-groups. However, we choose to work with congruences, for two reasons: to make a convincing point that our definition of quotient is the right one; and also because it makes proofs cleaner.

In section 6.1 we define congruences and the quotient 2-group by a congruence. In section 6.2 we prove that every congruence is deeply connected to a 2-group homomorphisms. In section 6.3 we define normal 2-subgroups and prove that they induce congruences; furthermore, we prove that normal 2-subgroups are precisely the kernels of 2-group homomorphisms, and that all congruences are induced by normal 2-subgroups. In section 6.4, we translate normal 2-subgroups into crossed module language. In section 6.5, we provide some examples, expanding some of the examples in sections 2.4 and 5.3. We close with section 6.6, where we define, describe and construct the normal closure of a 2-subgroup of a given 2-group.

6.1 Quotient 2-groups by congruences

Definition 6.1. *Given a 2-group $\mathcal{G}$, a* congruence $\equiv$ *on $\mathcal{G}$ is a pair of equivalence relations $(\equiv_0, \equiv_1)$ on G_0 and G_1, which obey the following properties:*

- *If $g \equiv_0 g'$ and $h \equiv_0 h'$, then $g \otimes h \equiv_0 g' \otimes h'$;*
- *Given morphisms $\chi : g \to h, \chi' : g' \to h'$, if $\chi \equiv_1 \chi'$, then $g \equiv_0 g'$ and $h \equiv_0 h'$;*
- *If $g \equiv_0 g'$, then $\mathrm{Id}_g \equiv_1 \mathrm{Id}_{g'}$;*

- *Given morphisms $\chi : g \to h, \chi' : g' \to h', \eta : k \to l, \eta' : k \to l'$ such that $\chi \equiv_1 \chi'$ and $\eta \equiv_1 \eta'$, we have $\chi \otimes \eta \equiv_1 \chi' \otimes \eta'$; furthermore, if $g = l$ and $g' = l'$, then $\chi \cdot \eta \equiv_1 \chi' \cdot \eta'$.*

When no confusion is possible, we can abbreviate $g \equiv_0 g'$ by $g \equiv g'$ for objects g, g' and $\chi \equiv_1 \chi'$ by $\chi \equiv \chi$ for morphisms χ, χ'.

Proposition 6.2. *Let $\mathcal{G}$ be a 2-group and $\equiv$ a congruence on $\mathcal{G}$. We denote the class of $g \in G_0$ by $[g]$ and the class of $\chi \in G_1$ by $[\chi]$. There is a 2-group, denoted $\mathcal{G}/\equiv$ and called the* quotient 2-group *of $\mathcal{G}$ by $\equiv$, which we proceed to describe.*

The object group is the set of equivalence classes of G_0 under $\equiv_0$ and the morphism group the set of equivalence classes of G_1 under $\equiv_1$; given a morphism $\chi : g \to h$, we say that $[\chi] : [g] \to [h]$; the identity on $[g]$ is $[\mathrm{Id}_g]$.

The monoidal product is defined on objects by $[g] \otimes [h] = [g \otimes h]$ and on morphisms by $[\chi] \otimes [\eta] = [\chi \otimes \eta]$.

The composition is defined as follows: let $[\chi] : [g] \to [h], [\eta] : [k] \to [l]$; if $[g] = [l]$, then there are $\chi_0 \equiv_1 \chi$ and $\eta_0 \equiv_1 \eta$ such that χ_0 and η_0 are composable, and $[\chi] \cdot [\eta] = [\chi_0 \cdot \eta_0]$.

Proof. We start by checking that everything is well defined; that is, since the definitions mention representatives of equivalence classes, one needs to show that they don't depend on the particular choice of representatives.. That the source, target and identity maps are well defined follows immediately from the definition of a congruence. The monoidal product is well defined: if $g, h, g', h' \in G_0$ are such that $g \equiv_0 g'$ and $h \equiv_0 h'$, then $g \otimes h \equiv_0 g' \otimes h'$, therefore $[g] \otimes [h] = [g'] \otimes [h']$, as desired; a similar argument works for morphisms.

Let us prove that composition is well defined. This is more intricate: we need to prove the existence of χ_0, η_0 as claimed, that the definition doesn't depend on these χ_0, η_0, and finally that it doesn't depend on χ, η. We begin by proving that given morphisms $\chi : g \to h, \eta : k \to l$ with $g \equiv_0 l$, there exist χ_0, η_0 as claimed. We claim that setting $\chi_0 = \mathrm{Id}_{l \otimes g^{-1}} \otimes \chi$ and $\eta_0 = \eta$ works. Indeed, we have $\chi_0 : l \to l \otimes g^{-1} \otimes h$ and $\eta_0 : k \to l$; furthermore, obviously $\eta_0 \equiv_1 \eta$; also, $\chi_0 \equiv_1 \chi$: since $g \equiv_0 l$, we have $1 = g \otimes g^{-1} \equiv_0 l \otimes g^{-1}$, and therefore $\mathrm{Id}_1 \equiv_0 \mathrm{Id}_{l \otimes g^{-1}}$, which means $\chi_0 = \mathrm{Id}_{l \otimes g^{-1}} \otimes \chi \equiv_1 \mathrm{Id}_1 \otimes \chi = \chi$, as desired.

We now prove that the definition doesn't depend on the choice of χ_0, η_0: let χ_0', η_0' be other such that $\chi \equiv_1 \chi_0'$ and $\eta \equiv_1 \eta_0'$; then $\chi_0 \equiv_1 \chi_0'$ and $\eta_0 \equiv_1 \eta_0'$, and thus $\chi_0 \cdot \eta_0 \equiv_1 \chi_0' \cdot \eta_0'$; this means that $[\chi_0 \cdot \eta_0] = [\chi_0' \cdot \eta_0']$, as desired. Showing that the definition doesn't depend on the choice of χ, η is as simple: if $\chi' \equiv_1 \chi$ and $\eta' \equiv_1 \eta$, then $\chi' \equiv_1 \chi_0$ and $\eta' \equiv_1 \eta_0$, thus $[\chi'] \cdot [\eta'] = [\chi_0 \cdot \eta_0] = [\chi] \cdot [\eta]$, as desired.

It remains to show that these definitions obey the properties demanded by the definition of a 2-group. The proofs that $\mathcal{G}/\equiv$ obeys the properties dealing only with the monoidal product are straightforward and left as an exercise to the reader; we note only that the horizontal inverse of $[\chi]$ is $[\chi^{-h}]$. We now provide the proofs of the ones that deal with composition. We start with associativity: let $\chi : g \to h, \eta : h' \to k', \zeta : k'' \to l''$ be morphisms such that $[h] = [h']$ and $[k'] = [k'']$. Taking $\zeta_0 = \zeta, \eta_0 = \mathrm{Id}_{k'' \otimes k'^{-1}} \otimes \eta$ and $\chi_0 = \mathrm{Id}_{k'' \otimes k'^{-1} \otimes h' \otimes h^{-1}} \otimes \chi$, we have $\zeta_0 \equiv_1 \zeta, \eta_0 \equiv_1 \eta, \chi_0 \equiv_1 \chi$, thus

$$([\zeta]\cdot[\eta])\cdot[\chi] = [\zeta_0\cdot\eta_0]\cdot[\chi] = [(\zeta_0\cdot\eta_0)\cdot\chi_0] = [\zeta_0\cdot(\eta_0\cdot\chi_0)] = [\zeta_0]\cdot[\eta_0\cdot\chi_0] = [\zeta_0]\cdot([\eta_0]\cdot[\chi_0]) = [\zeta]\cdot([\eta]\cdot[\chi]),$$

as desired.

We now prove the identity properties: let $\chi : g \to h$; then $[\chi] : [g] \to [h]$, and

$$[\chi] \cdot \mathrm{Id}_{[g]} = [\chi] \cdot [\mathrm{Id}_g] = [\chi \cdot \mathrm{Id}_g] = [\chi] = [\mathrm{Id}_h \cdot \chi] = [\mathrm{Id}_h] \cdot [\chi] = \mathrm{Id}_{[h]} \cdot [\chi],$$

as desired.

We finish by proving the interchange law: let $[\chi_1] : [h'_1] \to [k_1], [\eta_1] : [g_1] \to [h_1], [\chi_2] :, [h'_2] \to [k_2], [\eta_2] : [g_2] \to [h_2]$, such that $[h_1] = [h'_1], [h_2] = [h'_2]$. Then there are $\chi'_1 \equiv_1 \chi_1, \chi'_2 \equiv_1 \chi_2, \eta'_1 \equiv_1 \eta_1, \eta'_2 \equiv_1 \eta_2$, such that $\chi'_1 \cdot \eta'_1$ and $\chi'_2 \cdot \eta'_2$ makes sense. Thus

$$\begin{aligned}([\chi_1] \cdot [\eta_1]) \otimes ([\chi_2] \cdot [\eta_2]) &= [\chi'_1 \cdot \eta'_1] \otimes [\chi'_2 \cdot \eta'_2] = [(\chi'_1 \cdot \eta'_1) \otimes (\chi'_2 \cdot \eta'_2)] \\ &= [(\chi'_1 \otimes \chi'_2) \cdot (\eta'_1 \otimes \eta'_2)] = ([\chi'_1] \otimes [\chi'_2]) \cdot ([\eta'_1] \otimes [\eta'_2]) \\ &= ([\chi_1] \otimes [\chi_2]) \cdot ([\eta_1] \otimes [\eta_2]),\end{aligned}$$

as desired. □

2-Group congruences are closely related to 2-group homomorphisms, as we now show.

Lemma 6.3. *Let $F : \mathcal{G} \to \mathcal{H}$ be a 2-group homomorphism. There is a congruence $\equiv_F$* induced by F*, defined on $\mathcal{G}$ as the pair $(\equiv_0, \equiv_1)$ by $g \equiv_0 h$ if $F(g) = F(h)$ and $\chi \equiv_1 \eta$ if $F(\chi) = F(\eta)$.*

Proof. It's obvious that $\equiv_0$ and $\equiv_1$ are equivalence relations.

If $g \equiv_0 g'$ and $h \equiv_0 h'$, then $F(g) = F(g')$ and $F(h) = F(h')$, thus $F(g \otimes h) = F(g) \otimes F(h) = F(g') \otimes F(h') = F(g' \otimes h')$, and so $g \otimes h \equiv_0 g' \otimes h'$.

