Group and 2-group presentations

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Abstract

We can define a group by giving a set of its generators and a set of relators involving those generators, that is, by identifying the group with some group presentation. The problem of determining whether two group presentations are equivalent or not (the isomorphism problem) does not have a general solution, so we will go through some partial solutions. Moreover, at the same time, we will use a language of squares to give another overview of the concepts we will talk about. Then we talk a little bit about 2-groups; more specifically, about crossed modules, finishing with a description of the free crossed module..

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1.1 Free Groups

Given a group G and a subset E of G, the subgroup generated by E is

$$\langle E\rangle = \{g_1^{n_1} g_2^{n_2} ... g_m^{n_m} : g_1, ..., g_m \in E; n_1, ..., n_m \in \mathbb{Z}; m \in \mathbb{N}\}$$

and is the smallest subgroup of G that contains E. If $\langle E\rangle = G$, we say that E is a generating set of G and its elements are the generators of G.

Definition 1.1.1 *A generating set E of a group G is called a* free basis *if, for any group H, any function $\phi : E \to H$ can be extended into an homomorphism of G into H. (This homomorphism is unique since E is a generating set of G).*

If G has a free basis, G is called a free group*. In this case, the* rank *of G is the cardinality of its free basis.*

We now show how to obtain free groups:

Consider a set $\mathcal{A}$ (*alphabet*) of symbols $a, b, c, ...$ (*letters*). We call an element of the form a^n, with $a \in \mathcal{A}$ and n an integer, a *syllable*, and we call a finite sequence of syllables a *word* (we denote de empty word by 1). Then, the set $W(\mathcal{A})$ formed by all the words together with the operation of concatenation of words is clearly a semi group.

Now we introduce elementary contractions and expansions on the words. Given a word $u = w_1 a^0 w_2$, we say that the word $v = w_1 w_2$ is obtained by an elementary contraction of type I on u, or that u is obtained by an elementary expansion of type I on v. Also, given a word $u = w_1 a^n a^m w_2$, we say that the word $v = w_1 a^{n+m} w_2$ is obtained by an elementary contraction of type II on u, or that u is obtained by an elementary expansion of type II on v.

Two words are said to be equivalent if one can be obtained from the other by finitely many elementary contractions and expansions. This is clearly an equivalence relation. We now denote by $F[\mathcal{A}]$ the set of equivalence classes of words. It is easy to see that $F[\mathcal{A}]$ inherits the operation of $W(A)$: $[u][v] = [uv]$ (where $[u]$ denotes the equivalence class of the word u). It is also easy to check that $F[\mathcal{A}]$ is a group. Note also that $[\mathcal{A}] := \{[a] : a \in \mathcal{A}\}$ is a generating set of $F[\mathcal{A}]$.

We now see that this is intimately related to free groups.

Proposition 1.1.2 *A group is free if and only if it is isomorphic to $F[\mathcal{A}]$, for some $\mathcal{A}$.*

Proof: ($\Leftarrow$): First we prove that $F[\mathcal{A}]$ is in fact free by showing that $[\mathcal{A}]$ is a free basis for it. This is immediate because, for any function $\phi : [\mathcal{A}] \to H$, its extension $\tilde{\phi} : F[\mathcal{A}] \to H$ given by

$$\tilde{\phi}([a^n b^m c^p \cdots]) = \phi([a])^n \phi([b])^m \phi([c])^p \cdots$$

is clearly an homomorphism (notice it is well defined for if $[u] = [v]$ then $\tilde{\phi}([u]) = \tilde{\phi}([v])$. Now, if f is any isomorphism from a group G to $F[\mathcal{A}]$, then $f^{-1}([\mathcal{A}])$ is a free basis of G, so G is free.

($\Rightarrow$): Suppose now G is free group with free basis E. Let $\mathcal{A}$ be any alphabet with the same cardinality as E. We can identify $\mathcal{A}$ with $[\mathcal{A}]$ (because if a, b are distinct elements of $\mathcal{A}$ then $[a] \neq [b]$), so there is a bijection $f : E \to [\mathcal{A}]$.

Because E is a free basis of G, f can be extended to an homomorphism $\phi : G \to F[\mathcal{A}]$. In the same way, because $[\mathcal{A}]$ is a free basis of $F[\mathcal{A}]$, $f^{-1} : [\mathcal{A}] \to E$ can be extended to a homomorphism $\psi : F[\mathcal{A}] \to G$.

This way, the homomorphism $\phi\psi : F[\mathcal{A}] \to F[\mathcal{A}]$ is an extension of the identity $ff^{-1} : [\mathcal{A}] \to [\mathcal{A}]$ and the homomorphism $\psi\phi : G \to G$ is an extension of the identity $f^{-1}f : E \to E$.

Finally, because ϕ and ψ are unique extensions of f and f^{-1}, we conclude that $\phi\psi$ and $\psi\phi$ are the identity automorphisms. Hence G and $F[\mathcal{A}]$ are isomorphic.□

The next proposition will lead us to group presentations:

Proposition 1.1.3 *Any group is the homomorphic image of some free group.*

Proof: Let G be a group and E be any generating set of it. Consider an alphabet $\mathcal{A}$ such that there is a bijection $f : [\mathcal{A}] \to E$. Because $[\mathcal{A}]$ is a free basis for $F[\mathcal{A}]$, f extends to an homomorphism $\phi : F[\mathcal{A}] \to G$. This homomorphism has image G, because E a generates G. $\square$

Remark 1.1.4 *From this and the first isomorphism theorem, we see that any group is isomorphic to some quotient group of some free group.*

1.2 Group presentations

We start with the definition of consequence:

Definition 1.2.1 *Given elements $g_1, g_2, ...$ of a group G, we say that $g \in G$ is a* consequence *of $g_1, g_2, ...$ if every time $g_1, g_2, ...$ are mapped to 1 by an homomorphism, g is also mapped to 1.*

The consequence *of $S \subseteq G$ is the set of all the consequences of the elements of S, and we denote it by $\mathcal{C}(S)$.*

Remark 1.2.2 *We see that, for every homomorphism $\phi : G \to H$ such that $S \subseteq \ker \phi$, we have $\mathcal{C}(S) \subseteq \ker \phi$. Since $\ker \phi$ is a normal subgroup of G and since for every normal subgroup N of G there exists some ϕ such that $\ker \phi = N$, we conclude that $\mathcal{C}(S)$ is just the smallest normal subgroup of G that contains S (that is, it is the intersection of all normal subgroups that contain S).*

Remark 1.2.3 *Any element of the form $\prod_{i=1}^{m} h_i g_{\sigma(i)}^{\tau(i)} h_i^{-1}$ (that is, any product of finite conjugates of powers of the g_i's) is clearly a consequence of $g_1, g_2, ...$ Moreover, it is not difficult to see that all the elements of this form form a normal subgroup and that they are contained in every normal subgroup that contains $g_1, g_2, ...$ So, the consequence of $\{g_1, g_2 ...\}$ is just the normal subgroup that consists of all the elements of this form.*

So we would like to describe a group G by giving a set $\{g_1, g_2, ...\}$ of its generators and a set of equations of the form $f_i(g_1, g_2, ...) = 1$, in a way that every true relation that subsists among the elements $g_2, g_2, ...$ is a consequence of the given equations.

