A new class of 2-D TQFT from finite 2-groups

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Abstract

In this article, I briefly define what a TQFT (Topological Quantum Field Theory) is and then I present a 2-D TQFT model using finite groups (the Dijkgraaf-Witten model). Then, I define the concept of a 2-group and show how it naturally fits to geometry. Finally, using 2-groups, I build a TQFT and study its properties to obtain invariants of surfaces with boundary and of closed surfaces.

I. Brief Introduction to TQFT

Every Quantum Field Theory (QFT) is characterized by an evolution operator that controls how a certain quantum state evolves with time. Usually, the evolution operator takes into account many physical properties of space-time, such as electromagnetic fields, gravitational fields and space configurations (e.g: barriers). Therefore, it may be extremely difficult to calculate, and leads, not so rarely, to mathematical complications.

A TQFT avoid these complications, since it is a QFT where the evolution operator only depends on the topology of space-time, in other words on whether it has holes in it (e.g: torus) or not (e.g: sphere).

1.1 Mathematical definition

A $(d+1)$-dimensional TQFT (where d is the number of spatial dimensions) is a functor from the category of $(d+1)$-Cobord to Vect($\mathbb{K}$) preserving certain structures. Vect($\mathbb{K}$) is the category of finite dimensional vector spaces over $\mathbb{K}$.

The functor is built the following way: to each d-dimensional closed space Σ (objects of $(d+1)$-Cobord), assign a vector space over $\mathbb{K}$, V_Σ (objects of Vect($\mathbb{K}$)), and to each $(d+1)$-dimensional space-time from Σ to Σ' (morphisms of $(d+1)$-Cobord), assign a linear transformation $Z_M : V_\Sigma \to V_{\Sigma'}$. Z_M is the evolution operator and only depends on the topological class of M. Finally, the functor assigns to each "gluing" of two$(d+1)$-dimensional space-times (which corresponds to the composition of cobordisms) the composition of linear transformations.

Furthermore, a TQFT has to verify some axioms including:

$$V_\varnothing = \mathbb{K} \tag{1}$$
$$V_{\Sigma_1 \amalg \Sigma_2} = V_{\Sigma_1} \otimes V_{\Sigma_2} \tag{2}$$

where $\amalg$ is the disjoint union, $\varnothing$ is the empty set and $\otimes$ is the ordinary tensor product.

A TQFT therefore defines invariants for any closed $(d+1)$-dimensional manifold M, since it can be considered as a cobordism from $\varnothing$ to $\varnothing$. Now, $Z_M \in Hom(\mathbb{K}, \mathbb{K}) \sim \mathbb{K}$. Z_M can then be represented by a numerical constant which depends only on M. For a manifold M with boundary, Z_M is an operator represented by a tensor and depends on $V_{\partial M}$.

II. 2-D TQFT from finite groups - Dijkgraaf-Witten model

From now on, my discussion will be based on 2-D TQFT's, that is, $d = 1$.

2.1 Triangulated surfaces and polygons

Consider the case where any Σ is a polygon and where M is a triangulated surface. Now, I will assign to each vertex of any boundary polygon Σ a positive integer in order to induce an edge orientation. Then, to each oriented edge, I assign an element of a given finite group G (those group elements can be viewed as transports). Let's call such an assignment a colouring of Σ. Then, V_Σ is the vector space generated by all possible colourings of Σ. This way, if Σ has n edges, V_Σ has dimension $|G|^n$, where $|G|$ is the order of G, that is, the number of elements of G. If we have a triangulated surface M whose boundaries are a triangle Σ_1 ($n = 3$) and a pentagon Σ_2 ($n = 5$), then Z_M is represented by a $|G|^5 \times |G|^3$ matrix, because V_{Σ_1}, chosen as the starting vector space, has dimension $|G|^3$ and V_{Σ_2}, chosen as the ending vector space, has dimension $|G|^5$.

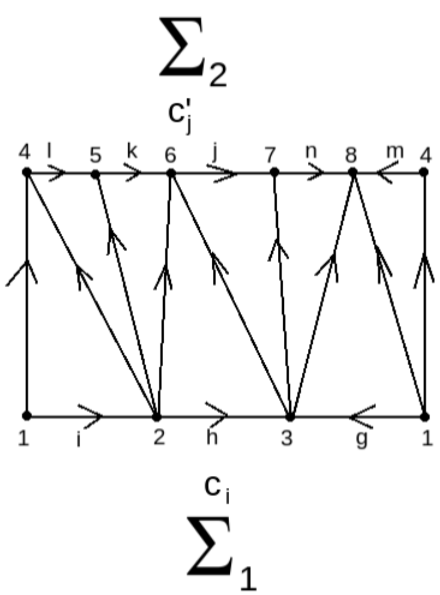

Figure 1: *Example of a triangulated surface M with two polygons as boundaries, Σ_1 and Σ_2. $g, h, i, j, k, l, m, n \in G$. The surface has been open along edge 1-4 so that the figure is clearer*

Just as a matter of notation, suppose c_i, $1 \leq i \leq |G|^3$, and c'_j, $1 \leq j \leq |G|^5$, are two colourings of, respectively, Σ_1 and Σ_2. Then the corresponding evolution operator matrix element in the ith row and jth column is written $\langle c_i \, | Z_M | \, c'_j \rangle$, adopting Dirac notation.

Let us now look more closely at triangulated surfaces. Two triangulated surfaces may be topologically equivalent but their triangulations may differ (e.g.: any Platonic solid is topologically equivalent to a sphere, but may present different triangulations between them, like a tetrahedron and an octahedron). For this kind of triangulated surfaces there exist two moves that allow us to pass from one triangulation to another: the Pachner moves (after Udo Pachner). The two Pachner moves are the following:

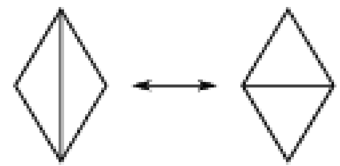

Figure 2: *First Pachner move*

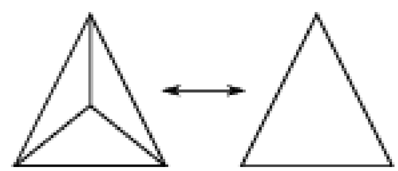

Figure 3: *Second Pachner move*

With those two moves, we define an equivalence between triangulated surfaces that have the same boundary/boundaries. For the theory's consistency, the evolution operator should be invariant under Pachner moves.

2.2 Explicit expression for the evolution operator Z_M

It remains to find an explicit expression for the evolution operator. First, let us extend the concept of colouring to a triangulated surface. If we have a triangulated surface M, we assign a positive integer to every vertex of M, in order to induce an edge orientation. Then, to each oriented edge (not in the boundary), we assign a G element such that a multiplicative constraint holds for each triangle. Such a set of assignments we call colouring.