If $\chi : g \to h, \chi' : g' \to h'$ are morphisms such that $\chi \equiv_1 \chi'$, then $F(\chi) : F(g) \to F(h)$ and $F(\chi') : F(g') \to F(h')$ are the same, so $F(g) = F(g')$ and $F(h) = F(h')$; that is, $g \equiv_0 g'$ and $h \equiv_0 h'$.

If $g \equiv_0 g'$, then $F(\mathrm{Id}_g) = \mathrm{Id}_{F(g)} = \mathrm{Id}_{F(g')} = F(\mathrm{Id}'_g)$, thus $\mathrm{Id}_g \equiv_1 \mathrm{Id}_{g'}$.

Given morphisms $\chi : g \to h, \chi' : g' \to h', \eta : k \to l, \eta' : k' \to l'$ such that $\chi \equiv_1 \chi'$ and $\eta \equiv_1 \eta'$, then $F(\chi \otimes \eta) = F(\chi) \otimes F(\eta) = F(\chi') \otimes F(\eta') = F(\chi' \otimes \eta')$, thus $\chi \otimes \eta \equiv_1 \chi' \otimes \eta'$. If, additionally, $g = l$ and $g' = l'$, then $F(\chi \cdot \eta) = F(\chi) \cdot F(\eta) = F(\chi') \cdot F(\eta') = F(\chi' \cdot \eta')$. This concludes our proof. □

Proposition 6.4. *Let $\mathcal{G}$ be a 2-group. Then the congruences on $\mathcal{G}$ are precisely the ones induced by 2-group homomorphisms going from $\mathcal{G}$.*

Proof. In light of the previous lemma, it only remains to be shown that if $\equiv$ is a congruence on $\mathcal{G}$, then there is a 2-group $\mathcal{H}$ and a 2-group homomorphism $F_\equiv : \mathcal{G} \to \mathcal{H}$ such that $\equiv$ is induced by $F_\equiv$. Indeed, if we take $\mathcal{H} = (\mathcal{G}/\equiv)$ and define $F_\equiv$ by $F_\equiv(g) = [g]$ on objects and $F_\equiv(\chi) = [\chi]$ on morphisms, then it is straightforward to prove that $F_\equiv$ is a 2-group homomorphism, and that $\equiv$ is induced by $F_\equiv$. □

6.2 Normal 2-subgroups and quotient 2-groups

Definition 6.5. *Let $\mathcal{H} \leq \mathcal{G}$ be 2-groups. We say that $\mathcal{H}$ is a* normal 2-subgroup *of $\mathcal{G}$ (denoted, as in group theory, $\mathcal{H} \lhd \mathcal{G}$) if $H_0 \lhd G_0$ and $H_1 \lhd G_1$.*

Remark 6.6. *Notice that a pair of subsets (H_0, H_1) with $H_0 \subseteq G_0, H_1 \subseteq G_1$ can only be a normal 2-subgroup if it is a 2-subgroup; in general, it is not enough that H_0, H_1 be normal subgroups of G_0, G_1, respectively; for example, take the pair (G_0, Id_1), for 2-groups $\mathcal{G}$ with nontrivial G_0.*

Lemma 6.7. *Let $\mathcal{H} \leq \mathcal{G}$ be 2-groups. There is a congruence $\equiv_\mathcal{H}$* induced by $\mathcal{H}$ on $\mathcal{G}$*, defined as the pair $(\equiv_0, \equiv_1)$, where $g \equiv_0 h$ if $g \otimes H_0 = h \otimes H_0$, and $\chi \equiv_1 \eta$ if $\chi \otimes H_1 = \eta \otimes H_1$, if and only if $\mathcal{H} \lhd \mathcal{G}$.*

Proof. Suppose that $\equiv_{\mathcal{H}}$ is a congruence. From the properties of $\equiv_0, \equiv_1$ under the monoidal product follows that $H_0 \triangleleft G_0$ and $H_1 \triangleleft G_1$, thus $\mathcal{H} \triangleleft \mathcal{G}$.

Suppose now that $\mathcal{H} \triangleleft \mathcal{G}$. It is well known that $\equiv_0, \equiv_1$ are equivalence relations, and obey the necessary properties on the monoidal product to be a congruence.

Let $\chi : g \to h, \chi' : g' \to h'$ be morphisms of $\mathcal{G}$. If $\chi \equiv_1 \chi'$, then $\chi' \otimes \chi^{-h} \in H_1$, and $\chi' \otimes \chi^{-h} : g' \otimes g^{-1} \to h' \otimes h^{-1}$, therefore $g' \otimes g^{-1}, h' \otimes h^{-1} \in H_0$, and thus $g \equiv_0 g', h \equiv_0 h'$.

If $g \equiv_0 g'$, then $g' \otimes g^{-1} \in H_0$, thus $\mathrm{Id}_g' \otimes \mathrm{Id}_g^{-h} = \mathrm{Id}_{g' \otimes g^{-1}} \in H_1$; therefore $\mathrm{Id}_g \equiv_1 \mathrm{Id}_{g'}$.

Finally, given $\chi : g \to h, \chi' : g' \to h', \eta : k \to g, \eta' : k' \to g'$ such that $\chi \equiv_1 \chi'$ and $\eta \equiv_1 \eta'$, we have

$$(\chi' \cdot \eta') \otimes (\chi \cdot \eta)^{-h} = (\chi' \cdot \eta') \otimes (\chi^{-h} \cdot \eta^{-h}) = (\chi' \otimes \chi^{-h}) \cdot (\eta' \otimes \eta^{-h}) \in H_1,$$

since $\chi' \otimes \chi^{-h}, \eta' \otimes \eta^{-h} \in H_1$, and H_1 is closed under vertical composition, thus $(\chi \cdot \eta) \equiv_1 (\chi' \cdot \eta')$. □

Definition 6.8. *Let $\mathcal{H} \triangleleft \mathcal{G}$ be 2-groups. We define the* 2-group quotient $\mathcal{G}/\mathcal{H}$ *as the 2-group $\mathcal{G}/\equiv_{\mathcal{H}}$. We denote $[g] = gH_0$ and $[\chi] = \chi H_1$ for $g \in G_0, \chi \in G_1$.*

Notice that $\mathrm{Obj}(\mathcal{G}/\mathcal{H}) = \mathrm{Obj}(\mathcal{G})/\mathrm{Obj}(\mathcal{H})$ and $\mathrm{Mor}(\mathcal{G}/\mathcal{H}) = \mathrm{Mor}(\mathcal{G})/\mathrm{Mor}(\mathcal{H})$.

Lemma 6.9. *Let $F : \mathcal{G} \to \mathcal{H}$ be a 2-group homomorphism. Then the congruence induced by F coincides with the congruence induced by* $\ker F$*; that is, $\equiv_F$ and $\equiv_{\ker F}$ are the same.*

Proof. Let $g, g' \in G_0$. Then

$$\begin{aligned} g \equiv_F g' &\Leftrightarrow F(g) = F(g') \Leftrightarrow F(g) \otimes F(g)^{-1} = F(g') \otimes F(g)^{-1} \\ &\Leftrightarrow 1 = F(g' \otimes g^{-1}) \Leftrightarrow g' \otimes g^{-1} \in \mathrm{Obj}(\ker F) \Leftrightarrow g \equiv_{\ker F} g'; \end{aligned}$$

thus $\equiv_F$ and $\equiv_{\ker F}$ coincide on objects.

The proof that $\equiv_F$ and $\equiv_{\ker F}$ coincide on morphisms is similar. □

Corollary 6.10. *Let $F : \mathcal{G} \to \mathcal{H}$ be a 2-group homomorphism. Then* $\ker F$ *is a normal 2-subgroup of $\mathcal{G}$.*

Proof. In light of the previous lemma, since $\equiv_{\ker F}$ coincides with $\equiv_F$, it follows that $\equiv_{\ker F}$ is a congruence; thus $\ker F \triangleleft \mathcal{G}$, by Lemma 6.7. □

Proposition 6.11. *Let $\mathcal{H} \leq \mathcal{G}$ be 2-groups. Then $\mathcal{H}$ is a normal 2-subgroup of $\mathcal{G}$ if and only if there exist a 2-group $\mathcal{K}$ and a 2-group homomorphism $F : \mathcal{G} \to \mathcal{K}$ such that* $\ker F = \mathcal{H}$.

Proof. In light of the previous corollary, we need only prove that normal 2-subgroups are kernels of 2-group homomorphisms. Indeed, let $\mathcal{K} = (\mathcal{G}/\mathcal{H})$ and $F : \mathcal{G} \to (\mathcal{G}/\mathcal{H})$ be the projection, defined as in Proposition 6.4. It is easy to see that $\ker F = \mathcal{H}$. □

We are now in conditions to reach our desired description of congruences.

Proposition 6.12. *Congruences are exactly the congruences induced by normal 2-subgroups.*

Proof. In light of Lemma 6.7, normal 2-subgroups induce congruences, so we now need to prove that, given a congruence $\equiv$ on a 2-group $\mathcal{G}$, there is a normal 2-subgroup $\mathcal{H} \triangleleft \mathcal{G}$ such that $\equiv$ coincides with $\equiv_{\mathcal{G}}$. From Proposition 6.4 follows that there is a 2-group homomorphism $F : \mathcal{G} \to \mathcal{K}$ such that $\equiv$ and $\equiv_F$ coincide; from the previous proposition follows that $\equiv_F$ and $\equiv_{\ker F}$ coincide; thus $\equiv$ and $\equiv_{\ker F}$ coincide; since $\ker F \triangleleft \mathcal{G}$, this proof is complete. □

6.3 Normal crossed submodules and crossed module quotients

Definition 6.13. *Let $\mathcal{G} = (G, E, \partial, \triangleright)$ be a crossed module, and $\mathcal{H} = (H, F)$ be a crossed submodule. We say that $\mathcal{H}$ is a* normal crossed submodule *of $\mathcal{G}$ if the following conditions are met: $H \triangleleft G$, G fixes F, and for all $\eta \in E, h \in H$ we have $(h \triangleright \eta)\eta^{-1} \in F$. This is denoted $\mathcal{H} \triangleleft \mathcal{G}$.*

Proposition 6.14. *Let $(H, F) = \mathcal{H} \triangleleft \mathcal{G}$ be crossed modules. Then the* quotient crossed module *$\mathcal{G}/\mathcal{H}$, defined by $(G/H, E/F, \partial', \triangleright')$, where $\partial', \triangleright'$ are defined by $\partial'(\eta F) = \partial'(\eta)H$ and $gH \triangleright \eta F = (g \triangleright \eta)F$ for all $gH \in G/H, \eta F \in E/F$, is indeed a crossed module.*

Furthermore, $\mathcal{H} \triangleleft \mathcal{G}$ if and only if $\xi(\mathcal{H}) \triangleleft \xi(\mathcal{G})$; in that case, we also have $\xi(\mathcal{G}/\mathcal{H}) \cong \xi(\mathcal{G})/\xi(\mathcal{H})$.