Consider a free group $F[X]$ with free basis $\{x_1, x_2, ...\}$ in one to one correspondence with $\{g_1, g_2, ...\}$. Note that there is an induced homomorphism $\phi : F[X] \to G$. We are going to impose the following relations on $F[X]$:

$$r_i = f_i(x_1, x_2, ...) = 1$$

(obtained by substituting each g_i by x_i on $f_i(g_1, g_2, ...) = 1$). To these r_i we will call *relators.*

If it is true that every element of G is a consequence of the $f_i(g_1, g_2, ...)$'s, then the kernel of ϕ will be $\mathcal{C}(R)$, where R is the set of all relators. So $F[X]/\mathcal{C}(R)$ will be isomorphic to G.

Now we have the desired setup to define presentation:

Definition 1.2.4 *A* group presentation, *denoted by $(\boldsymbol{x} : \boldsymbol{r})$, is an object that consists of a set $\boldsymbol{x}$ of* generators *and a subset $\boldsymbol{r}$ of $F[\boldsymbol{x}]$. To $\boldsymbol{r}$ we call the set of* relators.

The group presented *by the group presentation* $(\boldsymbol{x} : \boldsymbol{r})$ *is the group* $|\boldsymbol{x} : \boldsymbol{r}| := F[\boldsymbol{x}]/\mathcal{C}(\boldsymbol{r})$.

A presentation of a group G *consists of a group presentation* $(\boldsymbol{x} : \boldsymbol{r})$ *and an isomorphism* $\psi : |\boldsymbol{x} : \boldsymbol{r}| \to G$.

A trivial example is the presentation $(\mathbf{x} :)$, which presents the free group $F[\mathbf{x}]$. We also denote by $(:)$ the presentation of the trivial group.

A group presentation $(\mathbf{x} : \mathbf{r})$ is called *finitely generated* if $\mathbf{x}$ is finite, *finitely related* if $\mathbf{r}$ is finite and *finite* if it is both finitely generated and finitely related. We will turn our interest to finite presentations.

Notice that Proposition 2.1.3 implies that every group has a presentation. Also, every surjective homomorphism $\phi : F[\mathbf{x}] \to G$ whose kernel is the consequence of $\mathbf{r}$ determines a presentation of G.

$$\begin{tikzcd} F[\mathbf{x}] \arrow[d, "\gamma"'] \arrow[dr, "\phi"] & \\ |\mathbf{x} : \mathbf{r}| \arrow[r, "\psi"] & G \end{tikzcd}$$

(in this diagram γ represents the canonical homomorphism).

Definition 1.2.5 *Two group presentations are said to be* equivalent *if they present isomorphic groups.*

A group has much more than just one presentation. For example, $(x, y : x^3, y^2, (xy)^2)$ and $(y, z : y^2, z^2, (yz)^3)$ are both presentations of the group of symmetries of the triangle. Latter we will see how these presentations are equivalent.

The problem of deciding whether two given presentations are equivalent or not (the isomorphism problem) does not have a general solution (as well as, for example, the problem of deciding if a word is a consequence of some other words). We will go through some methods of finding partial solutions to it. More precisely, the objective is to present the Tietze theorem, which gives us the basic moves we can make to go from a certain presentation to an equivalent one.

Definition 1.2.6 *A* presentation mapping $f : (\boldsymbol{x} : \boldsymbol{r}) \to (\boldsymbol{y} : \boldsymbol{s})$ *consists of two group presentations* $(\boldsymbol{x} : \boldsymbol{r})$ *and* $(\boldsymbol{y} : \boldsymbol{s})$ *and a homomorphism* $f : F[\boldsymbol{x}] \to F[\boldsymbol{y}]$ *such that* $f(\boldsymbol{r}) \subseteq \mathcal{C}(\boldsymbol{s})$.

Lets denote both the canonical homomorphisms $F[\mathbf{x}] \to |\mathbf{x} : \mathbf{r}|$ and $F[\mathbf{y}] \to |\mathbf{y} : \mathbf{s}|$ by γ. Then the fact that $f : (\mathbf{x} : \mathbf{r}) \to (\mathbf{y} : \mathbf{s})$ is a presentation map says that $\ker \gamma \subseteq \ker \gamma f$, so there is a unique induced homomorphism $f_* : |\mathbf{x} : \mathbf{r}| \to |\mathbf{y} : \mathbf{s}|$ such that the following diagram commutes:

$$\begin{tikzcd} F[\mathbf{x}] \arrow[r, "f"] \arrow[d, "\gamma"'] & F[\mathbf{y}] \arrow[d, "\gamma"] \\ |\mathbf{x} : \mathbf{r}| \arrow[r, "f_*"'] & |\mathbf{y} : \mathbf{s}| \end{tikzcd}$$

Now, given presentation maps $f : (\mathbf{x} : \mathbf{r}) \to (\mathbf{y} : \mathbf{s})$ and $g : (\mathbf{y} : \mathbf{s}) \to (\mathbf{z} : \mathbf{t})$, we define their *composition* to be the presentation map that consists of the two presentations

$(\mathbf{x} : \mathbf{r})$ and $(\mathbf{z} : \mathbf{t})$ and the homomorphism $gf : (\mathbf{x} : \mathbf{r}) \to (\mathbf{z} : \mathbf{t})$ (it is easy to see that this is indeed a presentation map). It is also easy to check that $1_* = 1$ and $(gf)_* = g_* f_*$.

Definition 1.2.7 *Two presentation maps $f_1, f_2 : (\boldsymbol{x} : \boldsymbol{r}) \to (\boldsymbol{y} : \boldsymbol{s})$ are said to be* homotopic*, written $f_1 \simeq f_2$, if $f_1(x) f_2(x^{-1}) \in \mathcal{C}(\boldsymbol{s})$ for every $x \in F[\boldsymbol{x}]$.*

Proposition 1.2.8 *Let $f_1, f_2 : (\boldsymbol{x} : \boldsymbol{r}) \to (\boldsymbol{y} : \boldsymbol{s})$ be two presentation maps. Then $f_1 \simeq f_2$ if and only if $f_{1*} = f_{2*}$.*

Proof: Saying that $f_1(x) f_2(x^{-1}) \in \mathcal{C}(\mathbf{s})$ for every $x \in F[\mathbf{x}]$ is equivalent to saying that $\gamma f_1 = \gamma f_2$, and so $\gamma f_1 = f_{2*}\gamma$. But since f_{1*} was uniquely determined by f_1, this is also the same as saying that $f_{1*} = f_{2*}$. □

It is also easy to see that if $f_1 \simeq f_2$ (i.e. $f_{1*} = f_{2*}$) and $g_1 \simeq g_2$ (i.e. $g_{1*} = g_{2*}$), then $g_1 f_1 \simeq g_2 f_2$ (just notice that $(g_1 f_1)_* = g_{1*} f_{1*} = g_{2*} f_{2*} = (g_2 f_2)_*$).