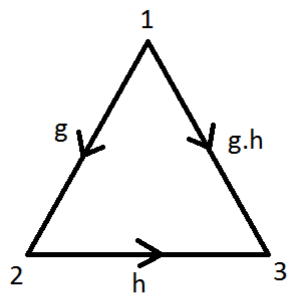

Figure 4: *Condition for a triangulated surface colouring*

Now, the explicit expression for the invariant is:

$$\langle c_i \,| Z_M |\, c'_j \rangle = \frac{1}{|G|^v} \sum_{colourings} \tag{3}$$

where c_i and c'_j are the boundary colourings, v is the total number of vertices in M (including vertices on the boundary), and $\sum_{colourings}$ is the sum over all possible M colourings consistent with c_i and c'_j. Let us now verify if this evolution operator remains invariant under Pachner moves.

Suppose we have a triangulated surface M, and we fix colourings of the boundary c_i and c'_j, and a surface colouring, c_1. Now, we focus on a specific triangle of M, with edge colourings g_1, g_2 and g_3 (which depends on g_1 and g_2). Using the second Pachner move, we add another vertex and three more edges (with colourings j_1, j_2 and j_3) inside that triangle. How many free combinations of j_1, j_2 and j_3 are there? Our rule for triangle colourings implies that, if two egde colourings are fixed, then there is only one possibility for the third edge colouring. For instance, if we fix j_1, then, because g_1 and g_2 are fixed, there is only one possibility for j_2 and j_3. Therefore, if we add a vertex, we add one more degree of freedom for colourings. Thus the number of colourings will be multiplied by $|G|$. For the new invariant Z'_M, we have:

$$\begin{aligned}
\langle c_i \,| Z'_M |\, c'_j \rangle &= \frac{1}{|G|^{v+1}} \sum_{colourings}{}' && (4)\\
&= \frac{1}{|G|^{v+1}} |G| \sum_{colourings} && (5)\\
&= \frac{1}{|G|^{v}} \sum_{colourings} && (6)\\
&= \langle c_i \,| Z_M |\, c'_j \rangle && (7)
\end{aligned}$$

where v is the number of vertices of the original triangulation, $\sum_{colourings}$ is the number of colourings of the original triangulation and $\sum_{colourings}{}'$ is the number of colourings of the "new" triangulation (after the Pachner move).

For the first Pachner move, the invariance of the evolution operator is trivial (it suffices to show that the number of possible colourings is the same).

2.3 Calculus of the evolution operator for a triangulated closed surface - the tetrahedron

To make these examples more concrete, let us calculate the evolution operator Z_t for a tetrahedron. Notice that, as the tetrahedron is a closed surface, Z_t will be a scalar (Z_t is, strictly speaking, only a shorthand for $\langle \varnothing \,| Z_M |\, \varnothing \rangle$). First, we assign an orientation to the tetrahedron. Let g_{ij} ($1 \leq i, j \leq 4$) be the colouring of the edge that goes from vertex i to vertex j ($i < j$). The conditions for having a colouring are given by:

$$\begin{aligned}
g_{13} &= g_{12}.g_{23} && (8)\\
g_{14} &= g_{13}.g_{34} && (9)\\
g_{24} &= g_{23}.g_{34} && (10)\\
g_{14} &= g_{12}.g_{24} && (11)
\end{aligned}$$

Nevertheless, only three of these equations are independent (for instance, if you replace, in the last equation, g_{24} by the result of the third equation, $g_{23}.g_{34}$, and then use the first equation to replace $g_{12}.g_{23}$ by g_{13}, you get the second equation. We have six free choices and three independent conditions, therefore the number of possible colourings is:

$$\sum_{colourings} = |G|^{6-3} = |G|^3 \tag{12}$$

The evolution operator Z_t is, therefore:

$$Z_t = \frac{1}{|G|^4} \sum_{colourings} = \frac{|G|^3}{|G|^4} = \frac{1}{|G|} \tag{13}$$

Z_t is a characteristic invariant of the tetrahedron, and should be also invariant for all topologically equivalent surfaces (e.g.: sphere). However, a sphere is not a triangulated surface, so we cannot calculate its invariant with the tools presented above. We need other tools that will allow us to consider any kind of piecewise smooth 2-manifold for calculating the invariant. We introduce those tools in the next section.

2.4 CW complexes

A CW-complex is a geometrical structure made of basic blocks, the cells. For instance, to "build" a two dimensional surface, one needs three types of cells. The first type of cell, 0-cell, is a discrete point. The set of all 0-cells is called the 0-skeleton. Then, the 1-cells, or one dimensional "lines" are attached to the 0-cells. We then define the 1-skeleton to be the union of the 0-skeleton and the set of 1-cells. Finally, we attach 2-cells, which are two dimensional disks, to the 1-skeleton (that is, we identify the boundaries of each 2-cell with part of the 1-skeleton). Thus, we get a two dimensional CW complex made of 0-cells, 1-cells and 2-cells.

Note that triangulated surfaces are also CW complexes. In fact, to build a triangle like the ones used above, we need three 0-cells that form the 0-skeleton, then we attach three 1-cells to the 0-skeleton and finally we "glue" a 2-cell in order to fill the triangle.

Therefore, we may say that two CW decompositions are equivalent if we can go from one to another using the following set of moves:

- add/remove vertices (0-cells) and edges (1-cells) until we get a triangulated surface
- do the necessary Pachner moves
- add/remove vertices and edges until we get the other CW decomposition

Even though there are some subtleties in the applicability of those moves, we will use this definition of equivalence of decompositions. It is not difficult to show that the evolution operator Z_M (defined in exactly the same way as for triangulated surfaces) for a CW complex remains invariant under these moves.

To show the invariance of Z_M for equivalent decompositions, we are going to compute it for a sphere decomposed into two 0-cells (vertices), two 1-cells and two 2-cells (see figure).

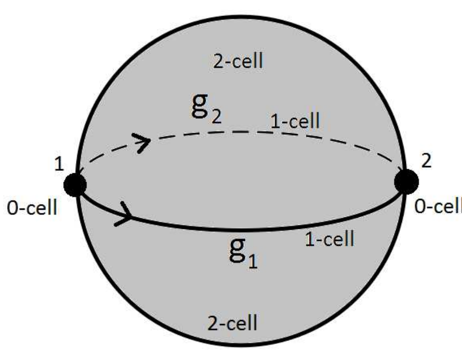

Figure 5: *Sphere decomposition in cells*

The multiplicative constraint implies that $g_1 = g_2$. Therefore, we can freely choose one edge colouring, and the other will have to be equal in order to have a valid colouring. This means that $\sum_{colourings} = |G|$. Thus:

$$Z_{sphere} = \frac{1}{|G|^v} \sum_{colourings} = \frac{1}{|G|^2}|G| = \frac{1}{|G|} \qquad (14)$$

Note that $Z_{sphere} = Z_{tetrahedron}$, which makes sense since the two closed surfaces are topologically equivalent! The results obtained with CW complexes are therefore consistent with the results obtained with triangulated surfaces. However, CW complexes are much simpler and more elegant than triangulated surfaces, and allow us to compute the evolution operator of surfaces in a much easier way.