Proof. First notice that if $\mathcal{H} \triangleleft \mathcal{G}$, then $F \triangleleft E$: if $e \in E, f \in F$, then $efe^{-1} = \partial(e) \triangleright f \in F$, since G fixes F; thus G/H and E/F are actually groups. Now it is easy to check that the crossed module properties hold in $\mathcal{G}/\mathcal{H}$, since they hold in $\mathcal{G}$; i.e. it is easy to check that ∂' is a group homomorphism, $\triangleright'$ is a group action by isomorphisms, and that the Peiffer Laws hold; thus $\mathcal{G}/\mathcal{H}$ is a crossed module.

Let us prove that $\mathcal{H} \triangleleft \mathcal{G}$ if and only if $\xi(\mathcal{H}) \triangleleft \xi(\mathcal{G})$. Suppose first that $\mathcal{H} \triangleleft \mathcal{G}$. Since $\xi(\mathcal{H}) \leq \xi(\mathcal{G})$, we need only prove that the object and morphism groups of $\xi(\mathcal{H})$ are normal subgroups of the object and morphism groups of $\xi(\mathcal{G})$. We have $\mathrm{Obj}(\xi(\mathcal{H})) = H \triangleleft G = \mathrm{Obj}(\xi(\mathcal{G}))$. Given $(h, f) \in \mathrm{Mor}(\xi(\mathcal{H}))$ and $(g, e) \in \mathrm{Mor}(\xi(\mathcal{G}))$, we have

$$
\begin{aligned}
(g, e) \otimes (h, f) \otimes (g, e)^{-h} &= (gh, e(g \triangleright f)) \otimes (g^{-1}, g^{-1} \triangleright e^{-1}) \\
&= (ghg^{-1}, e(g \triangleright f)[gh \triangleright (g^{-1} \triangleright e^{-1})]) \\
&= (ghg^{-1}, e(g \triangleright f)e^{-1}[(e(ghg^{-1} \triangleright e)^{-1}]) \in \mathrm{Mor}(\xi(\mathcal{H})),
\end{aligned}
$$

since $ghg^{-1} \in H$ we have $e(ghg^{-1} \triangleright e)^{-1} \in F$; and also $e(g \triangleright f)e^{-1} \in F$. Thus $\xi(\mathcal{H}) \triangleleft \xi(\mathcal{G})$.

Suppose now that $\xi(\mathcal{H}) \triangleleft \xi(\mathcal{G})$. Similarly to the previous case, $H \triangleleft G$. If $g \in G$ and $f \in F$, then

$$
(g, 1) \otimes (1, f) \otimes (g, 1)^{-h} = (g, g \triangleright f) \otimes (g^{-1}, 1) = (1, g \triangleright f),
$$

thus $g \triangleright f \in F$; that is, G fixes F. Finally, if $h \in H$ and $e \in E$, then

$$
(1, e) \otimes (h, 1) \otimes (1, e)^{-1} = (h, e) \otimes (1, e^{-1}) = (h, e(h \triangleright e^{-1})) = (h, e(h \triangleright e)^{-1}),
$$

thus $(h \triangleright e)e^{-1} = (e(h \triangleright e)^{-1})^{-1} \in F$. This finishes the proof that $\mathcal{H} \triangleleft \mathcal{G}$.

Finally, we prove that $\xi(\mathcal{G}/\mathcal{H}) \cong \xi(\mathcal{G})/\xi(\mathcal{H})$. In $\xi(\mathcal{G})/\xi(\mathcal{H})$ denote the congruence classes of $g \in \mathrm{Obj}(\xi(\mathcal{G}))$ and $(g, e) \in \mathrm{Mor}(\xi(\mathcal{G}))$ by $[g], [(g, e)]$, respectively. We claim that $\phi : \xi(\mathcal{G}/\mathcal{H}) \to \xi(\mathcal{G})/\xi(\mathcal{H})$ defined as follows is a 2-group isomorphism: given $gH \in \mathrm{Obj}(\xi(\mathcal{G}/\mathcal{H}))$, define $\phi(gH) = [g]$; given $(gH, eF) \in \mathrm{Mor}(\xi(\mathcal{G}/\mathcal{H}))$, define $\phi(gH, eF) = [(g, e)]$. Let us check that F is well-defined: if $gH = g'H$, then $g'g^{-1} \in H = \mathrm{Obj}(\xi(\mathcal{H}))$, thus $[g'] = [g]$, and so $\phi(gH) = \phi(g'H)$; if $(gH, eF) = (g'H, e'F)$, then

$$
\begin{aligned}
(g', e') \otimes (g, e)^{-h} &= (g', e') \otimes (g^{-1}, g^{-1} \triangleright e^{-1}) = (g'g^{-1}, e'(g' \triangleright (g^{-1} \triangleright e^{-1})) = (g'g^{-1}, e'(g'g^{-1} \triangleright e)^{-1}) \\
&= (g'g^{-1}, e'(g'g^{-1} \triangleright e')^{-1}(g'g^{-1} \triangleright e'e^{-1})) \in H \ltimes F,
\end{aligned}
$$

since $g'g^{-1} \in H$ and $e'e^{-1} \in F$, therefore $e'(g'g^{-1} \triangleright e')^{-1}(g'g^{-1} \triangleright e'e^{-1}) \in F$; thus $[(g, e)] = [(g', e')]$, as needed.

It is now easy to check that ϕ is a 2-group homomorphism, and that it is bijective on objects and morphisms, and thus it is a 2-group isomorphism, as claimed. $\square$

6.4 Examples; Homotopy groups, the Postnikov decomposition and the Postnikov series

The details of the following examples are left to the reader.

Example 6.15. *If $H \leq G$ are groups and $F \leq E$ are abelian groups, then $(H[0] \times F[1]) \triangleleft (G[0] \times E[1])$ if and only if $H \triangleleft G$ and $F \triangleleft E$.*

Example 6.16. *If $H \leq G$ are groups, then $H[\mathrm{Ad}] \triangleleft G[\mathrm{Ad}]$ if and only if $H \triangleleft G$.*

The next example expands on Example 5.13.

Example 6.17. *Let $\mathcal{G}$ be a 2-group. If $H \triangleleft G_0$, then $\mathcal{G}|H \triangleleft \mathcal{G}$: if $\eta \in \mathrm{Mor}(\mathcal{G}|H)$ and $\chi \in \mathrm{Mor}(\mathcal{G})$, then $\eta : h \to h'$ and $\chi : g \to g'$, where $h, h' \in H$ and $g, g' \in G_0$, and $\chi \otimes \eta \otimes \chi^{-h} : g \otimes h \otimes g^{-1} \to g' \otimes h' \otimes g'^{-1}$; since $g \otimes h \otimes g^{-1}, g' \otimes h' \otimes g'^{-1} \in H$, we have $\chi \otimes \eta \otimes \chi^{-h} \in \mathrm{Mor}(\mathcal{G}|H)$; thus $\mathrm{Mor}(\mathcal{G}|H) \triangleleft G_1$, as needed.*

The previous example has two easy corollaries: given a 2-group $\mathcal{G}$ we have $\pi_1(\mathcal{G})[1] \triangleleft \mathcal{G}$, since $\pi_1(\mathcal{G}) = \mathcal{G}|1$ and $1 \triangleleft G_0$; also $\mathcal{O}(\mathcal{G}) \triangleleft \mathcal{G}$, since $\mathcal{O}(\mathcal{G}) = \mathcal{G}|t(\ker s)$ and $t(\ker s) \triangleleft G_0$, because $\ker s \triangleleft G_1$ and t is surjective.

Thus we have a "central series"

$$\mathrm{Id}_1 \triangleleft \pi_1(\mathcal{G})[1] \triangleleft \mathcal{O}(\mathcal{G}) \triangleleft \mathcal{G},$$

which we call the *Postnikov series*.

Let us calculate the "2-group factors" of this series. We claim that $\mathcal{G}/\mathcal{O}(\mathcal{G}) \cong \pi_0(\mathcal{G})[0]$ and that $\mathcal{O}(\mathcal{G})/\pi_1(\mathcal{G})[1] \cong P(\mathcal{G})$, where $P(\mathcal{G}) = t(\ker s)$.

Since every morphism in $\mathcal{G}$ with source 1 is in $\mathcal{O}(\mathcal{G})$, it is clear that $\mathcal{G}/\mathcal{O}(\mathcal{G})$ must be a 2-group whose morphisms are all identities. By definition, its object group is $\pi_0(\mathcal{G})$, thus $\mathcal{G}/\mathcal{O}(\mathcal{G}) \cong \pi_0(\mathcal{G})[0]$.

Since $\mathcal{O}(\mathcal{G})$ is connected, so must be $\mathcal{O}(\mathcal{G})/\pi_1(\mathcal{G})[1]$. Notice that the congruence induced by $\pi_1(\mathcal{G})[1]$ on objects reduces to the identity. If $\chi : g \to h$ and $\chi' : g \to h$ are morphisms in $\mathcal{O}(\mathcal{G})$, then $\chi' \otimes \chi^{-h} : 1 \to 1$ is a morphism in $\pi_1(\mathcal{G})[1]$, thus χ and χ' are congruent morphisms. This proves that there is exactly one morphism in every hom-set of $\mathcal{O}(\mathcal{G})/\pi_1(\mathcal{G})[1]$. Since the object group of $\mathcal{O}(\mathcal{G})/\pi_1(\mathcal{G})[1]$ is $P(\mathcal{G})$, it is thus proven that $\mathcal{O}(\mathcal{G})/\pi_1(\mathcal{G})[1] \cong P(\mathcal{G})[\mathrm{Ad}]$.