We saw that every presentation map f determines an homomorphism $f_* : |\mathbf{x} : \mathbf{r}| \to |\mathbf{y} : \mathbf{s}|$. The next result will allow us to conclude that there is in fact a one to one correspondence between the homotopy classes of the presentations maps and the homomorphisms on the groups presented.

Proposition 1.2.9 *Let $\theta : |\boldsymbol{x} : \boldsymbol{r}| \to |\boldsymbol{y} : \boldsymbol{s}|$ be an homomorphism. Then there exists a presentation map $f : (\boldsymbol{x} : \boldsymbol{r}) \to (\boldsymbol{y} : \boldsymbol{s})$ such that $f_* = \theta$. Also, any two such presentation maps are homotopic.*

Proof: We start by choosing a function $f : \mathbf{x} \to F[\mathbf{y}]$ such that $\gamma(f(x)) = \theta(\gamma(x))$ for every generator $x \in \mathbf{x}$. Because $\mathbf{x}$ is a free basis for $F[\mathbf{x}]$, f extends to an homomorphism $f : F[\mathbf{x}] \to F[\mathbf{y}]$. Clearly, f is such that $\gamma f = \theta\gamma$, so it remains to check that f is a presentation map, that is, that $f(r) \in \mathcal{C}(\mathbf{s})$ for every $r \in \mathcal{C}(\mathbf{r})$. Indeed, since for every $r \in \mathcal{C}(\mathbf{r})$ we have that $\gamma(f(r)) = \theta(\gamma(r)) = \theta(1) = 1$, we conclude immediately that $f(r) \in \mathcal{C}(\mathbf{s})$.

Finally, by the last proposition, we conclude that any two such presentation maps must be homotopic, because they determine the same homomorphism θ. □

Now we get to our first result on presentation equivalence:

Proposition 1.2.10 *Two presentations $(\boldsymbol{x} : \boldsymbol{r})$ and $(\boldsymbol{y} : \boldsymbol{s})$ are equivalent if and only if there exists two presentation maps $f : (\boldsymbol{x} : \boldsymbol{r}) \to (\boldsymbol{y} : \boldsymbol{s})$ and $g : (\boldsymbol{y} : \boldsymbol{s}) \to (\boldsymbol{x} : \boldsymbol{r})$ such that $gf \simeq 1$ and $fg \simeq 1$.*

Proof: $(\Rightarrow)$:

If $(\mathbf{x} : \mathbf{r})$ and $(\mathbf{y} : \mathbf{s})$ are equivalent, there exists an isomorphism $\theta : |\mathbf{x} : \mathbf{r}| \to |\mathbf{y} : \mathbf{s}|$. By the previous Proposition, there are presentation maps $f : (\mathbf{x} : \mathbf{r}) \to (\mathbf{y} : \mathbf{s})$ and $g : (\mathbf{y} : \mathbf{s}) \to (\mathbf{x} : \mathbf{r})$ such that $f_* = \theta$ and $g_* = \theta^{-1}$. So we have $(fg)_* = f_* g_* = \theta\theta^{-1} = 1$, from which $fg \simeq 1$. Similarly, $gf \simeq 1$.

$(\Leftarrow)$:

If there exists two presentation maps $f : (\mathbf{x} : \mathbf{r}) \to (\mathbf{y} : \mathbf{s})$ and $g : (\mathbf{y} : \mathbf{s}) \to (\mathbf{x} : \mathbf{r})$ such that $gf \simeq 1$ and $fg \simeq 1$, then $g_* f_* = (gf)_* = 1_* = 1$ and $f_* g_* = (fg)_* = 1_* = 1$. Hence $f_* : |\mathbf{x} : \mathbf{r}| \to |\mathbf{y} : \mathbf{s}|$ is an invertible homomorphism, and so $|\mathbf{x} : \mathbf{r}|$ and $|\mathbf{y} : \mathbf{s}|$ are isomorphic. □

Definition 1.2.11 *A presentation map* $f : (\boldsymbol{x} : \boldsymbol{r}) \to (\boldsymbol{y} : \boldsymbol{s})$ *is called a* presentation equivalence *if there exists a presentation map* $g : (\boldsymbol{y} : \boldsymbol{s}) \to (\boldsymbol{x} : \boldsymbol{r})$ *such that* $gf \simeq 1$ *and* $fg \simeq 1$*. In this case we also call the pair* f, g *a presentation equivalence.*

1.3 Squares

We are also a bit concerned about working with another kind of objects: squares. The point is that, hopefully, these objects will give us another perspective of what is going on in terms of equivalence of words in a free group, what it means for a word to be consequence of others, etc. They will become another way for us to visualizing things, and that might be useful sometimes.

So, given a group G, a square is an object of the form

$$\begin{array}{c|c|c} & g_3 & \\ \cline{2-2} g_4 & \eta & g_2 \\ \cline{2-2} & g_1 & \end{array} \tag{1}$$

where g_1, g_2, g_3, g_4, η are elements of G which must satisfy the *square equation*: $\eta = g_1 g_2 g_3^{-1} g_4^{-1}$. To η we call the *center* of the square. In case the center is 1, we call the square a *commutative square*. If the center or side of a square is 1, we will simply omit it.

When squares share a common side, they can be multiplied horizontally or vertically in the following way, respectively:

$$\begin{array}{c|c|c|c} & g_3 & g_7 & \\ \cline{2-3} g_4 & \eta_1 \quad g_2 & \eta_2 & g_6 \\ \cline{2-3} & g_1 & g_5 & \end{array} \;=\; \begin{array}{c|c|c} & g_3 g_7 & \\ \cline{2-2} g_4 & g_1 \eta_2 g_1^{-1} \eta_1 & g_6 \\ \cline{2-2} & g_1 g_5 & \end{array}$$

$$\begin{array}{c|c|c} & g_3 & \\ \cline{2-2} g_4 & \eta_1 & g_2 \\ \cline{2-2} & g_1 & \\ \cline{2-2} g_5 & \eta_2 & g_7 \\ \cline{2-2} & g_6 & \end{array} \;=\; \begin{array}{c|c|c} & g_3 & \\ \cline{2-2} g_5 g_4 & \eta_2 g_5 \eta_1 g_5^{-1} & g_7 g_2 \\ \cline{2-2} & g_6 & \end{array}$$

The reader can check that the resulting squares satisfy the required square equation. These multiplications also satisfy the *interchange law*, that is, given a rectangular Array of squares, the result is the same no matter the order in which we make the multiplications (first horizontally, then vertically, or vice-versa).

Also, given a square S as in (1), we define its horizontal and vertical inverses, respectively, by:

$$S^{-h} = \begin{array}{ccc} & g_3^{-1} & \\ g_2 & \boxed{\quad g_1^{-1}\eta^{-1}g_1 \quad} & g_4 \\ & g_1^{-1} & \end{array} \qquad \text{and} \qquad S^{-v} = \begin{array}{ccc} & g_1 & \\ g_4^{-1} & \boxed{\quad g_4^{-1}\eta^{-1}g_4 \quad} & g_2^{-1} \\ & g_3 & \end{array}$$

The next proposition shows a way to identify if an element is consequence of others using squares.