2.5 Computation of the evolution operator for many surfaces

We will now try and provide a kind of "database" with the evolution operator for many closed and open surfaces.

2.5.1 Closed surfaces

We already computed the evolution operator for a sphere and got:

$$Z_{sphere} = \frac{1}{|G|} \qquad (15)$$

We will now find out what is the evolution operator for a torus (which is not topologically equivalent to a sphere). A torus colouring can be represented using a square whose right and left edges are identified and whose upper and lower edge are also identified. Moreover, the four vertices are identified as one (see figure).

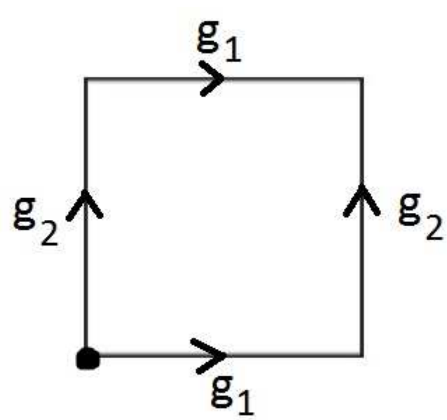

Figure 6: *Torus representation*

Therefore, the condition for having a valid colouring is:

$$g_1 g_2 g_1^{-1} g_2^{-1} = 1 \Leftrightarrow g_1 g_2 = g_2 g_1 \tag{16}$$

Thus, since $v = 1$, the invariant for the torus is given by:

$$Z_{torus} = \frac{1}{|G|^v} \sum_{colourings} \tag{17}$$

$$\Leftrightarrow Z_{torus} = \frac{1}{|G|} \# \{g_1, g_2 \in G : g_1 g_2 = g_2 g_1\} \tag{18}$$

Now, as we have computed the invariant for two closed surfaces, we will compute the evolution operator for open surfaces. Note that, for open surfaces, the evolution operator will also depend on the fixed boundary Σ of the surface.

2.5.2 Open surfaces

Let us begin by computing the evolution operator for a disk. Let us just point out that it is impossible to build a disk out of simplicial complexes (or triangulated surfaces), since a disk has a single curved boundary edge. However, using CW complexes, it is easy to find a decomposition for a disk. In fact, one needs one 0-cell (one vertex), one 1-cell (one edge) that starts and finishes at the same point, and one 2-cell (disk surface) that fills the circle (see figure). However, one may say that, as a disk is topologically equivalent to a triangle, the evolution operator will be the same. This is not the case, because the two surfaces differ on the respective boundary Σ. In fact, while $V_{\Sigma_{Disk}}$ has dimension $|G|$ (just one edge), $V_{\Sigma_{Triangle}}$ has dimension $|G|^3$ and therefore the evolution operators cannot be equal. So, even though a different number of vertices and edges in the interior of the surface keeps the evolution operator unchanged (see Pachner moves), if the number of vertices and edges is different on the boundary of the surface the evolution operator will be changed.

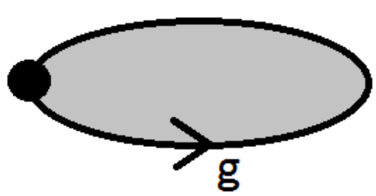

Figure 7: *Disk decomposition*

To find the evolution operator $\langle g \,| Z_{disk} |\, \varnothing \rangle$ (as a function of g, the boundary colouring) we will use the multiplicative constraint in a slightly different way. We remember that, for triangulated surfaces, when one did a closed loop, the multiplication of all oriented group colourings had to be the identity in order to have a valid colouring. This comes from the multiplicative constraint. Extending this idea to CW complexes, we notice that, in order to have a valid colouring, g has to be the identity. So, $\langle g \,| Z_{disk} |\, \varnothing \rangle = 0$ if $g \neq 1$. If $g = 1$, there is only one possible colouring (because in our decomposition there are no inside edges) and there is one vertex, therefore:

$$\langle 1 \,| Z_{disk} |\, \varnothing \rangle = \frac{1}{|G|} \tag{19}$$

We will now turn our attention to a "filled" triangle. Even if a triangle is topologically equivalent to a disk, the evolution operator will be different because the two objects have different boundaries (and therefore different vector spaces associated with them). The CW decomposition for a triangle is three vertices, three edges and a surface to fill the triangle. We induce an orientation to each edge by assigning to each vertex an integer. Let us call the boundary edge colourings g, h and i. The multiplicative constraint implies that $g = ih$ in order to have valid colourings. Therefore, $\langle g, h, i \,| Z_{triangle} |\, \varnothing \rangle = 0$ if $g \neq ih$. Moreover, as there are three vertices, if $g = ih$:

$$\langle g, h, i \,| Z_{triangle} |\, \varnothing \rangle = \frac{1}{|G|^3} \tag{20}$$

Note that, comparing with the disk, there is a $|G|^3$ in the denominator instead of $|G|$. This is due to the fact that the triangle has two more vertices than the disk, and therefore the evolution operator is multiplied by $\frac{1}{|G|^2}$.

Let's now focus our attention on the cylinder. A CW decomposition of a cylinder is two vertices, three edges (two corresponding to the boundaries, and one linking the two vertices), and one surface (the cylinder surface). Let g and h be the two boundary edge colourings (they are "fixed") and k be the third edge colouring (it is "free"), as shown in the figure. We will also use the boundary edge orientation shown in the figure (without loss of generality). To find the colouring condition, we cut the cylinder along the "k" edge and get a rectangle where we identify the "k" edges.

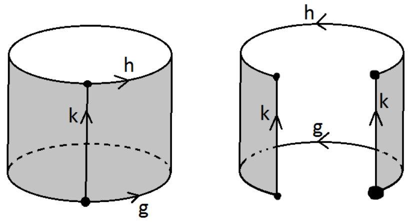

Figure 8: *The cylinder and the "cut" to get a rectangle*

If one does a loop around the rectangle, one has to get the identity. Hence, the condition is: $gkh^{-1}k^{-1} = 1$ which is the same as $kh = gk$. Therefore, since $v = 2$, we get for the evolution operator:

$$\langle h \left| Z_{cylinder} \right| g \rangle = \frac{1}{|G|^2} \# \{k \in G : kh = gk\} = \frac{1}{|G|^2} C(g,h) \tag{21}$$

where we have introduced the notation $C(g,h) := \# \left\{k \in G : khk^{-1} = g\right\}$.