We call $(\pi_1(\mathcal{G})[1], P(\mathcal{G})[\mathrm{Ad}], \pi_0(\mathcal{G})[0])$ the *Postnikov factor 2-groups* of $\mathcal{G}$, and $(\pi_1(\mathcal{G}), P(\mathcal{G}), \pi_0(\mathcal{G}))$ the *Postnikov factor groups* of $\mathcal{G}$. Notice how naturally the homotopy groups come up as Postnikov factor groups.

Given a 2-group $\mathcal{G}$, its *Postnikov decomposition* is the following sequence of 2-group homomorphisms:

$$\mathrm{Id}_1 \to \pi_1(\mathcal{G})[1] \xrightarrow{I} \mathcal{G} \xrightarrow{P} \pi_0(\mathcal{G})[0] \to \mathrm{Id}_1,$$

where I is the inclusion and P is the canonical projection of $\mathcal{G}$ onto $\mathcal{G}/\mathcal{O}(\mathcal{G})$. Elgueta calls this sequence "2-exact" and provides an explanation in [4]; in section 7.1 we will propose a precise definition of 2-exact sequences.

Both the Postnikov decomposition and Postnikov series seem to be enlightening when looking at a particular 2-group, as the following examples show.

Example 6.18. *It is easy to check that given a group G, the Postnikov factors of $\mathrm{Aut}(G)$ are $(Z(G), \mathrm{Inn}(G), \mathrm{Out}(G))$. The Postnikov decomposition of $\mathrm{Aut}(G)$ is*

$$\mathrm{Id}_1 \to Z(G)[1] \xrightarrow{i} \mathrm{Aut}(G) \xrightarrow{\pi} \mathrm{Out}(G)[0] \to \mathrm{Id}_1.$$

In fact, this shows how one can get information about the group $\mathrm{Aut}(G)$ *by analyzing the 2-group* $\mathrm{Aut}(G)$. *It is only natural that to better understand the 2-group* $\mathrm{Aut}(\mathcal{G})$ *one should analyze the 3-group* $\mathrm{Aut}(\mathcal{G})$*; this idea will be explored in section 8.*

Example 6.19. *Given a topological group G, the Postnikov factors of* $G[\mathrm{Top}]$ *are* $(\pi_1(G), \Gamma_1, G/\Gamma_1)$, *where* $\pi_1(G)$ *is the fundamental group of G and Γ_1 is the connected component of* 1. *The Postnikov decomposition of* $G[\mathrm{Top}]$ *is*

$$\mathrm{Id}_1 \to \pi_1(G)[1] \xrightarrow{i} G[\mathrm{Top}] \xrightarrow{\pi} (G/\Gamma_1)[0].$$

In particular, the Postnikov factors of $O(2)[\mathrm{Top}]$ *are isomorphic to* $(\mathbb{Z}, SO(2), \mathbb{Z}_2)$, *and its Postnikov decomposition is* $\mathrm{Id}_1 \to \mathbb{Z}[1] \xrightarrow{i} O(2)[\mathrm{Top}] \xrightarrow{\pi} \mathbb{Z}_2[0] \to \mathrm{Id}_1$.

We finish this section by defining and classifying simple 2-groups. This is done as a mere curiosity, since simple 2-groups do not seem to be as important for studying 2-groups as simple groups are for studying groups.

Definition 6.20. *A 2-group $\mathcal{G}$ is* simple *if its only normal 2-subgroups are* Id_1 *and* $\mathcal{G}$.

Theorem 6.21. *A 2-group $\mathcal{G}$ is simple if and only if it is isomorphic to one of the following:*

1. $G[0]$, *where G is a simple group;*

2. $\mathbb{Z}_p[1]$, *where p is a prime number;*

3. $G[\mathrm{Ad}]$, *where G is a simple nonabelian group.*

Proof. We begin by proving that any simple 2-group must be one of the above. Let $\mathcal{G}$ be a simple 2-group. Since $\pi_1(\mathcal{G})[1] \lhd \mathcal{G}$, either $\pi_1(\mathcal{G})[1] = \mathrm{Id}_1$ or $\pi_1(\mathcal{G})[1] = \mathcal{G}$.

If $\pi_1(\mathcal{G})[1] = \mathcal{G}$, then $\mathcal{G} \cong G_1[1]$, thus G_1 is abelian; furthermore, from example 6.15 follows that G_1 is simple. Since G_1 is a simple abelian group, it is isomorphic to $\mathbb{Z}_p$ for some prime number p, and thus $\mathcal{G} \cong \mathbb{Z}_p[1]$.

If $\pi_1(\mathcal{G})[1] = \mathrm{Id}_1$, then $\mathcal{O}(\mathcal{G})$ is either Id_1 or $\mathcal{G}$. If $\mathcal{O}(\mathcal{G}) = \mathrm{Id}_1$, then $\mathcal{G} = G_0[0]$; from example 6.15 follows that G_0 is simple. If $\mathcal{O}(\mathcal{G}) = \mathcal{G}$, then $\mathcal{G} = G_0[\mathrm{Ad}]$; from example 6.16 follows that G_0 is simple. If G_0 is abelian, taking $\mathcal{H} = \mathrm{Id}(\mathbb{Z}_p)$, it is easy to check that $\mathcal{H} \lhd \mathbb{Z}_p[\mathrm{Ad}]$, therefore $\mathcal{G}$ is not simple. This concludes the first part of our proof.

We now prove that the given 2-groups are simple. That $G[0]$ and $\mathbb{Z}_p[1]$ are simple for G simple or p prime follows from examples 6.15 and 6.16. To see that $G[\mathrm{Ad}]$ is simple whenever G is simple and nonabelian, we will prove that $(G, G, \mathrm{Id}, \mathrm{Ad})$ is simple; this suffices, since $(G, G, \mathrm{Id}, \mathrm{Ad}) \cong \gamma(G[\mathrm{Ad}])$. Indeed, a normal 2-subgroup of $(G, G, \mathrm{Id}, \mathrm{Ad})$ would be a pair (H, K) such that $H, K \lhd G$; if $(H, K) \neq (1, 1), (G, G)$, then either $(H, K) = (G, 1)$ or $(H, K) = (1, G)$. The latter case is impossible, since $\partial(G) = G \not\leq 1$, thus $(1, G)$ is not even a 2-subgroup; the former case is also impossible: we need that $[h, g] = (h\mathrm{Ad}g)g^{-1} \in 1$; this is impossible, since G is nonabelian. This finishes our proof. $\square$

6.5 The normal closure of a 2-group

Definition 6.22. *Given 2-groups $\mathcal{H} \leq \mathcal{G}$, the* normal closure $\mathcal{H}^{\mathcal{G}}$ *of $\mathcal{H}$ is the "smallest" normal 2-subgroup $\mathcal{K} \lhd \mathcal{G}$ such that $\mathcal{H} \leq \mathcal{K}$; i.e. if $\mathcal{K}'$ is any other normal 2-subgroup of $\mathcal{G}$ is such that $\mathcal{H} \leq \mathcal{K}$ then $\mathcal{K} \leq \mathcal{K}'$.*

The following propositions are easy to check.

Proposition 6.23. *Given 2-groups $\mathcal{H} \leq \mathcal{G}$, the normal closure $\mathcal{H}^{\mathcal{G}}$ exists; furthermore,*

$$\mathcal{H}^{\mathcal{G}} = (H_0^{G_0}, H_1^{G_1}).$$

Proposition 6.24. *Let S be a set of normal 2-subgroups of a given 2-group $\mathcal{G}$. Then*

$$\bigcap_{\mathcal{H} \in S} \mathcal{H}$$

is a normal 2-subgroup of $\mathcal{G}$.

Notice that given a set of morphisms $X \subseteq G_1$ there is also a "smallest" normal 2-subgroup of $\mathcal{G}$ containing X, which is $\langle X \rangle^{\mathcal{G}}$.

6.6 The isomorphism theorems

In this section we provide analogues to the isomorphism theorems in group theory and their proofs. There is nothing too surprising in this section. Analogues of statements such as the Zassenhaus lemma or the Jordan-Hölder theorem should be easy to state and prove as well, but we do not go over them since we do not feel they are as important for 2-groups as they are for groups.

We start with the first isomorphism theorem.

Theorem 6.25. *Let $F : \mathcal{G} \to \mathcal{H}$ be a 2-group homomorphism. Then*

$$\mathcal{G}/\ker F \cong \mathrm{im}\ f.$$

Proof. Define $\phi : \mathcal{G}/\ker F \to \mathrm{im}\ F$ by $\phi([g]) = F(g)$ and $\phi([\chi]) = F(\chi)$ for $g \in G_0$ and $\chi \in G_1$. It's easy to see that F is a well defined 2-group isomorphism, as needed. □

Before we can state and prove the second isomorphism theorem, we will need to introduce the internal product of two 2-subgroups of a given 2-group.

Definition 6.26. *Given 2-subgroups $\mathcal{H}, \mathcal{K} \leq \mathcal{G}$, define $\mathcal{HK} = (H_0K_0, H_1K_1)$.*

Remark 6.27. *Notice that, in general, $\mathcal{HK}$ doesn't have to be a 2-subgroup of $\mathcal{G}$.*

Proposition 6.28. *Given 2-subgroups $\mathcal{H}, \mathcal{K} \leq \mathcal{G}$ such that $\mathcal{H} \triangleleft \mathcal{G}$, we have $\mathcal{HK} \leq \mathcal{G}$. Furthermore, if $\mathcal{K} \triangleleft \mathcal{G}$, then $\mathcal{HK} \triangleleft \mathcal{G}$.*

Proof. We have $\mathcal{HK} = (H_0K_0, H_1K_1)$. It is known from basic group theory that $H_1K_1 \leq G_1$, since $H_1 \triangleleft G_1$ and $K_1 \leq G_1$. Also, since $s(H_1) \leq H_0$ and $s(K_1) \leq K_0$ we have $s(H_1K_1) = s(H_1)s(K_1) \leq H_0K_0$; similarly $t(H_1K_1) = t(H_1)t(K_1)$. Finally, since $\mathrm{Id}(H_0) \leq H_1$ and $\mathrm{Id}(K_0) \leq K_1$ we have $\mathrm{Id}(H_0K_0) = \mathrm{Id}(H_0)\mathrm{Id}(K_0) \leq H_1K_1$. Thus $\mathcal{HK} \leq \mathcal{G}$.