Proposition 1.3.1 *An element $s \in G$ is a consequence of $r_1, r_2, ...$ if and only if every square with center s can be written as an evaluation of a rectangular Array of squares with centers from* $\{1, r_1, r_2, ..., r_1^{-1}, r_2^{-1}, ...\}$.

Proof: We will take the advantage of the fact that any consequence of $r_1, r_2, ...$ is a finite product of conjugates of powers of $r_1, r_2, ...$, i.e, of conjugates of positive powers of $r_1, r_2, ..., r_1^{-1}, r_2^{-1}, ...$.

($\Leftarrow$): When we evaluate a rectangular Array of squares with centers from the set $\{1, r_1, r_2, ..., r_1^{-1}, r_2^{-1}, ...\}$, we will obtain a square whose center involves conjugates and powers of $r_1, r_2, ..., r_1^{-1}, r_2^{-1}, ...$ (this comes from the definition of square multiplications). Therefore, this implication is proved.

($\Rightarrow$): We want to write the square

$$\begin{array}{ccc} & c & \\ d & \boxed{\quad s \quad} & b \\ & a & \end{array}$$

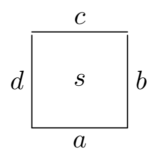
as a product of squares with centers from $\{1, r_1, r_2, ..., r_1^{-1}, r_2^{-1}, ...\}$. We will only need horizontal multiplication. Note that for every $r \in \{1, r_1, r_2, ..., r_1^{-1}, r_2^{-1}, ...\}$ we can write

$$\begin{array}{c} \\ \boxed{hr^nh^{-1}} \\ hr^nh^{-1} \end{array} \quad \text{as} \quad \begin{array}{cc} \multicolumn{2}{c}{h} \\ \boxed{\quad\quad} & \boxed{\quad r \quad} \\ h & r \end{array} \cdots \begin{array}{cc} \multicolumn{2}{c}{h^{-1}} \\ \boxed{\quad r \quad} & \boxed{\quad\quad} \\ r & h^{-1} \end{array} .$$

Now, since $s = abc^{-1}d^{-1}$ can be written as a finite product of conjugates of positive powers of $r_1, r_2, ..., r_1^{-1}, r_2^{-1}, ...$, we just have to multiply horizontaly squares of this type to obtain

$$\begin{array}{c} \\ \boxed{\quad s \quad} \\ abc^{-1}d^{-1} \end{array} .$$

All that is left is to obtain



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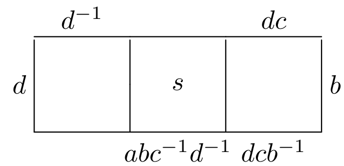

and we are done. □

Definition 1.3.2 *For any* $\mathbf{r} \subset G$*, we will call* square consequence of $\mathbf{r}$ *to the set of squares which can be written as a rectangular Array of squares which centers are elements of* $\mathbf{r}$*, inverses of elements of* $\mathbf{r}$ *or 1, and we denote it by* $\mathcal{SC}(\mathbf{r})$*.*

Remark 1.3.3 By the previous proof we can say the following: if $r \in \mathcal{C}(\mathbf{r})$, then every square with center r is in $\mathcal{SC}(\mathbf{r})$; also, if $\mathcal{SC}(\mathbf{r})$ contains at least a square with center r, then $r \in \mathcal{C}(\mathbf{r})$.

Proposition 1.3.4 $\mathcal{SC}(\mathbf{r})$ *is closed for horizontal and vertical multiplication.*

Proof: Given two squares $S_1, S_2 \in \mathcal{SC}(\mathbf{r})$ with centers r_1 and r_2 respectively, we know that $r_1, r_2 \in \mathcal{C}(\mathbf{r})$. Now, the horizontal multiplication of S_1 and S_2 is a square S whose center is an element r of $\mathcal{C}(\mathbf{r})$ (this comes from the definition of square multiplication and the fact that $\mathcal{C}(\mathbf{r})$ is a normal subgroup of G). Therefore $S \in \mathcal{SC}(\mathbf{r})$.

Similar reasoning applies to vertical multiplication. □

Within the context of a free group $F[X]$ we can now say that two words u and v are equivalent iff there is a rectangular Arrayment of commutative squares in which u is written on the upper side, v is written on the lower side, and on the other sides is written 1.

In a similar way, if we impose relations $r_1 = 1, r_2 = 1, ...$ on $F[X]$, u and v will now be equivalent iff $vu^{-1} \in \mathcal{C}(\{r_1, r_2, ...\})$, that is, iff there is a rectangular Arrayment of squares with centers in $\{1, r_1, r_2, ..., r_1^{-1}, r_2^{-1}, ...\}$, in which u is written on the upper side, v is written on the lower side, and on the other sides is written 1.

1.4 The Tietze equivalences and the Tietze theorem

We will now start by introducing some quite simple types of presentation equivalences, called the *Tietze equivalences*. They are of four types: **I,I',II** and **II'**. Intuitively, **I** says that if we add to the set of relators an element that is a consequence of the previous elements, we dont alter the presented group; on the other hand $\mathbf{I}'$ says that we can remove a relator if that relator is a consequence of the others. **II** says that, in a presentation $(\mathbf{x} : \mathbf{r})$ we can always add a new generator y, but in the sense that y is just going to "give another name" to some word $w \in F[\mathbf{x}]$, i.e, we also add the relator yw^{-1}. Also, if there is a generator y that is just substituting a word w written with the other generators (that is, the only relator involving y is yw^{-1}), then we can remove from the presentation the generator y as well as the relator yw^{-1} .

- Let $(\mathbf{x} : \mathbf{r})$ be a presentation and let s be a consequence of $\mathbf{r}$. Let also $\mathbf{s} = \mathbf{r} \cup s$. Then $\mathbf{I} : (\mathbf{x} : \mathbf{r}) \to (\mathbf{x} : \mathbf{s})$ is defined to be the presentation map consisting of the two presentations $(\mathbf{x} : \mathbf{r})$ and $(\mathbf{x} : \mathbf{s})$ and the identity homomorphism $1 : F[\mathbf{x}] \to F[\mathbf{x}]$. Similarly, the presentation map $\mathbf{I'} : (\mathbf{x} : \mathbf{s}) \to (\mathbf{x} : \mathbf{r})$ consists of the two presentations and the identity homomorphism.

Note that, since $\mathcal{C}(\mathbf{r}) = \mathcal{C}(\mathbf{s})$, these are clearly presentation maps.