This method to compute the evolution operator will be used from now on: we "colour" the surface, we cut it in order to get a plane polygon (in this case a rectangle), and then we do a closed loop around the polygon to find the condition.

We will now use this method to compute the evolution operator for another open surface, the pair of pants (see figure).

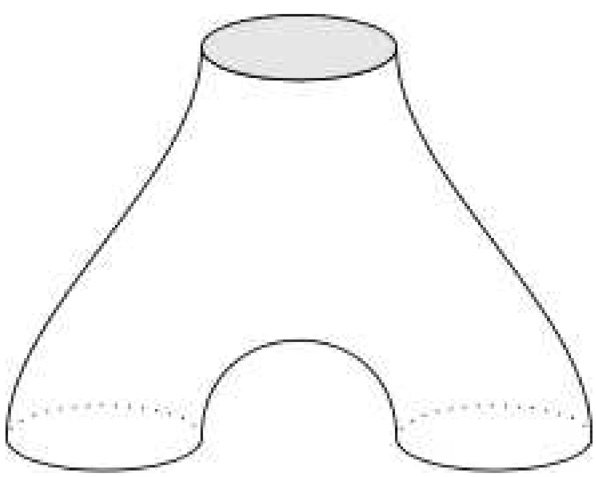

Figure 9: *A pair of pants*

A CW decomposition of a pair of pants is three vertices (one for each boundary), five edges (one for each boundary and two for linking the "upper" edge with the two "lower" edges) and one surface to fill the pair of pants. The colourings are g for the upper edge, i and h for the two lower edges and j_1 and j_2 for the inner edges. If we cut the pair of pants along the edges coloured by j_1 and j_2, we get an heptagon (polygon with seven edges).

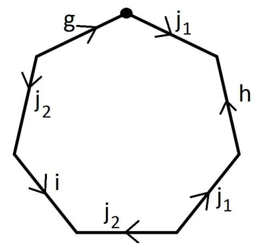

Figure 10: *The heptagon formed by cutting the pair of pants*

The condition is, therefore, given by:

$$j_1 h^{-1} j_1^{-1} j_2 i^{-1} j_2^{-1} g = 1 \tag{22}$$

Hence, since $v = 3$, we get for the evolution operator:

$$\langle i, h \left| Z_{cylinder} \right| g \rangle = \frac{1}{|G|^3} \# \left\{ j_1 h^{-1} j_1^{-1} j_2 i^{-1} j_2^{-1} g = 1 \right\} \tag{23}$$

With all those types of evolution operators, we may now find a formula for the composition of surfaces in order to verify the consistency of the theory.

2.6 Composition of surfaces

For the composition of surfaces, one would say that it is represented by the composition of the respective evolution operators. In fact, that is mostly it. However, there is a small detail that has to be specified in order to get this right. Let us illustrate that with an example. Suppose we are gluing a disk (evolution operator Z_D) and a cylinder (evolution operator Z_C), so that we get another disk, whose evolution operator is $Z_D{}'$ (see figure). The colouring of the cylinder's left edge, which is the same as the disk's edge is h and the colouring of the cylinder's right edge is g.

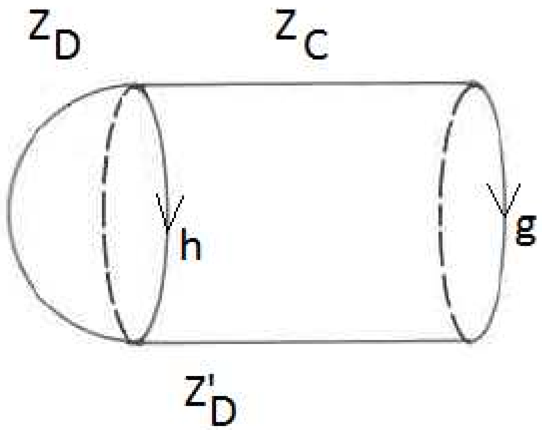

Figure 11: *An illustration of the gluing disk+cylinder*

Let us calculate the ordinary composition, which will be named Z_{comp} of the two evolution operators Z_D and Z_C:

$$\begin{aligned}
\langle \varnothing | Z_{comp} | g \rangle &= \sum_{h \in G} \langle \varnothing | Z_D | h \rangle \langle h | Z_C | g \rangle \\
\Rightarrow \quad \langle \varnothing | Z_{comp} | g \rangle &= \langle \varnothing | Z_D | 1 \rangle \langle 1 | Z_C | g \rangle \\
\Rightarrow \quad \langle \varnothing | Z_{comp} | g \rangle &= \frac{1}{|G|} \cdot \frac{1}{|G|^2} \# \{ k \in G : k = gk \} \\
\Rightarrow \quad \langle \varnothing | Z_{comp} | g \rangle &= \begin{cases} \frac{1}{|G|^2}, & g = 1 \\ 0, & g \neq 1 \end{cases} \qquad (24)
\end{aligned}$$

Since, if $g = 1$, $\# \{ k \in G : k = gk \} = |G|$ and if $g \neq 1$, $\# \{ k \in G : k = gk \} = 0$.

This way, we almost obtained the evolution operator for the resulting disk. The only difference is a factor of $\frac{1}{|G|}$. So, regarding the formula for composition of surfaces, it would be great if we could insert a factor of $|G|$ to offset the extra $\frac{1}{|G|}$ factor that appears. However, we need to justify why this extra factor of $\frac{1}{|G|}$ appears.

When two surfaces are glued along some boundary edges, the resulting glued edge counts as a single edge for computing purposes, whereas, before the surfaces were glued, there were two boundary edges. That is why, if we don't account for that edge (and vertex) loss, we won't get valid results, because the evolution operator won't take into account the loss of an edge and the loss of a vertex. As the expression for the evolution operator depends on $\frac{1}{|G|^v}$, we have to insert an extra factor of $|G|$ in the composition formula that represents this vertex loss when we glue two surfaces along an edge. If we glue two surfaces along n edges, we have n extra vertices, and to offset this we have to multiply the formula for composition by $|G|^n$.

In our original gluing problem, we therefore have to multiply Z_{comp} by $|G|$ in order to get the right $Z_D{}'$, because we are gluing along one edge. Hence:

$$\langle \varnothing | Z_D{}' | g \rangle = |G| \cdot \langle \varnothing | Z_{comp} | g \rangle = \begin{cases} \frac{1}{|G|}, & g = 1 \\ 0, & g \neq 1 \end{cases} \qquad (25)$$

Thus, the formula for gluing two surfaces M_1 and M_2 along n edges is:

$$Z_M = |G|^n \cdot Z_{comp} \qquad (26)$$

where Z_{comp} is the ordinary composition of the linear tranformations Z_{M_1} and Z_{M_2} (matrix multiplication).