If, additionally, $\mathcal{K} \triangleleft \mathcal{G}$, then $H_0, K_0 \triangleleft G_0$ and $H_1, K_1 \triangleleft G_1$; from group theory follows that $H_0K_0 \triangleleft G_0$ and $H_1K_1 \triangleleft G_1$, thus $\mathcal{HK} \triangleleft \mathcal{G}$. □

We are now in conditions to deal with the second isomorphism theorem.

Theorem 6.29. *Let $\mathcal{H}, \mathcal{K} \leq \mathcal{G}$ be 2-subgroups such that $\mathcal{H} \triangleleft \mathcal{G}$. Then $(\mathcal{H} \cap \mathcal{K}) \triangleleft \mathcal{K}$ and*

$$\mathcal{HK}/\mathcal{H} \cong \mathcal{K}/(\mathcal{H} \cap \mathcal{K}).$$

Proof. Define the 2-group homomorphism $F : \mathcal{K} \to (\mathcal{H}\mathcal{K}/\mathcal{H})$ by $F(k) = [k]$ and $F(\chi) = [\chi]$, where $[k]$ and $[\chi]$ denote the congruence classes of k and χ under $\equiv_{\mathcal{H}}$. It is easy to check that F is a 2-group homomorphism, with kernel $\mathcal{H} \cap \mathcal{K}$ and image $\mathcal{H}\mathcal{K}/\mathcal{H}$; by the first isomorphism theorem, we have the desired result. □

We finish the "classic" isomorphism theorems with the third one.

Theorem 6.30. *Let $\mathcal{H}, \mathcal{K} \lhd \mathcal{G}$. Then $\mathcal{K}/\mathcal{H} \lhd \mathcal{G}/\mathcal{H}$ and*

$$(\mathcal{G}/\mathcal{H})/(\mathcal{K}/\mathcal{H}) \cong \mathcal{G}/\mathcal{K}.$$

Proof. Define the 2-group homomorphism $F : \mathcal{G}/\mathcal{H} \to \mathcal{G}/\mathcal{K}$ by $F(gH_0) = gK_0$ and $F(\chi H_1) = \chi K_1$ for all $gH_0 \in G_0/H_0, \chi H_1 \in G_1/H_1$. It is easy to check that F is a well defined 2-group homomorphism with kernel $\mathcal{K}/\mathcal{H}$ and image $\mathcal{G}/\mathcal{K}$; by the first isomorphism theorem, we have the desired result. □

We finish with the correspondence theorem.

Theorem 6.31. *Let $\mathcal{H} \lhd \mathcal{G}$ be 2-groups. There is a bijection between 2-subgroups $\mathcal{K}$ such that $\mathcal{H} \leq \mathcal{K} \leq \mathcal{G}$ and 2-subgroups $\mathcal{K}^*$ of $\mathcal{G}/\mathcal{H}$, given by $\mathcal{K} \mapsto \mathcal{K}/\mathcal{H}$. Furthermore, $\mathcal{K}_1 \leq \mathcal{K}_2 \leq \mathcal{G}$ if and only if $\mathcal{K}_1^* \leq \mathcal{K}_2^*$, and $\mathcal{K}_1 \lhd \mathcal{K}_2$ if and only if $\mathcal{K}_1^* \lhd \mathcal{K}_2^*$, in which case $\mathcal{K}_2/\mathcal{K}_1 \cong \mathcal{K}_2^*/\mathcal{K}_1^*$.*

Proof. Denote the set of 2-subgroups $\mathcal{K}$ such that $\mathcal{H} \leq \mathcal{K} \leq \mathcal{G}$ by $\mathrm{Sub}(G, H)$, and the set of 2-subgroups of $\mathcal{G}/\mathcal{H}$ by $\mathrm{Sub}(\mathcal{G}/\mathcal{H})$.

Let $P : \mathcal{G} \to \mathcal{G}/\mathcal{H}$ be the canonical projection. Define $f : \mathrm{Sub}(\mathcal{G}, \mathcal{H}) \to \mathrm{Sub}(\mathcal{G}/\mathcal{H})$ by $f : \mathcal{K} \mapsto \mathcal{K}/\mathcal{H}$, and $g : \mathrm{Sub}(\mathcal{G}/\mathcal{H}) \to \mathrm{Sub}(\mathcal{G}, \mathcal{H})$ by $g : \mathcal{K}* \mapsto P^{-1}(\mathcal{K}*)$. It is easy to check that f and g are inverse functions, thus bijections.

The second part of the statement is easy to check.

For the final part of the statement, it is enough to apply the third isomorphism theorem of groups to the object and morphism groups of each group involved: notice that $\mathrm{Obj}(\mathcal{K}/\mathcal{H}) = \mathrm{Obj}(\mathcal{K})/\mathrm{Obj}(\mathcal{H})$ and $\mathrm{Mor}(\mathcal{K}/\mathcal{H}) = \mathrm{Mor}(\mathcal{K})/\mathrm{Mor}(\mathcal{H})$. □

7 Equivalence of 2-groups: strict homotopy invariants and homotopically minimal 2-subgroups

In this section we analyze the problem of classifying 2-groups up to equivalence.

In section 7.1 we propose a definition of 2-exact sequences and give an interpretation of the Postnikov invariant as an equivalence class of 2-exact sequences. In section 7.2 we analyze homotopy invariants and how they are insufficient for strict 2-groups, and propose a *strict Postnikov invariant*, which together with the homotopy module is a complete invariant. We also provide some partial invariants related to splitness of some sequences. In section 7.3 we introduce another complete invariant: the homotopically minimal 2-subgroup. We compute the homotopically minimal 2-subgroup of $\mathrm{Aut}(D_{2n})$.

Keep in mind that both complete invariants are much more unsatisfying and much less useful than the homotopy invariants for weak 2-groups. However, it does not seem that better invariants can be found for strict 2-groups.

7.1 2-Exact sequences and the Postnikov invariant

We begin by proposing a definition of 2-exact sequence.

Definition 7.1. *A* 2-exact sequence *of 2-groups is a sequence*

$$\mathcal{G}_0 \xrightarrow{F_1} \mathcal{G}_1 \xrightarrow{F_2} \cdots \xrightarrow{F_n} \mathcal{G}_n$$

of 2-groups and 2-group homomorphisms such that:

- $F_{k+1}F_k$ *is the trivial homomorphism for* $k = 1, \ldots, n-1$*;*
- $\mathrm{im}\ F_k \triangleleft \ker F_{k+1}$ *and* $\ker F_{k+1}/\mathrm{im}\ F_k \simeq \mathrm{Id}_1$ *for* $k = 1, \ldots, n-1$.

Notice that, as promised, the Postnikov decomposition is a 2-exact sequence.

Another possible (stronger) definition could be one demanding that $F_{k+1}F_k$ be the trivial homomorphism and that the inclusion $\mathrm{im}\ F_k \to \ker F_{k+1}$ be an equivalence of 2-groups. The two definitions aren't the same: if $\mathcal{H} \triangleleft \mathcal{G}$ are two 2-groups such that $\mathcal{H} \not\simeq \mathcal{G}$ but $\mathcal{G}/\mathcal{H} \simeq \mathrm{Id}_1$, then the sequence $\mathrm{Id}_1 \to \mathcal{G} \xrightarrow{\Pi} \mathcal{G}/\mathcal{H}$ is 2-exact in the former definition, but not in the latter.

Proposition 7.2. *Let A, G be groups such that A is abelian. If the sequence*

$$\mathrm{Id}_1 \to A[1] \xrightarrow{J} \mathcal{G} \xrightarrow{P} G[0] \to \mathrm{Id}_1$$

is 2-exact, then J is injective, P is surjective, there is an induced action of G on A given by

$$P(g) \triangleright a = J^{-1}(\mathrm{Id}_g \otimes J(a) \otimes \mathrm{Id}_{g^{-1}}),$$

and there is a module isomorphism between (G, A) and $(\pi_0(\mathcal{G}), \pi_1(\mathcal{G}))$.

We now give another description of the Postnikov invariant; it is easy to see that it is isomorphic to the group $\mathcal{E}^3(G, A)$ seen in section 3.3; also, it resembles a 2-group version of $\mathcal{E}^2(G, A)$.

Definition 7.3. *Given a module $(G, A, \triangleright)$, consider all 2-exact sequences*

$$\mathrm{Id}_1 \to A[1] \xrightarrow{J} \mathcal{G} \xrightarrow{P} G[0] \to \mathrm{Id}_1,$$

such that the induced action of G on A is the given one. We define an equivalence relation between such sequences, which is the smallest equivalence relation such that

$$\mathrm{Id}_1 \to A[1] \xrightarrow{J} \mathcal{G} \xrightarrow{P} G[0] \to \mathrm{Id}_1$$

and

$$\mathrm{Id}_1 \to A[1] \xrightarrow{J'} \mathcal{G}' \xrightarrow{P'} G[0] \to \mathrm{Id}_1$$

are equivalent *whenever there is a 2-group homomorphism $F : \mathcal{G} \to \mathcal{G}'$ such that the diagram*

$$\begin{tikzcd}
 & & \mathcal{G} \arrow[dd, "F"] \arrow[dr, "P"] & & \\
\mathrm{Id}_1 \arrow[r] & A[1] \arrow[ur, "J"] \arrow[dr, "J'"'] & & G[0] \arrow[r] & \mathrm{Id}_1 \\
 & & \mathcal{G}' \arrow[ur, "P'"'] & &
\end{tikzcd}$$

commutes (notice that F doesn't need to be an isomorphism). We denote the set of such equivalence classes by $\mathcal{E}^2(G[0], A[1])$.

We give $\mathcal{E}^2(G[0], A[1])$ a product*, as follows: the product of two equivalence classes of sequences above is the equivalence class of the sequence*

$$\mathrm{Id}_1 \to A[1] \stackrel{J\times J'}{\to} (\mathcal{G} \times_{G[0]} \mathcal{G}')/\mathcal{K} \stackrel{\Pi}{\to} G[0] \to \mathrm{Id}_1,$$

where $\mathcal{G} \times_{G[0]} \mathcal{G}'$ $\mathcal{K} = (1, \{(\alpha, \alpha^{-h}) : \alpha \in \mathrm{Mor}(J(A))\})$ and $J \times J', \Pi$ are defined in the obvious way.