- Let $(\mathbf{x} : \mathbf{r})$ be a presentation and let y be a generator that does not belong to $\mathbf{x}$. Let ξ be any element of $F[\mathbf{x}]$. Take $\mathbf{y} = \mathbf{x} \cup y$ and $\mathbf{s} = \mathbf{r} \cup y\xi^{-1}$. Then $\mathbf{II} : (\mathbf{x} : \mathbf{r}) \to (\mathbf{y} : \mathbf{s})$ is defined to be the presentation map consisting of the two presentations $(\mathbf{x} : \mathbf{r})$ and $(\mathbf{y} : \mathbf{s})$ and the homomorphism $\mathbf{II} : F[\mathbf{x}] \to F[\mathbf{y}]$ defined by $\mathbf{II}(x) = x$. The presentation map $\mathbf{II'} : (\mathbf{y} : \mathbf{s}) \to (\mathbf{x} : \mathbf{r})$ is the map that consists of the two presentations and the homomorphism $\mathbf{II'} : F[\mathbf{y}] \to F[\mathbf{x}]$ such that $\mathbf{II'}(x) = x$ for all $x \in \mathbf{x}$ and $\mathbf{II'}(y) = \xi$.

  Let's check that $\mathbf{II}$ and $\mathbf{II'}$ are indeed presentation maps. By seeing $\mathbf{II}(\mathbf{r}) = \mathbf{r} \subset \mathcal{C}(\mathbf{r} \cup y\xi^{-1}) = \mathcal{C}(\mathbf{s})$ we have checked that $\mathbf{II}$ is a presentation map. Now, for $\mathbf{II'}$, we see that $\mathbf{II'}(\mathbf{r}) = \mathbf{r} \subset \mathcal{C}(\mathbf{r})$ and that $\mathbf{II'}(y\xi^{-1}) = \mathbf{II'}(y)\mathbf{II'}(\xi)^{-1} = \xi\xi^{-1} = 1 \in \mathcal{C}(\mathbf{r})$, so $\mathbf{II'}$ is also a presentation map.

Proposition 1.4.1 *The pairs* $\boldsymbol{I}, \boldsymbol{I'}$ *and* $\boldsymbol{II}, \boldsymbol{II'}$ *are presentation equivalences.*

Proof: In order to check that $\mathbf{I}, \mathbf{I'}$ is a presentation equivalence, we have to check that $\mathbf{I'} \circ \mathbf{I} \simeq 1$ and $\mathbf{I} \circ \mathbf{I'} \simeq 1$. To see that $\mathbf{I'} \circ \mathbf{I} \simeq 1$ just note that $\mathbf{I'} \circ \mathbf{I}(x)x^{-1} = xx^{-1} = 1 \in \mathcal{C}(\mathbf{r})$ for every $x \in F[\mathbf{x}]$. To see that $\mathbf{I} \circ \mathbf{I'} \simeq 1$ just note that $\mathbf{I} \circ \mathbf{I'}(x)x^{-1} = xx^{-1} = 1 \in \mathcal{C}(\mathbf{r} \cup s) = \mathcal{C}(\mathbf{s})$ for every $x \in F[\mathbf{x}]$.

Let's now check that $\mathbf{II}, \mathbf{II'}$ is a presentation equivalence. In order to see that $\mathbf{II'} \circ \mathbf{II} \simeq 1$ we just have to note that $\mathbf{II'} \circ \mathbf{II}(x)x^{-1} = xx^{-1} = 1 \in \mathcal{C}(\mathbf{r})$ for every $x \in F[\mathbf{x}]$. To check that $\mathbf{II} \circ \mathbf{II'} \simeq 1$, we just have to see that $\mathbf{II} \circ \mathbf{II'}(x)x^{-1} = xx^{-1} = 1 \in \mathcal{C}(s)$ for every $x \in \mathbf{x}$ and that $\mathbf{II} \circ \mathbf{II'}(y)y^{-1} = \mathbf{II}(\xi)y^{-1} = \xi y^{-1} = (y\xi^{-1})^{-1} \in \mathcal{C}(\mathbf{r} \cup y\xi^{-1}) = \mathcal{C}(\mathbf{s})$. $\square$

Next we are going to prove the Tietze theorem, which states that the Tietze equivalences are the building blocks for any presentation equivalence. To do so we will need the following lemma:

Lemma 1.4.2 *Let* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *be disjoint sets of generators. Let* $(\boldsymbol{x} : \boldsymbol{r})$ *be a presentation of a group* G *induced by an homomorphism* $\phi : F[\boldsymbol{x}] \to G$ *(so that* $\ker\phi = \mathcal{C}(\boldsymbol{r})$*). Let also* $\theta = F[\boldsymbol{x} \cup \boldsymbol{y}] \to F[\boldsymbol{x}]$ *be a retraction, i.e, an homomorphism such that* $\theta(x) = x$ *for every* $x \in F[\boldsymbol{x}]$.

Then the kernel of $\phi\theta$ *is the set* $C = \mathcal{C}(\boldsymbol{r} \cup \{y\theta(y)^{-1} : y \in \boldsymbol{y}\})$.

Proof: Lets begin by proving that $C \subset \ker(\phi\theta)$. We just have to see that, for every $r \in \mathbf{r}$, $\phi\theta(r) = \phi(r) = 1$ and that, for every $y \in \mathbf{y}$, $\phi\theta(y\theta(y^{-1})) = \phi(\theta(y))\phi(\theta(\theta(y^{-1})) = \phi(\theta(y))\phi(\theta(y^{-1})) = 1$ (because $\theta\theta = \theta$). So $C \subset \ker\phi\theta$.

Lets now prove that $C \supset \ker(\phi\theta)$. Let $Y = \{y\theta(y)^{-1} : y \in \mathbf{y}\}$. To do this we will use squares.

Take $u \in \ker(\phi\theta)$, that is, $u \in F[\mathbf{x} \cup \mathbf{y}]$ such that $\theta(u) \in \mathcal{C}(\mathbf{r})$. We wish to prove that $u \in C = \mathcal{C}(\mathbf{r} \cup Y)$, i.e, that the square $\begin{array}{c} \boxed{\;u\;} \\ u \end{array}$ is in $\mathcal{SC}(\mathbf{r} \cup Y)$. Notice that

$$\begin{array}{c} \boxed{\;u\;} \\ u \end{array} = \begin{array}{c} \boxed{\;\theta(u)\;} \\ \theta(u) \\ \boxed{u\theta(u)^{-1}\;\;} \\ u \end{array}.$$