Now that we have obtained the formula for composition, let us work on some examples to verify the consistency of the theory.

2.6.1 Other examples of composition of surfaces

To begin with, we will perform the gluing of two disks along their boundary edge in order to get a sphere (see figure).

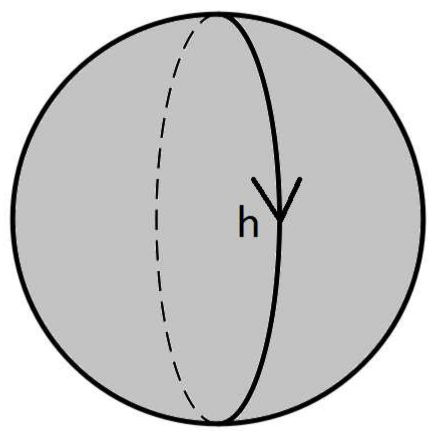

Figure 12: *Illustration of the gluing of two disks in order to get a sphere*

According to our formula, if we call Z_S the evolution operator of the resulting surface and Z_D the evolution operator for a disk, we get:

$$Z_S = |G| \cdot \sum_{h \in G} \langle \varnothing | Z_D | h \rangle \langle h | Z_D | \varnothing \rangle \quad (27)$$

$$\Rightarrow \quad Z_S = |G| \cdot \langle \varnothing | Z_D | 1 \rangle \langle 1 | Z_D | \varnothing \rangle \quad (28)$$

$$\Rightarrow \quad Z_S = |G| \cdot \frac{1}{|G|} \cdot \frac{1}{|G|} \quad (29)$$

$$\Rightarrow \quad Z_S = \frac{1}{|G|} \quad (30)$$

We recover the invariant for a sphere, as expected.

Now, we are going to perform the gluing of two cylinders, obtaining one resulting cylinder (see figure).

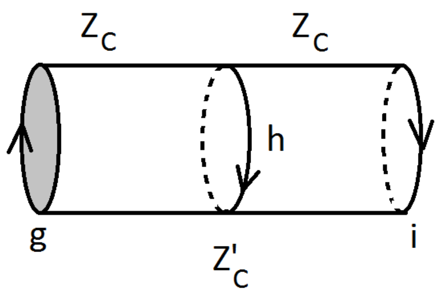

Figure 13: *Illustration of the gluing of two cylinders in order to get a cylinder*

Calling $Z_C{}'$ the evolution operator of the resulting surface, and Z_C the evolution operator for a cylinder, we get:

$$\langle g | Z_C{}' | i \rangle = |G| \cdot \sum_{h \in G} \langle g | Z_C | h \rangle \langle h | Z_C | i \rangle \quad (31)$$

$$\langle g | Z_C{}' | i \rangle = |G| \cdot \sum_{h \in G} \frac{1}{|G|^2} C(g,h) \frac{1}{|G|^2} C(h,i) \quad (32)$$

$$\langle g | Z_C{}' | i \rangle = \frac{1}{|G|^3} \cdot \sum_{h \in G} C(g,h) C(h,i) \quad (33)$$

$$(34)$$

However, we should get:

$$\langle g | Z_C{}' | i \rangle = \frac{1}{|G|^2} C(g,i) \quad (35)$$

Therefore, for the theory to be valid and consistent, we must have:

$$\sum_{h \in G} C(g,h) C(h,i) = |G| \cdot C(g,i) \quad (36)$$

This is a group theory result that can be proven (see the appendix) and that will be used from now on. This way, TQFT provided a motivation to prove a group theory result, and that shows the importance of the study of such constructions.

To finish, we will glue a disk, a pair of pants, another pair of pants and another disk to obtain a torus. This time there are four gluing edges.

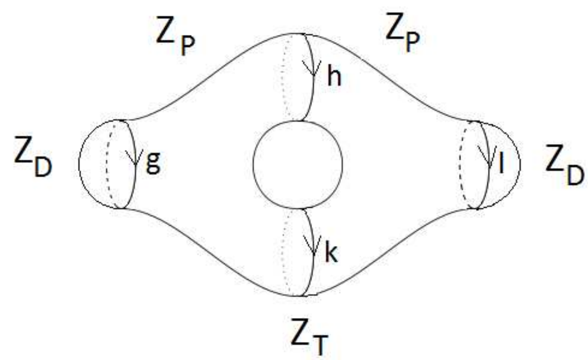

Figure 14: *Illustration of the gluing of two disks and two pair of pants in order to get a torus*

Calling Z_T the resulting surface evolution operator and Z_P the evolution operator for a pair of pants, we get:

$$\begin{aligned}
Z_T = & \ |G|^4 \sum_{g,h,k,l \in G} \langle \varnothing \,| Z_D |\, g \rangle \langle g \,| Z_P |\, h, k \rangle \langle h, k \,| Z_P |\, l \rangle \\
& \langle l \,| Z_D |\, \varnothing \rangle \\
\Rightarrow \ Z_T = & \ |G|^4 \sum_{h,k \in G} \frac{1}{|G|} \langle 1 \,| Z_P |\, h, k \rangle \langle h, k \,| Z_P |\, 1 \rangle \frac{1}{|G|} \\
\Rightarrow \ Z_T = & \ |G|^2 \sum_{h,k \in G} \frac{\# \left\{ j_1, j_2 \in G : j_1 h^{-1} j_1^{-1} j_2 k^{-1} j_2^{-1} = 1 \right\}^2}{|G|^6} \\
\Rightarrow \ Z_T = & \ \frac{1}{|G|^4} \sum_{h,k \in G} \# \left\{ j_1, j_2 \in G : h^{-1} = j_1^{-1} j_2 k j_2^{-1} j_1 \right\}^2 \\
\Rightarrow \ Z_T = & \ \frac{1}{|G|^4} \sum_{h,k \in G} \left[|G| \# \left\{ j \in G : h^{-1} = j k j^{-1} \right\} \right]^2 \\
\Rightarrow \ Z_T = & \ \frac{1}{|G|^2} \sum_{h,k \in G} C(h^{-1}, k) \cdot C(h^{-1}, k) \\
\Rightarrow \ Z_T = & \ \frac{1}{|G|^2} \sum_{k \in G} \left[\sum_{h \in G} C(h^{-1}, k) \cdot C(h^{-1}, k) \right] \\
\Rightarrow \ Z_T = & \ \frac{1}{|G|^2} \sum_{k \in G} |G| C(k, k) \\
\Rightarrow \ Z_T = & \ \frac{1}{|G|} \sum_{k \in G} \# \{ j \in G : kj = jk \} \\
\Rightarrow \ Z_T = & \ \frac{1}{|G|} \# \{ j, k \in G : kj = jk \}
\end{aligned} \tag{37}$$

where we have used the group theory relation found earlier.