Thus the Postnikov invariant can be interpreted as the class of the Postnikov decomposition

$$\mathrm{Id}_1 \to \pi_1(\mathcal{G})[1] \stackrel{J}{\to} \mathcal{G} \stackrel{\pi}{\to} \pi_0(\mathcal{G})[0] \to \mathrm{Id}_1$$

in $\mathcal{E}^2(\pi_0(\mathcal{G})[0], \pi_1(\mathcal{G})[1])$, in addition to the interpretation as a cohomology class in $H^3(\pi_0(\mathcal{G}), \pi_1(\mathcal{G}))$ and a equivalence class of a sequence in $\mathcal{E}^3(\pi_0(\mathcal{G}), \pi_1(\mathcal{G}))$.

7.2 The strict Postnikov invariant; splitness criteria

In this section we propose a new homotopy invariant for strict 2-groups: the *strict Postnikov invariant $S\alpha(\mathcal{G})$*. Together with the homotopy module $\pi(\mathcal{G})$, these two form a complete invariant for finite 2-groups.

Definition 7.4. *Given groups G, A with A an abelian G-module, consider all 2-exact sequences*

$$\mathrm{Id}_1 \to A[1] \stackrel{J}{\to} \mathcal{G} \stackrel{P}{\to} G[0] \to \mathrm{Id}_1,$$

such that the induced action of G on A is the given one. We define an equivalence relation between such sequences as follows: the sequences

$$\mathrm{Id}_1 \to A[1] \stackrel{J}{\to} \mathcal{G} \stackrel{P}{\to} G[0] \to \mathrm{Id}_1$$

and

$$\mathrm{Id}_1 \to A[1] \stackrel{J'}{\to} \mathcal{G}' \stackrel{P'}{\to} G[0] \to \mathrm{Id}_1,$$

are strictly equivalent *if there are 2-group homomorphisms $F : \mathcal{G} \to \mathcal{G}', F' : \mathcal{G}' \to \mathcal{G}$ such that the diagram*

$$\begin{tikzcd}
 & & \mathcal{G} \arrow[dr, "P"] \arrow[dd, bend left, "F"] & & \\
\mathrm{Id}_1 \arrow[r] & A[1] \arrow[ur, "J"] \arrow[dr, "J'"'] & & G[0] \arrow[r] & \mathrm{Id}_1 \\
 & & \mathcal{G}' \arrow[ur, "P'"'] \arrow[uu, bend left, "F'"] & &
\end{tikzcd}$$

commutes. We denote the set of such equivalence classes by $S\mathcal{E}^2(G[0], A[1])$.

Given a 2-group $\mathcal{G}$, its strict Postnikov invariant *is the equivalence class $S\alpha(\mathcal{G})$ in $S\mathcal{E}^2(\pi_0(\mathcal{G})[0], \pi_1(\mathcal{G})[1])$ of the Postnikov decomposition of $\mathcal{G}$.*

Remark 7.5. *An important feature of the usual Postnikov invariant $\alpha(\mathcal{G})$ is that it can be seen as a 3-cocycle in $H^3(\pi_0(\mathcal{G}),\pi_1(\mathcal{G}))$. As described in section 3.3, it can also be seen as a equivalence class of six-term exact sequence of groups. Under the equivalence relation between such exact sequences considered, two sequences give rise to the same 3-cocycle if and only if they are equivalent; of course this does not hold true for strict equivalence. Furthermore, the product of sequences described in section 3.3 does not make $S\mathcal{E}^2(G[0],A[1])$ into a group, but only a monoid, so there should be no way to make $S\alpha(\mathcal{G})$ into a 3-cocycle or anything similar.*

Lemma 7.6. *Let $\mathcal{G}$ be a finite 2-group and $F:\mathcal{G}\to\mathcal{G}$ be a 2-group homomorphism such that $\pi_0(F)=\mathrm{Id}_{\pi_0(\mathcal{G})}$ and $\pi_1(F)=\mathrm{Id}_{\pi_1(\mathcal{G})}$. Then there is a positive integer n such that F^n is isomorphic to $\mathrm{Id}_{\mathcal{G}}$.*

Proof. Since $\mathcal{G}$ is finite, there is m such that $F^m(\mathcal{G})=F^{m+1}(\mathcal{G})$; denote $F^m(\mathcal{G})=\mathcal{H}$; since $F^m(\mathcal{H})=\mathcal{H}$ and $\mathcal{H}$ is finite, the restriction of F^m to $\mathcal{H}$ is an automorphism; as such, there is a positive integer k such that the restriction of F^{mk} to $\mathcal{H}$ is the identity; let $n=mk$ and $G=F^n$. Define $\tau:\mathrm{Id}_{\mathcal{G}}\to G$ as follows: given $g\in G_0$ and $\chi:h\to g$ such that $h\in H_0$ (which exists, since $\pi_0(F)=\mathrm{Id}_{\pi_0(\mathcal{G})}$), the correspondence is given by $\tau(g)=G(\chi)\cdot\chi^{-v}$. Let us check that τ is indeed a 2-homomorphism.

Firstly, lets check that τ is well-defined. Let $\chi:h\to g$ and $\chi':h'\to g$ be morphisms with $h,h'\in H_0$. Then

$$(G(\chi')\cdot\chi'^{-v})\otimes(G(\chi)\cdot\chi^{-v})^{-h}=G(\chi'\otimes\chi^{-h})\cdot(\chi'\otimes\chi^{-h})^{-v}=\mathrm{Id}_1,$$

since $\chi'\otimes\chi^{-h}:h'\otimes h^{-1}\to 1$ is a morphism in $\mathcal{H}$.

Now, we prove that τ is a natural transformation. Obviously $\tau(g):g\to G(g)$ for all $g\in G_0$. If $\chi:g\to g'$ is a morphism in G_1, then there are $h\in H_0$ and $\eta:h\to g$; notice that $\chi\cdot\eta:h\to g'$. By definition, $\tau(g)=G(\eta)\cdot\eta^{-v}$ and $\tau(g')=G(\chi\cdot\eta)\cdot(\chi\cdot\eta)^{-v}$, thus

$$G(\chi)\cdot\tau(g)=G(\chi)\cdot(G(\eta)\cdot\eta^{-v})=(G(\chi\cdot\eta)\cdot(\chi\cdot\eta)^{-v})\cdot\chi=\tau(g')\cdot\chi.$$

Finally, we prove that τ is multiplicative: if $g,g'\in G_0$, then there are $h,h'\in H_0$ and morphisms $\chi:h\to g$ and $\chi':h'\to g'$; since $\chi\otimes\chi':h\otimes h'\to g\otimes g'$, we have

$$\tau(g\otimes h)=G(\chi\otimes\chi')\cdot(\chi\otimes\chi')^{-v}=(G(\chi)\cdot\chi^{-v})\otimes(G(\chi')\cdot\chi'^{-v})=\tau(g)\otimes\tau(h).\quad\square$$

Corollary 7.7. *Given finite 2-groups $\mathcal{G},\mathcal{H}$ with $(\pi_0(\mathcal{G}),\pi_1(\mathcal{G}))$ and $(\pi_0(\mathcal{H}),\pi_1(\mathcal{H}))$ as isomorphic modules and isomorphism (σ_0,σ_1), there is a bijection $S\mathcal{E}^2(\pi_0(\mathcal{G})[0],\pi_1(\mathcal{G})[1])\to S\mathcal{E}^2(\pi_0(\mathcal{G})[0],\pi_1(\mathcal{G})[1])$ defined as follows: the strict equivalence class of a sequence*

$$\mathrm{Id}_1\to\pi_1(\mathcal{G})[1]\xrightarrow{J}\mathcal{K}\xrightarrow{P}\pi_0(\mathcal{G})[0]\to\mathrm{Id}_1$$

is mapped to the strict equivalence class of the sequence

$$\mathrm{Id}_1\to\pi_1(\mathcal{H})[1]\xrightarrow{J\sigma_1^{-1}[1]}\mathcal{K}\xrightarrow{\sigma_0[0]P}\pi_0(\mathcal{H})[0]\to\mathrm{Id}_1.$$

If $S\alpha(\mathcal{G})$ is mapped to $S\alpha(\mathcal{H})$, then $\mathcal{G}\simeq\mathcal{H}$.

Proof. Since $S\alpha(\mathcal{G})$ is mapped to $S\alpha(\mathcal{H})$, there are morphisms $F:\mathcal{G}\to\mathcal{H}$ and $G:\mathcal{H}\to\mathcal{G}$ such that the diagram

$$\begin{tikzcd}
 & \pi_1(\mathcal{G})[1] \arrow[r, "J"] \arrow[dd, bend left, "\sigma_1"] & \mathcal{G} \arrow[r, "P"] \arrow[dd, bend left, "F"] & \pi_0(\mathcal{G})[0] \arrow[dr] \arrow[dd, bend left, "\sigma_0"] & \\
\mathrm{Id}_1 \arrow[ur] \arrow[dr] & & & & \mathrm{Id}_1 \\
 & \pi_1(\mathcal{H})[1] \arrow[r, "J'"] \arrow[uu, bend left, "\sigma_1^{-1}"] & \mathcal{H} \arrow[r, "P'"] \arrow[uu, bend left, "G"] & \pi_0(\mathcal{H})[0] \arrow[ur] \arrow[uu, bend left, "\sigma_0^{-1}"] &
\end{tikzcd}$$

commutes. Both GF and FG obey the hypothesis of the previous lemma, so there are integers k, m such that $(GF)^k \cong \mathrm{Id}_{\mathcal{G}}$ and $(FG)^m \cong \mathrm{Id}_{\mathcal{H}}$. Let $n = mk$ and $F' = F(GF)^{n-1}$. Then $GF' = (GF)^n \cong \mathrm{Id}_{\mathcal{G}}$ and $F'G = (FG)^n \cong \mathrm{Id}_{\mathcal{H}}$. Thus F', G give rise to the equivalence $\mathcal{G} \simeq \mathcal{H}$. □

Theorem 7.8. *The homotopy module (up to isomorphism) and the strict Postnikov invariant together are a complete invariant for finite 2-groups.*

Proof. From the previous corollary, two finite 2-groups with the same homotopy module and the same strict Postnikov invariant are equivalent. We know that equivalent 2-groups have isomorphic homotopy modules from section 3.3, and it is easy to see that they have the same strict Postnikov invariant. □

7.3 Splitness criteria

We start by giving splitness criteria for the equivalence of 2-groups.