Because $\theta(u) \in \mathcal{C}(\mathbf{r})$, we also have $\theta(u) \in \mathcal{C}(\mathbf{r} \cup Y)$, so $\begin{array}{c} \boxed{\;\theta(u)\;} \\ \theta(u) \end{array}$ is in $\mathcal{SC}(\mathbf{r} \cup Y)$. Also, since u can be written as a product of elements of $\mathbf{x} \cup \mathbf{y}$, $\begin{array}{c} \theta(u) \\ \boxed{u\theta(u)^{-1}\;\;} \\ u \end{array}$ can be written as a horizontal product of squares of the form $\begin{array}{c} \theta(x) \\ \boxed{x\theta(x)^{-1}\;\;} \\ x \end{array} = \begin{array}{c} x \\ \boxed{\;\;\;\;} \\ x \end{array}$, with $x \in \mathbf{x}$, and $\begin{array}{c} \theta(y) \\ \boxed{y\theta(y)^{-1}\;\;} \\ y \end{array}$, with $y \in \mathbf{y}$. Therefore $\begin{array}{c} \theta(u) \\ \boxed{u\theta(u)^{-1}\;\;} \\ u \end{array}$ is in $\mathcal{SC}(\mathbf{r} \cup Y)$. Finally, by Proposition 2.3.2, we have that $\begin{array}{c} \boxed{\;u\;} \\ u \end{array}$ is in $\mathcal{SC}(\mathbf{r} \cup Y)$, and we conclude that $u \in \mathcal{C}(\mathbf{r} \cup Y)$. $\square$

Theorem 1.4.3 *If* $(\boldsymbol{x} : \boldsymbol{r}) \underset{g}{\overset{f}{\rightleftarrows}} (\boldsymbol{y} : \boldsymbol{s})$ *is an equivalence between finite presentations, then f and g can be written as finite compositions of Tietze equivalences.*

Proof: We begin by doing the proof assuming $\mathbf{x}$ and $\mathbf{y}$ are disjoint. Consider the following diagram:

$$\begin{tikzcd}
& F[\mathbf{x} \cup \mathbf{y}] \arrow[dl, shift left, "\rho"] \arrow[dr, shift left, "\sigma"] & \\
F[\mathbf{x}] \arrow[ur, shift left, "\tau"] \arrow[rr, shift left, "f"] \arrow[d, "\gamma"'] & & F[\mathbf{y}] \arrow[ul, shift left, "\upsilon"] \arrow[ll, shift left, "g"] \arrow[d, "\gamma"] \\
|\mathbf{x} : \mathbf{r}| \arrow[rr, shift left, "f_*"] & & |\mathbf{y} : \mathbf{s}| \arrow[ll, shift left, "g_*"]
\end{tikzcd}$$

where τ and υ are inclusions and ρ and σ are retractions defined in such a way that $\rho(y) = g(y)$ for every $y \in \mathbf{y}$ and $\sigma(x) = f(x)$ for every $x \in \mathbf{x}$. This way $f = \sigma\tau$ and $g = \rho\upsilon$.

Let $\mathbf{a} = \{x\sigma(x)^{-1} : x \in \mathbf{x}\}$ and $\mathbf{b} = \{y\rho(y)^{-1} : y \in \mathbf{y}\}$.

It is clear that ρ and τ induce presentation maps $(\mathbf{x} : \mathbf{r}) \underset{\rho}{\overset{\tau}{\rightleftarrows}} (\mathbf{x} \cup \mathbf{y} : \mathbf{s} \cup \mathbf{b})$ which can be factored into a finite number of Tietze equivalences of types $\mathbf{II}'$ and $\mathbf{II}$, respectively.

Similarly, σ and υ induce presentation maps $(\mathbf{y} : \mathbf{s}) \underset{\sigma}{\overset{\upsilon}{\rightleftarrows}} (\mathbf{x} \cup \mathbf{y} : \mathbf{r} \cup \mathbf{a})$ which can be factored into a finite number of Tietze equivalences of types $\mathbf{I}'$ and $\mathbf{I}$, respectively.

Now lets check that $\mathbf{s} \cup \mathbf{a} \subset \mathcal{C}(\mathbf{r} \cup \mathbf{b})$:

Since, by the lemma, $\mathcal{C}(\mathbf{r} \cup \mathbf{b}) = \ker(\gamma\rho)$, we only have to check that $\gamma\rho(\mathbf{s} \cup \mathbf{a}) = \{1\}$. This is true because, for $s \in \mathbf{s}$, $\gamma\rho(s) = \gamma g\sigma(s) = \gamma g(s) = 1$ and, for $x \in \mathbf{x}$, $\gamma\rho(x\sigma(x)^{-1}) = \gamma g\sigma(x\sigma(x)^{-1}) = \gamma g(\sigma(x)\sigma(x)^{-1}) = 1$. So, as pretended, $\mathbf{s} \cup \mathbf{a} \subset \mathcal{C}(\mathbf{r} \cup \mathbf{b})$.

Similarly $\mathbf{r} \cup \mathbf{b} \subset \mathcal{C}(\mathbf{s} \cup \mathbf{a})$.

Hence, we have the following diagram:

$$\begin{tikzcd}
(\mathbf{x} \cup \mathbf{y} : \mathbf{r} \cup \mathbf{b}) \arrow[r, shift left, "\alpha"] \arrow[d, shift left, "\rho"] & (\mathbf{x} \cup \mathbf{y} : \mathbf{r} \cup \mathbf{b} \cup \mathbf{s} \cup \mathbf{a}) \arrow[l, shift left, "\alpha'"] \arrow[r, shift left, "\beta'"] & (\mathbf{x} \cup \mathbf{y} : \mathbf{s} \cup \mathbf{a}) \arrow[l, shift left, "\beta"] \arrow[d, shift left, "\sigma"'] \\
(\mathbf{x} : \mathbf{r}) \arrow[u, shift left, "\tau"] \arrow[rr, shift left, "f"] & & (\mathbf{y} : \mathbf{s}) \arrow[u, shift left, "\upsilon"'] \arrow[ll, shift left, "g"]
\end{tikzcd}$$

where $\alpha, \alpha', \beta, \beta'$ are presentation equivalences carried by the identity homomorphism. They are such that α and β can be decomposed in a finite number of Tietze $\mathbf{I}$ equivalences and α' and β' can be decomposed in a finite number of Tietze $\mathbf{I}'$ equivalences.

Observing that $f = \upsilon\beta\alpha'\rho$ and $g = \tau\alpha\beta'\sigma$, we see that f and g can be factored in a finite number of Tietze equivalences.

Now lets take care of the case where $\mathbf{x}$ and $\mathbf{y}$ are not disjoint. Take a set $\mathbf{z}$ of generators disjoint from $\mathbf{x} \cup \mathbf{y}$ and in one to one correspondence with $\mathbf{x}$. This correspondence induces an isomorphism h_1 between $F[\mathbf{x}]$ and $F[\mathbf{z}]$, with inverse $h_2 = h_1^{-1}$.