The last equation is the expression we had found for the torus invariant, confirming the validity of the theory. This concludes our approach to TQFT using finite groups. We have shown that it is a consistent theory, that it assigns topological invariants to closed surfaces, that it describes very well the composition of surfaces, inter alia.

However, something important we found in this section was the fact that the theory motivated the derivation of a group theory result. This brings us to study a TQFT using finite 2-groups, hoping to find some 2-group mathematical relations.

III. A 2-D TQFT from finite 2-groups

3.1 What is a 2-group ?

A finite 2-group $(G, H, \partial, \triangleright)$ is a mathematical structure constituted by two finite groups G and H, a group homomorphism $\partial : H \to G$ and a left action $\triangleright$ of G on H such that:

$$g \triangleright (h_1 h_2) = (g \triangleright h_1)(g \triangleright h_2) \tag{38}$$

$$g \triangleright 1_H = 1_H \tag{39}$$

where h_1 and h_2 are group H elements and g is a group G element. There are also two more conditions linking ∂ and $\triangleright$:

$$\partial(g \triangleright h) = g \partial(h) g^{-1} \tag{40}$$

$$\partial(h_1) \triangleright h_2 = h_1 h_2 h_1^{-1} \tag{41}$$

An example of a 2-group is $G = \mathbb{Z}_2$, $H = \mathbb{Z}_3$, the ∂ operation is defined as $\partial(n) = 0$ where $n \in \{0; 1; 2\}$ (n represents any H element), and the non-trivial action is defined as:

$$0 \triangleright n = n \tag{42}$$

$$1 \triangleright n = \begin{cases} 0, & n = 0 \\ 2, & n = 1 \\ 1, & n = 2 \end{cases} \tag{43}$$

In fact, one has:

$$n = 0 \triangleright n = (1 + 1) \triangleright n = 1 \triangleright (1 \triangleright n) \tag{44}$$

And the way the group action has been defined this last equation is always valid. The other conditions are also fulfilled.

3.2 A link between 2-groups and squares

Let $(G, H, \partial, \triangleright)$ be a 2-group. Let's define a type of "filled" squares where each edge has orientation left-right if it is horizontal and down-up if it is vertical. The interior of the square is coloured with an H element, e, and the edges are coloured with G elements, X, Y, W and Z (see figure), such that:

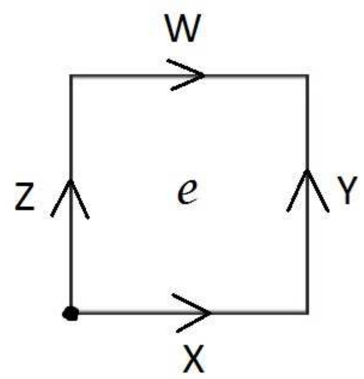

Figure 15: *Colouring of a square using 2-groups*

$$\partial(e) = XYW^{-1}Z^{-1} \tag{45}$$

Note that the inverses appear because of the edge orientation.

We also define two operations with squares. The first one is the horizontal composition:

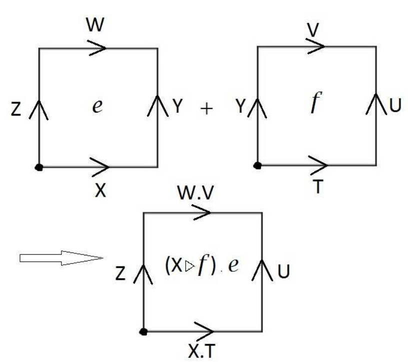

Figure 16: *Horizontal composition of squares and transformations of the colourings*

Note that:

$$\begin{aligned}
& \partial((X \triangleright f)e) &&= \partial(X \triangleright f)\partial(e) \\
\Rightarrow\ & \partial((X \triangleright f)e) &&= X\partial(f)X^{-1}\partial(e) \\
\Rightarrow\ & \partial((X \triangleright f)e) &&= XTUV^{-1}Y^{-1}X^{-1}XYW^{-1}Z^{-1} \\
\Rightarrow\ & \partial((X \triangleright f)e) &&= (XT)U(WV)^{-1}Z^{-1}
\end{aligned} \tag{46}$$

Therefore this way the square structure is preserved.

The second operation is the vertical composition:

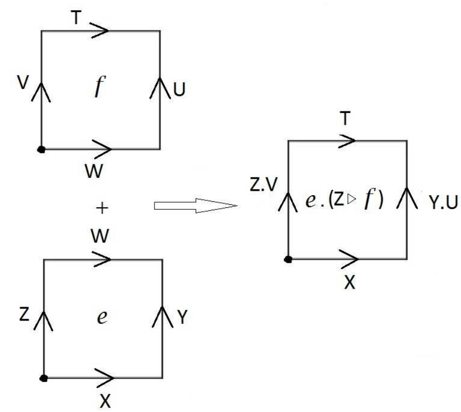

Figure 17: *Vertical composition of squares and transformations of the colourings*

Note that:

$$\begin{aligned}
& \partial(e(Z \triangleright f)) &&= \partial(e)\partial(Z \triangleright f) \\
\Rightarrow\ & \partial(e(Z \triangleright f)) &&= \partial(e)Z\partial(f)Z^{-1} \\
\Rightarrow\ & \partial(e(Z \triangleright f)) &&= XYW^{-1}Z^{-1}ZWUT^{-1}V^{-1}Z^{-1} \\
\Rightarrow\ & \partial(e(Z \triangleright f)) &&= X(YU)T^{-1}(ZV)^{-1}
\end{aligned} \tag{47}$$

Thus, in this operation the square structure is preserved too.

The remarkable fact is that the operations satisfy an "interchange" law, as explained by the figure.

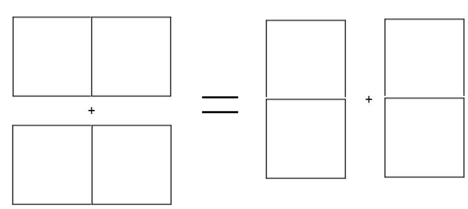

Figure 18: *The interchange law*

The verification of this fact is straightforward but requires some space, so we will leave it as an exercise for the reader.

This way, 2-groups proved to be a nice mathematical structure for the description of squares. Let us now try and use 2-groups to describe broader kinds of geometrical figures, in order to establish a solid 2-D TQFT.

3.3 Colouring and combining polygons

3.3.1 Colouring polygons

We will now extend this type of 2-group colouring to every polygon. Suppose we have a n-gon (polygon with n sides) with an arbitrary fixed edge orientation. However, in the case of squares we had the condition $\partial(e) = XYW^{-1}Z^{-1}$ that defined a circulation with a starting edge being the one coloured by X (or starting point being the intersection of the edges coloured by X and Z) and with an orientation (in this case it was a positive orientation). Therefore, in the case of n-gons, we also have to define a starting point and a circulation orientation, and we define them through the relation $\partial(h) = g_1 g_2 ... g_n$ (where $h \in H$ and $g_i \in G$, $1 \leq i \leq n$), as shown in the figure.