Definition 7.9. *Let $\mathcal{G}$ be a 2-group. The* quotient sequence *of $\mathcal{G}$ is the short exact sequence*

$$1 \to t(\ker s) \xrightarrow{i} G_0 \xrightarrow{\pi} \pi_0(\mathcal{G}) \to 1,$$

where i is the inclusion and π is the canonical projection; this is denoted $\chi(\mathcal{G})$. The inclusion sequence *of $\mathcal{G}$ is the short exact sequence*

$$1 \to \pi_1(\mathcal{G}) \xrightarrow{i} \ker s \xrightarrow{t} t(\ker s) \to 1,$$

where i is the inclusion; this is denoted $\iota(\mathcal{G})$

Remark 7.10. *If $\gamma(\mathcal{G}) = (G, E, \partial, \triangleright)$, then $\chi(\mathcal{G})$ and $\iota(\mathcal{G})$ are the sequences*

$$1 \to \mathrm{im}\ \partial \xrightarrow{i} G \xrightarrow{\pi} G/\mathrm{im}\ \partial \to 1$$

and

$$1 \to \ker \partial \xrightarrow{i} E \xrightarrow{\partial} \mathrm{im}\ \partial \to 1,$$

respectively.

Proposition 7.11. *Let $\mathcal{G}, \mathcal{H}$ be 2-groups. If $\chi(\mathcal{G})$ splits but $\chi(\mathcal{H})$ doesn't, then $\mathcal{G} \not\simeq \mathcal{H}$.*

Proof. Suppose that $F : \mathcal{G} \to \mathcal{H}, G : \mathcal{H} \to \mathcal{G}$ are 2-group homomorphisms which form an equivalence, and that a group homomorphism $s : \pi_0(\mathcal{G}) \to G_0$ splits $\chi(\mathcal{G})$.

We claim that the group homomorphism $s' : \pi_0(\mathcal{H}) \to H_0$ defined as $s' = f_0 s \pi_0(G)$ splits the sequence $\chi(\mathcal{H})$. We need to prove that if $h \in H_0$, then h is connected to $s'([h])$. Indeed,

$s'([h]) = f_0(s(\pi_0(G)([h]))) = f_0(s([g_0(h)]))$ is connected to $f_0(g_0(h))$, since $g_0(h)$ is connected to $s([g_0(h)])$. On the other hand, $f_0(g_0(h))$ is connected to h, since $FG \cong \mathrm{Id}_{\mathcal{H}}$. This finishes the proof. $\square$

Proposition 7.12. *Let $\mathcal{G}, \mathcal{H}$ be 2-groups. If $\iota(\mathcal{G})$ left splits but $\iota(\mathcal{H})$ doesn't, then $\mathcal{G} \not\simeq \mathcal{H}$.*

Proof. Similar to the previous proof; the details are left to the reader. $\square$

The following two examples show how this criterion can be used, and give counter examples for a number of things.

Example 7.13. *Let $\mathcal{G} = \xi(\mathbb{Z}_2, \mathbb{Z}_4, \partial, \triangleright)$, where $\partial([n]_4) = [n]_2$ for $n \in \mathbb{Z}$ and $\triangleright$ is trivial. Let $\mathcal{H} = \mathbb{Z}_2[1]$. We claim that $\mathcal{G}$ and $\mathcal{H}$ have the same (old) homotopy invariants, but are not equivalent.*

Notice that $\pi(\mathcal{G}) = (1, \mathbb{Z}_2, \triangleright) = \pi(\mathcal{H})$, where $\triangleright$ is the trivial action. Also, the inclusion 2-group homomorphism $F : \mathcal{H} \to \mathcal{G}$ is such that the diagram

$$\begin{tikzcd}
 & & \mathcal{H} \arrow[dd, "F"] \arrow[dr, "P"] & & \\
\mathrm{Id}_1 \arrow[r] & 1[1] \arrow[ur, "J"] \arrow[dr, "J'"'] & & \mathbb{Z}_2[0] \arrow[r] & \mathrm{Id}_1 \\
 & & \mathcal{G} \arrow[ur, "P'"'] & &
\end{tikzcd}$$

commutes, thus $\mathcal{G}$ and $\mathcal{H}$ have the same Postnikov invariant.

The 2-groups $\mathcal{G}$ and $\mathcal{H}$ are not equivalent: notice that $\iota(\mathcal{G})$ is the only short exact sequence $1 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 1$, which does not split, but $\iota(\mathcal{H})$ is the short exact sequence $1 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \to 1 \to 1$, which obviously left splits. From proposition 7.12 follows that $\mathcal{G} \not\simeq \mathcal{H}$.

In addition to showing that the usual invariants are not sufficient, this example also shows a number of other things:

- *2-groups which are equivalent as categories do not have to be equivalent as 2-groups;*
- *2-groups with trivial π_1 cannot be distinguished by the usual invariants;*
- *A 2-group does not need to be equivalent to a skeletal 2-group; that is, a 2-group whose underlying category is skeletal;*
- *It is possible that the inclusion $\mathcal{H} \triangleleft \mathcal{G}$ does not give rise to an equivalence, but $\mathcal{G}/\mathcal{H} \simeq \mathrm{Id}_1$ (notice that, in this case, $\mathcal{G}/\mathcal{H} \cong \mathbb{Z}_2[\mathrm{Ad}]$).*

Example 7.14. *Let $\mathcal{G} = \xi(\mathbb{Z}_4, \mathbb{Z}_2, \partial, \triangleright)$, where $\partial : [n]_2 \mapsto [2n]_4$ and $\triangleright$ is the trivial action. Let $\mathcal{H} = \mathbb{Z}_2[0]$. We claim that $\mathcal{G}, \mathcal{H}$ have the same homotopy invariants, but are not equivalent.*

Notice that $\pi(\mathcal{G}) = (\mathbb{Z}_2, 1, \triangleright) = \pi(\mathcal{H})$, where $\triangleright$ is the trivial action. Also, there is an obvious projection $F : \mathcal{G} \to \mathcal{H}$, which makes the diagram

$$\begin{tikzcd}
 & & \mathcal{G} \arrow[dd, "F"] \arrow[dr, "P"] & & \\
\mathrm{Id}_1 \arrow[r] & \mathbb{Z}_2[1] \arrow[ur, "J"] \arrow[dr, "J'"'] & & 1[0] \arrow[r] & \mathrm{Id}_1 \\
 & & \mathcal{H} \arrow[ur, "P'"'] & &
\end{tikzcd}$$

commute, and thus $\mathcal{G}$ and $\mathcal{H}$ have the same Postnikov invariant.

The 2-groups $\mathcal{G}$ and $\mathcal{H}$ are not equivalent: notice that $\chi(\mathcal{G})$ is the only short exact sequence $1 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 1$, which does not split, but $\chi(\mathcal{H})$ is the short exact sequence $1 \to 1 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \to 1$, which obviously splits. From proposition 7.11 follows that $\mathcal{G} \simeq \mathcal{H}$.

This example shows that 2-groups with trivial π_0 cannot be distinguished by the usual invariants.

7.4 Homotopically minimal 2-subgroups

We have seen in example 7.13 that a 2-group does not need to be equivalent to a skeletal 2-group. In this section we will see however that the family of finite 2-groups equivalent to a given finite 2-group has a minimal element.

Lemma 7.15. *Let $\mathcal{G}$ and $\mathcal{H}$ be 2-groups, and $F : \mathcal{G} \to \mathcal{H}, G : \mathcal{H} \to \mathcal{G}$ be 2-group homomorphisms which give rise to an equivalence $\mathcal{G} \simeq \mathcal{H}$. Then $\mathcal{G} \simeq GF(\mathcal{G})$ and $\mathcal{H} \simeq FG(\mathcal{H})$.*

Proof. We prove that $\mathcal{G} \simeq GF(\mathcal{G})$; the rest of the statement is analogous. It suffices to see that the 2-group homomorphisms $GF : \mathcal{G} \to GF(\mathcal{G})$ and the inclusion $I : GF(\mathcal{G}) \to \mathcal{G}$ form an equivalence of 2-groups. Let $\tau : \mathrm{Id}_{\mathcal{G}} \to GF(\mathcal{G})$ be a 2-homomorphism. The 2-homomorphism τ, with the appropriate restrictions, can be made into a 2-homomorphism $\mathrm{Id}_{\mathcal{G}} \to IGF$ and a 2-homomorphism $\mathrm{Id}_{GF(\mathcal{G})} \to GFI$. This proves the desired result. $\square$

Theorem 7.16. *Let $\mathcal{G}$ be a finite 2-group and $[\mathcal{G}]$ be the class of 2-groups equivalent to it. Then there is a* homotopically minimal *$\mathcal{H} \in [\mathcal{G}]$; that is, a 2-group $\mathcal{H} \in [\mathcal{G}]$ such that if $\mathcal{K} \in [\mathcal{G}]$, then there is an embedding $\phi : \mathcal{H} \to \mathcal{K}$. Furthermore, such $\mathcal{H}$ is unique up to isomorphism.*

Proof. Let $\mathcal{H}$ be some element of $[\mathcal{G}]$ with minimal $|H_1|$. If $\mathcal{K} \in \mathcal{G}$, then there are 2-group homomorphisms $F : \mathcal{H} \to \mathcal{K}$ and $G : \mathcal{K} \to \mathcal{H}$ which give rise to an equivalence. From the previous lemma, $\mathcal{H} \simeq GF(\mathcal{H})$; from our minimality hypothesis, $|\mathrm{Mor}(GF(\mathcal{H}))| = |H_1|$; thus GF is injective, and so F is injective; F is such an embedding.

The last part of the statement is straightforward: if $\mathcal{H}'$ is another such 2-group, then there are embeddings $F : \mathcal{H} \to \mathcal{H}'$ and $F' : \mathcal{H}' \to \mathcal{H}$; since $\mathcal{H}, \mathcal{H}'$ are finite, both embeddings must be in fact isomorphisms. $\square$

Definition 7.17. *Given a finite 2-group G and a 2-subgroup $\mathcal{H} \leq \mathcal{G}$, we say $\mathcal{H}$ is an* homotopically minimal 2-subgroup *if the inclusion $\mathcal{H} \to \mathcal{G}$ gives rise to an equivalence $\mathcal{H} \simeq \mathcal{G}$ and it has no proper 2-subgroup with this property. By the previous proposition, any two homotopically minimal 2-subgroups of a given 2-group are isomorphic, and thus we can define* the *homotopically minimal 2-subgroup of $\mathcal{G}$, denoted $\mu(\mathcal{G})$.*

If $\mathcal{G} = \mu(\mathcal{G})$, we say that $\mathcal{G}$ is homotopically minimal.