Let $\mathbf{t} = h_1(\mathbf{r})$. Define also the homomorphisms $k_1 = fh_2$ and $k_2 = h_1 g$.

$$
\begin{tikzcd}
 & (\mathbf{z} : \mathbf{t}) \arrow[dl, shift left, "h_2"] \arrow[dr, shift left, "k_1"] & \\
(\mathbf{x} : \mathbf{r}) \arrow[ur, shift left, "h_1"] \arrow[rr, shift left, "f"] & & (\mathbf{y} : \mathbf{s}) \arrow[ul, shift left, "k_2"] \arrow[ll, shift left, "g"]
\end{tikzcd}
$$

It is easy to see that the pair h_1, h_2 is an equivalence. Let's check that k_1, k_2 is also an equivalence. First, in order to see that k_1 and k_2 are indeed presentation maps, just see that $k_1(\mathbf{t}) = fh_2(\mathbf{t}) = fh_1^{-1}(\mathbf{t}) = f(\mathbf{r}) \subset \mathcal{C}(\mathbf{s})$ and $k_2(\mathbf{s}) = h_1 g(\mathbf{s}) \subset h_1(\mathcal{C}(\mathbf{r})) = \mathcal{C}(\mathbf{t})$. All that is left is to verify that $k_1 k_2 = fh_2 h_1 g \simeq f 1 g \simeq fg \simeq 1$ and that $k_2 k_1 = h_1 g f h_2 \simeq h_1 1 h_2 \simeq h_1 h_2 \simeq 1$. So k_1, k_2 is an equivalence.

Finally, apply the first part of the proof to the equivalences $(\mathbf{x} : \mathbf{r}) \underset{h_2}{\overset{h_1}{\rightleftarrows}} (\mathbf{z} : \mathbf{t})$ and $(\mathbf{y} : \mathbf{s}) \underset{k_1}{\overset{k_2}{\rightleftarrows}} (\mathbf{z} : \mathbf{t})$ to conclude that h_1, h_2, k_1, k_2 can be factored in a finite number of Tietze equivalences. Then, since $f = k_1 h_1$ and $g = h_2 k_2$, f and g can also be factored in a finite number of Tietze equivalences. $\square$

Now, as an example, we are going to show how two presentations are equivalent by using Tietze equivalences to get from one to another. Lets begin with $(x, y : x^3, y^2, (xy)^2)$ and show that $(y, z : y^2, z^2, (yz)^3)$ is an equivalent presentation. Notice that, setting $z = xy$, we have $x = zy^{-1} = zy$, therefore, from the relations $x^3 = 1$ and $(xy)^2 = 1$, we get $(yz)^3 = 1$ and $z^2 = 1$. Notice also that $x^3 = 1$ can be derived from $(yz)^3 = 1$ and $x = yz$ and that $z = xy$ can be derived from $x = zy$ and $y = y^{-1}$. This way we can go through with the following series of Tietze transformations:

$$
\begin{tikzcd}
(x, y : x^3, y^2, (xy)^2) \arrow[d, "\textsc{ii}"] \\
(x, y, z : x^3, y^2, (xy)^2, z(xy)^{-1}) \arrow[d, "\textsc{i} \text{ twice}"] \\
(x, y, z : x^3, y^2, (xy)^2, z(xy)^{-1}, z^2, x(zy)^{-1}) \arrow[d, "\textsc{i}' \text{ twice}"] \\
(x, y, z : x^3, y^2, z^2, x(zy)^{-1}) \arrow[d, "\textsc{i} \text{ and } \textsc{i}'"] \\
(x, y, z : y^2, z^2, x(zy)^{-1}, (zy)^3) \arrow[d, "\textsc{ii}'"] \\
(y, z : y^2, z^2, (zy)^3)
\end{tikzcd}
$$

These are all presentations of the group of symmetries of the triangle. In the first, x can be seen as a rotation and y as a reflection; as for the last one, y and z can be both seen as reflections.

2 2-Groups Presentations

We will now work with 2-groups; and from the various equivalent definitions that are of 2-groups, we will work from the point of view of crossed modules. The point here would be to find a way to describe '2-group presentations', which would first require a description of a 'free 2-group' and 'relations on a free 2-group'. We are still unable to go that far; we will stop with the definition and construction of free crossed modules, given by Whitehead. Then we will present some ideas that might lead to relations on a free crossed module.

2.1 A brief introduction to crossed modules

A group can be seen as a category with only one object in which all morphisms are invertible. So, the following definition makes sense:

Definition 2.1.1 *A* 2-group *is defined as a 2-category with only one object in which all morphisms/2-morphisms are invertible.*

For more details regarding this definition see [3]. We will now talk about crossed modules, which are also equivalent to 2-groups (for a proof of the equivalence between 2-groups and crossed modules see [7]).

Definition 2.1.2 *A* crossed module *consists of $(G, H, \partial, \triangleright)$, where G and H are groups, $\triangleright$ is an action of G on H by automorphisms, and $\partial : H \to G$ is an homomorphism, satisfying the following equations:*

$$\partial(g \triangleright \eta) = g\partial(\eta)g^{-1}$$

and

$$\partial(\eta) \triangleright \zeta = \eta\zeta\eta^{-1}$$

for all $g \in G$ and $\eta, \zeta \in H$.

It is easy to see that it is a consequence of the definition that $\partial(H)$ is a normal subgroup of G, and also that $\ker \partial$ is contained in the center of H. We now define *crossed module homomorphisms.*

Definition 2.1.3 *Let $C = (G, H, \partial, \triangleright)$ and $C' = (G', H', \partial', \triangleright')$ be crossed modules. A* crossed module homomorphism *$F : C \to C'$ consists of a pair (f_1, f_2) of group homomorphisms $f_1 : G \to G'$ and $f_2 : H \to H'$ such that the diagram*

$$\begin{tikzcd} H \arrow[r, "f_2"] \arrow[d, "\partial"'] & H' \arrow[d, "\partial'"] \\ G \arrow[r, "f_1"'] & G' \end{tikzcd}$$

commutes and such that $f_2(g \triangleright \eta) = f_1(g) \triangleright' f_2(\eta)$ for every $g \in G$ and $\eta \in H$.

Crossed module homomorphisms can be composed. If $(f_1, f_2) = F : C \to C'$ and $(f_1', f_2') = F' : C' \to C''$ are crossed module homomorphisms, their composition $FF' : C \to C''$ is the crossed module homomorphism that consists of the pair $(f_1 f_1', f_2 f_2')$.

A crossed module isomorphism is a crossed module homomorphism (f_1, f_2) in which f_1 and f_2 are both isomorphisms.

2.2 The double groupoid of a crossed module

A construction that we think might be of use is the *double groupoid* of a crossed module.

A double groupoid is a special case of a double category (its a double category in which all morphisms are invertible). However, we will not give a full definition of double category. Its enough to mention that, intuitively, a double category consists of a set of objects O, two sets $\mathcal{H}$ and $\mathcal{V}$ of morphisms, denoted horizontal and vertical, together with structural maps making then horizontal and vertical categories with objects O, and also a set S of squares which form the morphisms of two category structures whose objects are $\mathcal{H}$ and $\mathcal{V}$, respectively.

For us, it is enough to describe the double groupoid associated to a crossed module.