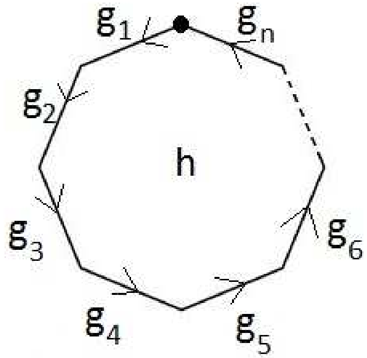

Figure 19: *An example of an n-gon colouring*

3.3.2 Combining polygons

To combine oriented and coloured polygons, we also extend the ideas of the combination of squares. Suppose we have two polygons, which have a common edge (in this case, that edge is coloured by g_m), and whose circulation orientation and starting point are defined through the following relations:

$$\partial(h_1) = g_1 g_2 ... g_m ... g_{n_1} \tag{48}$$
$$\partial(h_2) = g'_1 g'_2 ... g'_{r-1} g'_r g'_{r+1} ... g'_{n_2} \tag{49}$$

where $g_m = g'_r$, $h_1, h_2 \in H$, $g_i \in G$ for $1 \leq i \leq n_1$ and $g'_i \in G$, for $1 \leq i \leq n_2$.

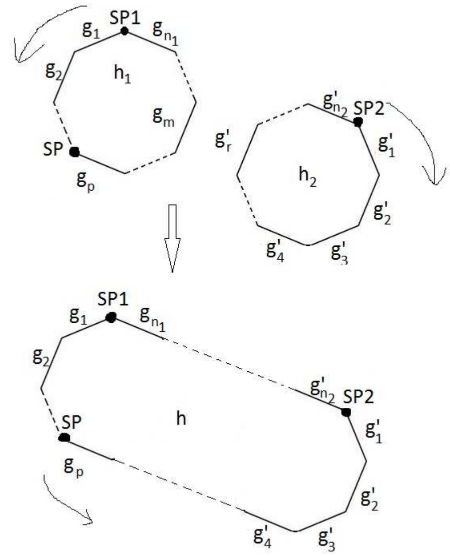

Figure 20: *Combining two polygons*

We want to find what is the H colouring of the obtained polygon. That polygon has a previously defined circulation orientation (here we take, without loss of generality, the same as the first polygon) and starting point taken to be the starting point of the edge coloured by g_p (for simplicity, $p < m$). We call that point SP and the starting points of polygons 1 and 2 $SP1$ and $SP2$. The H colouring of the obtained polygon, h, has to saisfy the following relation (see figure):

$$\begin{aligned}\partial(h) = \; & g_p g_{p+1} ... g_{m-1} {g'_{r-1}}^{-1} {g'_{r-2}}^{-1} ... {g'_1}^{-1} \\ & {g'_{n_2}}^{-1} {g'_{n_2-1}}^{-1} ... {g'_{r+1}}^{-1} g_{m+1} \\ & g_{m+2} ... g_{n_1} g_1 g_2 ... g_{p-1}\end{aligned} \tag{50}$$

For notation simplification we define $P_1 = g_1 g_2 ... g_{p-1}$ (P standing for "Path") and $P_2 = g'_1 g'_2 ... g'_{r-1} g^{-1}_{m-1} g^{-1}_{m-2} ... g_p^{-1}$, which are paths from $SP1$ to SP and from $SP2$ to SP. We also define $Q_1 = g_{m+1} ... g_{n_1}$ and $Q_2 = g'_{r+1} ... g'_{n_2}$. Then, the condition for h becomes:

$$\partial(h) = P_2^{-1} Q_2^{-1} Q_1 P_1 \tag{51}$$

Now, if we do some algebra, we get:

$$\begin{aligned} & \partial(h) = P_2^{-1} \partial(h_2)^{-1} P_2 g_p g_{p+1} ... g_m Q_1 P_1 \\ \Rightarrow \; & \partial(h) = P_2^{-1} \partial(h_2)^{-1} P_2 P_1^{-1} \partial(h_1) P_1 \\ \Rightarrow \; & \partial(h) = \partial(P_2^{-1} \triangleright h_2^{-1}) \partial(P_1^{-1} \triangleright h_1) \\ \Rightarrow \; & \partial(h) = \partial((P_2^{-1} \triangleright h_2^{-1})(P_1^{-1} \triangleright h_1)) \end{aligned} \tag{52}$$

This being valid for any 2-group, we conclude that:

$$h = (P_2^{-1} \triangleright h_2^{-1})(P_1^{-1} \triangleright h_1) \tag{53}$$

h_2 is inverted because polygon 2 has opposite circulation orientation compared to the obtained polygon. We have now arrived at a quite general relation for polygon combination and the corresponding transformation of colourings.

Now we are ready to build a TQFT from finite 2-groups.

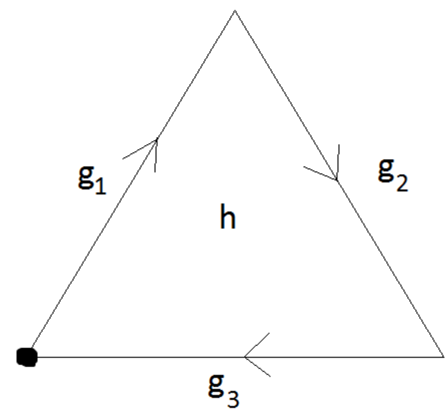

Figure 21: *A colouring of a triangle with a 2-group, all the edge colourings as well as the inside colouring are left free*

3.4 Explicit expression for the evolution operator Z_M

By analogy with the finite group case, we are going to test one expression for the evolution operator, found by Yetter, which must be invariant under the moves described before to pass from a CW decomposition to another, which is the following:

$$\langle c_i \,|Z_M|\, c_j' \rangle = \frac{|H|^v}{|G|^v |H|^e} \sum_{colourings} \tag{54}$$

where c_i and c_j' are the boundary colourings, v is the number of vertices, e is the number of edges, and $\sum_{colourings}$ is the sum over all possible M colourings compatible with c_i and c_j' and satisfying the 2-group condition (1-cell and 2-cell colourings are left free, provided they are not on the boundary).

As we said, this evolution operator must be invariant under the Pachner moves and the vertex/edge addition/removal move. For vertex and edge addition or removal, it is trivially invariant (left as an exercise). We are going to verify if it is invariant under the second Pachner move.