The following theorem follows immediately from the previous one, but never the less it is worth mentioning.

Theorem 7.18. *Any finite strict 2-groups $\mathcal{G}$ and $\mathcal{H}$ are equivalent if and only if $\mu(\mathcal{G})$ and $\mu(\mathcal{H})$ are isomorphic.*

The previous lemma and theorem prove that the problem of classifying finite 2-groups up to equivalence reduces to classifying homotopically minimal finite 2-groups up to isomorphism.

Proposition 7.19. *If a 2-group $\mathcal{G}$ is such that $\mathcal{O}(\mathcal{G})$ is homotopically minimal, then $\mathcal{G}$ is homotopically minimal.*

Proof. If $\mathcal{H} \leq \mathcal{G}$ is a 2-subgroup of $\mathcal{G}$ such that the inclusion $I : \mathcal{H} \to \mathcal{G}$ is an equivalence, then the restriction of this inclusion to $\mathcal{O}(\mathcal{H})$ gives a 2-group homomorphism $I' : \mathcal{O}(\mathcal{H}) \to \mathcal{O}(\mathcal{G})$ which gives rise to an equivalence. Since $\mathcal{O}(\mathcal{G})$ is homotopically minimal, it follows that $\mathcal{O}(\mathcal{G}) = \mathcal{O}(\mathcal{H})$. Since the inclusion $I : \mathcal{H} \to \mathcal{G}$ gives rise to an equivalence, $\pi_0(\mathcal{G}) = \pi_0(\mathcal{H})$, and thus $\mathcal{G} = \mathcal{H}$. Thus $\mathcal{G}$ is homotopically minimal. $\square$

The converse of the previous proposition is not true in general (see Example 7.14).

We finish this section by computing $\mu(\mathrm{Aut}(D_{2n}))$ for $n \geq 3$. We will use crossed module language.

Recall that D_{2n} is isomorphic to $\{\pm 1\} \ltimes \mathbb{Z}_n$, where $\{\pm 1\}$ acts on $\mathbb{Z}_n$ by multiplication. A presentation of D_{2n} is $\langle a, b | a^n = b^2 = baba = 1\rangle$, where $a = (1, 1)$ and $b = (-1, 0)$.

It is a well known fact that the automorphism group of D_{2n} is isomorphic to $U(\mathbb{Z}_n) \ltimes \mathbb{Z}_n$, where the action is by multiplication. One possible isomorphism is the following: the pair (u, j) is mapped to the automorphism σ which maps a and b to a^u and ba^j, respetively.

Finally, notice that $\gamma : \{\pm 1\} \ltimes \mathbb{Z}_n \to U(\mathbb{Z}_n) \ltimes \mathbb{Z}_n$ maps (u, j) to $(u, -2uj)$.

Lemma 7.20. *Given $n \geq 3$, let $n = 2^k q$, where q is odd and k is a nonnegative integer. We have* $\mathrm{Aut}(D_{2n}) \simeq \mathrm{Aut}(D_{2n})|M_n$, *where $M_n = U(\mathbb{Z}_n) \ltimes q\mathbb{Z}_n$.*

Proof. Let m be an integer such that $m \equiv_q 0$ and $m \equiv_{2^k} 1$, and define $f : D_{2n} \to D_{2n}$ to be the group homomorphism generated by $f(a) = a^m$ and $f(b) = b$. Notice that f is idempotent.

Now we define a crossed module homomorphism $F : \gamma(\mathrm{Aut}(D_{2n})) \to \gamma(\mathrm{Aut}(D_{2n})|M_n)$ as follows. Given an automorphism σ of D_{2n}, define $F(\sigma)$ as the automorphism generated by $F(\sigma) : a \mapsto \sigma(a)$ and $F(\sigma) : b \mapsto b(b^{-1}\sigma(a))^m$. Given $g \in D_{2n}$, define $F(g) = f(g)$. It is straightforward to check that F is indeed a crossed module homomorphism.

Furthermore, $\pi_0(F) = \mathrm{Id}_{\pi_0(\mathcal{G})}$ and $\pi_1(F) = \mathrm{Id}_{\pi_1(\mathcal{G})}$, and the desired conclusion follows from Lemma 7.6. $\square$

Lemma 7.21. *Let n be an odd integer. There is a subgroup of $U(\mathbb{Z}_n) \ltimes 0$ which contains exactly one element from every coset of $\gamma(\mathbb{Z}_2 \ltimes 0)$ if and only if -1 is not a quadratic residue modulo n.*

Proof. Suppose that -1 is a quadratic residue modulo n, and that there is such a subgroup H.

Every coset of $\gamma(\mathbb{Z}_2 \ltimes 0)$ is of the kind $\{(x, 0), (-x, 0)\}$. If -1 is a quadratic residue, then there is $x \in U(\mathbb{Z}_n)$ such that $x^2 = -1$. From the previous observation either $(x, 0)$ or $(-x, 0)$ belong to H; in any case, $(-1, 0) = (x, 0)^2 = (-x, 0)^2$ belongs to H, since it is a subgroup. Thus H contains the full coset of $(1, 0)$, contradicting the hypothesis.

Suppose now that -1 is not a quadratic residue modulo n. Then there is a prime p that divides n such that -1 is not a quadratic residue modulo n. Take H to be $S \ltimes 0$, where S is the subgroup of $U(\mathbb{Z}_n)$ whose elements are quadratic residues modulo p. It is easy to see that H fulfils the necessary conditions. $\square$

Proposition 7.22. *Given $n \geq 3$, let $n = 2^k q$, where q is odd and k is a nonnegative integer. We have $\mu(\mathrm{Aut}(D_{2n})) = \mathrm{Aut}(D_{2n})|M'_n$, where M'_n is the subgroup of $\mathrm{Aut}(D_{2n})$ given as follows:*

1. *If n is odd and -1 is a quadratic residue modulo n, then M'_n is the set of automorphisms which map $\langle b, a^q \rangle$ to itself;*

2. *If n is odd and -1 is not a quadratic residue modulo n, let $p \equiv_4 3$ be a prime which divides n; then M'_n is the set of automorphisms which map $\langle b, a^q \rangle$ to itself and a to a^u, where u is some quadratic residue modulo p;*

3. *If n is even, then M'_n is the set of automorphisms which map $\langle b, a^q \rangle$ to itself.*

Proof. From Lemma 7.20 follows that $\mu(\mathrm{Aut}(D_{2n}) = \mu(\mathrm{Aut}(D_{2n})|M_n)$.

If n is odd, then each coset of $\gamma(\mathbb{Z}_2 \ltimes q\mathbb{Z}_n)$ has exactly two elements. Thus there are two possibilities: either $\mathrm{Aut}(D_{2n})|M_n$ is homotopically minimal, or there is a subgroup H of M_n with index 2 such that the inclusion $\mathrm{Aut}(D_{2n})|H \to \mathrm{Aut}(D_{2n})|M_n$ gives rise to an equivalence. If -1 is a quadratic residue, then the second case is an impossibility: from Lemma 7.21 follows that there can be no such subgroup H with exactly one element from every connected component of $\mathrm{Aut}(D_{2n})|M_n$. So $\mathrm{Aut}(D_{2n})|M_n$ is homotopically minimal in this case.

On the other hand, if -1 is not a quadratic residue, take H to be a subgroup as described in Lemma 7.21. Define $f_1 : M_n \to H$ by

$$f_1(x, 0) = \left(x \left(\frac{x}{p} \right), 0 \right),$$

where $\left(\frac{x}{p} \right)$ is the Legendre symbol. Define $f_2 : \{\pm 1\} \ltimes 0 \to 1 \ltimes 0$ as the trivial homomorphism. It is easy to check that (f_1, f_2) is a crossed module homomorphism, and that the corresponding 2-group homomorphism F obeys $\pi_0(F) = \mathrm{Id}_{\pi_0(\mathcal{G})}$ and $\pi_1(F) = \mathrm{Id}_{\pi_1(\mathcal{G})}$, and the desired conclusion follows from Lemma 7.6.

If n is even, suppose that (H_1, H_2) is a crossed submodule of $\gamma(\mathrm{Aut}(D_{2n})|M_n)$ such that the inclusion gives rise to an equivalence. Let $(f_1, f_2) : \gamma(\mathrm{Aut}(D_{2n})|M_n) \to (H_1, H_2)$ be the crossed module homomorphism which, together with the inclusion, forms said equivalence. Then $f_2 : \{\pm 1\} \ltimes q\mathbb{Z}_n \to H_2$ must fix the center of $\{\pm 1\} \ltimes q\mathbb{Z}_n$; that is, $f_2(1, 2^{k-1}q) = (1, 2^{k-1}q)$. Since $f_2(1, 2^{k-1}q) = f_2(1, q)^{2^{k-1}}$, this means that $f_2(1, q) = (1, jq)$, where j is some odd number. We have $\langle (1, jq) \rangle = 1 \ltimes q\mathbb{Z}$, so $1 \ltimes q\mathbb{Z}_n \leq \mathrm{im}\ f_2$, and either $H_2 = 1 \ltimes q\mathbb{Z}_n$ or $H_2 = \{\pm 1\} \ltimes q\mathbb{Z}_n$. The former case is impossible: if $(-1, 0)$ is mapped to an element in $1 \ltimes q\mathbb{Z}_n$, it must be an element of order two: either $(1, 0)$ or $(1, 2^{k-1}q)$. The same applies to $(-1, q)$, so $(1, q) = (-1, q)(-1, 0)$ is mapped to $(1, 0)$ or $(1, 2^{k-1}q)$, which contradicts the fact that $(1, q)$ is mapped to some $(1, jq)$, where j is odd. So H_2 must be $\{\pm 1\} \ltimes q\mathbb{Z}_n$; in conclusion, $(H_1, H_2) = \gamma(\mathrm{Aut}(D_{2n})|M_n)$. This means that $\mathrm{Aut}(D_{2n})|M_n$ is homotopically minimal in this case, finishing our proof. $\square$

In particular, the set of examples $\mathrm{Aut}(D_{2^{k+1}})$ gives an infinite family of homotopically minimal 2-groups which are not discrete.

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