Definition 2.2.1 *Given a crossed module $C = (G, H, \partial, \triangleright)$, the double groupoid associated to C, denoted $\mathcal{D}(C)$, is the double groupoid consisting of:*

- *A unique object : $O = \{\star\}$;*
- *Horizontal and vertical morphisms $\mathcal{H} = \mathcal{V} = G$;*
- *The set of all squares of the form*

$$\begin{tikzcd} {} \arrow[r, dash, "g_3"] \arrow[d, dash, "g_4"'] \arrow[dr, phantom, "\eta"] & {} \arrow[d, dash, "g_2"] \\ {} \arrow[r, dash, "g_1"'] & {} \end{tikzcd}$$

  where $g_1, g_2, g_3, g_4 \in G$, $\eta \in H$ and $\partial(\eta) = g_1 g_2 g_3^{-1} g_4^{-1}$;
- *Horizontal and vertical composition of squares given by*

$$\begin{tikzcd} {} \arrow[r, dash, "g_3"] \arrow[d, dash, "g_4"'] \arrow[dr, phantom, "\eta_1"] & {} \arrow[r, dash, "g_7"] \arrow[d, dash, "g_2"] \arrow[dr, phantom, "\eta_2"] & {} \arrow[d, dash, "g_6"] \\ {} \arrow[r, dash, "g_1"'] & {} \arrow[r, dash, "g_5"'] & {} \end{tikzcd} = \begin{tikzcd} {} \arrow[r, dash, "g_3 g_7"] \arrow[d, dash, "g_4"'] \arrow[dr, phantom, "\eta_1(g_4 g_3 g_2^{-1}) \triangleright \eta_2"] & {} \arrow[d, dash, "g_6"] \\ {} \arrow[r, dash, "g_1 g_5"'] & {} \end{tikzcd} = \begin{tikzcd} {} \arrow[r, dash, "g_3 g_7"] \arrow[d, dash, "g_4"'] \arrow[dr, phantom, "(g_1 \triangleright \eta_2)\eta_1"] & {} \arrow[d, dash, "g_6"] \\ {} \arrow[r, dash, "g_1 g_5"'] & {} \end{tikzcd}$$

  (notice that the two expressions for the H element in the horizontal compositions are the same, using $\partial(\eta_1^{-1}) \triangleright \eta_2 = \eta_1^{-1} \eta_2 \eta_1$), and

$$\begin{tikzcd} {} \arrow[r, dash, "g_3"] \arrow[d, dash, "g_4"'] \arrow[dr, phantom, "\eta_1"] & {} \arrow[d, dash, "g_2"] \\ {} \arrow[r, dash, "g_1"] \arrow[d, dash, "g_5"'] \arrow[dr, phantom, "\eta_2"] & {} \arrow[d, dash, "g_7"] \\ {} \arrow[r, dash, "g_6"'] & {} \end{tikzcd} = \begin{tikzcd} {} \arrow[r, dash, "g_3"] \arrow[d, dash, "g_5 g_4"'] \arrow[dr, phantom, "\eta_2(g_5 \triangleright \eta_1)"] & {} \arrow[d, dash, "g_7 g_2"] \\ {} \arrow[r, dash, "g_6"'] & {} \end{tikzcd} .$$

  These operations do satisfy the interchange law.

Also, squares in $\mathcal{D}(\mathcal{G})$ have horizontal and vertical inverses, which are respectively given by (for the square in the definition):

$$\begin{tikzcd} \cdot \arrow[r, dash, "g_3^{-1}"] \arrow[d, dash, "g_2"'] \arrow[dr, phantom, "\eta^{-h}"] & \cdot \arrow[d, dash, "g_4"] \\ \cdot \arrow[r, dash, "g_1^{-1}"'] & \cdot \end{tikzcd} = \begin{tikzcd} \cdot \arrow[r, dash, "g_3^{-1}"] \arrow[d, dash, "g_2"'] \arrow[dr, phantom, "g_1^{-1} \rhd \eta^{-1}"] & \cdot \arrow[d, dash, "g_4"] \\ \cdot \arrow[r, dash, "g_1^{-1}"'] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, dash, "g_1"] \arrow[d, dash, "g_4^{-1}"'] \arrow[dr, phantom, "\eta^{-v}"] & \cdot \arrow[d, dash, "g_2^{-1}"] \\ \cdot \arrow[r, dash, "g_3"'] & \cdot \end{tikzcd} = \begin{tikzcd} \cdot \arrow[r, dash, "g_1"] \arrow[d, dash, "g_4^{-1}"'] \arrow[dr, phantom, "g_4^{-1} \rhd \eta^{-1}"] & \cdot \arrow[d, dash, "g_2^{-1}"] \\ \cdot \arrow[r, dash, "g_3"'] & \cdot \end{tikzcd}$$

2.3 Free crossed modules

Just like we did for groups, we will now give the definition of *free crossed module* and then present the construction of free crossed modules.

Definition 2.3.1 *Let $\mathcal{G} = (G, H, \partial, \rhd)$ be a crossed module. We say that a set $\{h_i : i \in I\}$ of elements of H is a free basis of $\mathcal{G}$ if, for every crossed module $\mathcal{G}' = (G', H', \partial', \rhd')$, every set $\{h'_i : i \in I\}$ of elements of H' and every homomorphism $\psi : G \to G'$ such that $\psi(\partial(h_i)) = \partial'(h'_i)$, there exists a unique homomorphism $\phi : H \to H'$ such that $\phi(h_i) = h'_i$ and (ψ, ϕ) is a crossed module homomorphism.*

$\mathcal{G}$ is called a free crossed module *if it has a free basis.*

We now describe how to obtain the free crossed module $\mathcal{F}(\omega : K \to G)$ generated by a function $\omega : K \to G$. Here G is any group and K any set.

Denote by $F^G(K)$ the free group generated by $K \times G$. Consider the (unique) action $\rhd'$ of G in $F^G(K)$ that satisfies $g_1 \rhd' (k, g_2) = (k, g_1 g_2)$, as well as the homomorphism $d : F^G(K) \to G$ given by $d(k, g) = g\omega(k)g^{-1}$. With this, $(G, F^G(K), d, \rhd')$ satisfies all but one required property to be a crossed module (it does not satisfy $d(u) \rhd' v = uvu^{-1} \quad \forall u, v \in F^G(K)$).

So we must introduce the set of relators $R = \{uvu^{-1}(d(u) \rhd' v)^{-1} : u, v \in F^G(K)\}$ in $F^G(K)$. Denote by $FC(\omega)$ the quotient group $F^G(K)/\mathcal{C}(R)$. Denote also by $\partial : FC(\omega) \to G$ the homomorphism induced by d, and by $\rhd$ the action of G on $FC(\omega)$ induced by $\rhd'$.

Now, $\mathcal{F}(\omega : K \to G) = (G, FC(\omega), \partial, \rhd)$ is a crossed module, called the free crossed module generated by $\omega : K \to G$.

Whitehead also proved that $\mathcal{F}(\omega : K \to G)$ is in fact a free crossed module with basis $\{p(k, 1) : k \in K\}$, where $p : F^G(K) \to FC(\omega)$ is the natural projection. A proof can be found in the literature.

We finish our discussion here. A possible approach in order to define a 'crossed module presentation' would be to describe the free double groupoid, associated to a free crossed module. The squares in that free double groupoid would be equivalence classes and it should be possible to show that the interchange law as well the square compositions would remain valid. Then it would be a question of defining relations among squares, and consequently, relations in crossed modules.

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