Suppose we have a certain triangle belonging to the interior of a surface. The condition a colouring has to satisfy to be valid is $\partial(h) = g_1 g_2 g_3$.

Here, we can freely choose h, g_1 and g_2, and this way g_3 is fixed through the colouring condition. The number of possible colourings is therefore:

$$\sum_{colourings} = |H| \cdot |G|^2 \tag{55}$$

Now, if we perform the second Pachner move, adding another vertex inside the triangle, we also get three more edges (see figure):

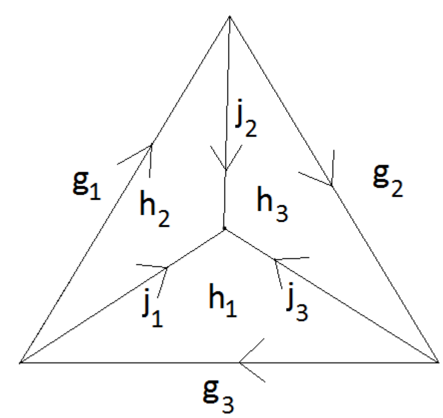

Figure 22: *The triangle after the second Pachner move was performed*

In this triangle, the colouring conditions are (we choose arbitrarily orientations and starting points):

$$\partial(h_1) = g_3 j_1 j_3^{-1}$$
$$\partial(h_2) = g_1 j_2 j_1^{-1}$$
$$\partial(h_3) = g_2 j_3 j_2^{-1}$$

Note that, if we want to express h as a function of h_1, h_2, h_3, g_1, g_2 and g_3, we only have to apply the rules for combining polygons.

If we fix h_1, g_3 and j_1, then j_3 is fixed by the first condition. Then, we can fix h_2 and g_1, and that fixes j_2 (because j_1 was already fixed). Finally, we fix h_3 and that fixes g_2 too. Therefore, the number of possible colourings in this triangle is ${\sum_{colourings}}' = |H|^3 \cdot |G|^3$ because we need to choose three H elements and three G elements for the remaining elements to be fixed. However, in this case we have one more vertex and three more edges, so, to compare this result with the one above, we have to multiply it by $\frac{|H|^1}{|G|^1|H|^3}$.

$$\frac{|H|}{|G||H|^3}\sum_{col.}{}' = \frac{1}{|G||H|^2}|H|^3 \cdot |G|^3 = |H| \cdot |G|^2 = \sum_{col.} \tag{56}$$

Therefore Z_M is invariant under the second Pachner move. For the first Pachner move, it suffices to say that the number of colourings is the same for both cases.

3.5 The image subgroup A

Before we start to compute the evolution operator of some surfaces, let us analyze a disk, with a free edge. The colouring condition is $\partial(h) = g$

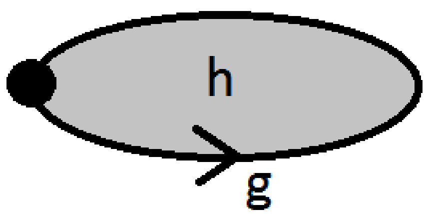

Figure 23: *A disk with a free edge*

The number of possible colourings is very easy to compute. In fact, once we have chosen h, g is fixed by the colouring condition. Hence, $\sum_{colourings} = |H|$.

However, we can think of it another way. If we choose g, then it may be impossible to fix h, because the homomorphism ∂ may not be surjective. Thus, to overcome the problem, we let A be the subgroup of G constituted by the image of ∂. Now, to compute the number of colourings, we can only choose colourings such that $g \in A$. Nevertheless, once $g \in A$ is chosen, h is still not fixed. In fact, ∂ may not be injective. As ∂ is a homomorphism, we have $|ker(\partial)|$ possibilities for choosing h, where $ker(\partial)$ is the kernel of ∂ (that is, the set of all $h \in H$ such that $\partial(h) = 1$). Therefore, the number of colourings is also given by $\sum_{colourings} = |A||ker(\partial)|$. Combining the two results for $\sum_{colourings}$, we get:

$$|ker(\partial)| = \frac{|H|}{|A|} \tag{57}$$

Although it is not hard to show this result using group theory, we showed it only by computing the number of possible colourings of a disk in two different ways.

Now that we have introduced the image group A, we are ready to compute the evolution operator for some surfaces.

3.6 Computation of the evolution operator for many surfaces

Now we are going to compute the evolution operator for some closed and open surfaces, like we did in section 2.5.

3.6.1 Closed surfaces

Let's start with the sphere. Using the same CW decomposition as before, with h_1 and h_2 being the colourings of the north and south hemisphere, we have the following colouring conditions:

$$\partial(h_1) = g_1 g_2$$
$$\partial(h_2) = g_1 g_2$$

Note that, from now on, we will arbitrarily choose a circulation orientation and starting point for each colouring condition. That does not affect the evolution operator.

Now, if we choose an h_1, then there are $|ker(\partial)|$ possibilities for h_2, and $|G|$ possibilities for g_1. With this, g_2 becomes fixed. Therefore, the number of possible colourings is $\sum_{col.} = |H| \cdot |ker(\partial)| \cdot |G|$ and the evolution operator is given by:

$$Z_{Sphere} = \frac{|H|^2}{|G|^2|H|^2}\sum_{col.}$$
$$\Rightarrow \quad Z_{Sphere} = \frac{1}{|G|^2}\frac{|H|^2|G|}{|A|}$$
$$\Rightarrow \quad Z_{Sphere} = \frac{|H|^2}{|G||A|} \tag{58}$$

Now, using the decomposition presented in 2.5.1, we are going to compute the evolution operator for a torus.

h is the colouring of the single 2-cell, and g_1 and g_2 are the colourings of the two 1-cells. Therefore, the colouring condition is:

$$\partial(h) = g_1 g_2 g_1^{-1} g_2^{-1}$$

So, the evolution operator is given by:

$$Z_{Torus} = \frac{1}{|G||H|} \#\{(h, g_1, g_2) \in H \times G^2 : \partial(h) = g_1 g_2 g_1^{-1} g_2^{-1}\} \quad (59)$$

That is all for examples of closed surfaces. Now we will treat the case of open surfaces.

3.6.2 Open surfaces

We start with a disk (with a fixed edge colouring g), decomposed as previously. We colour the 2-cell with h. The colouring condition is $\partial(h) = g$. If $g \notin A$, then there are no possible colourings. If $g \in A$, then there are $|ker(\partial)|$ possibilities for choosing h. Therefore, in this case, the evolution operator is:

$$\langle \varnothing | Z_{Disk} | g \in A \rangle = \frac{|H|}{|G||H|} |ker(\partial)|$$
$$\langle \varnothing | Z_{Disk} | g \in A \rangle = \frac{|H|}{|G||A|}$$

Following the ordering of section 2.5.2, we want now to compute the evolution operator for a triangle (which, for the same reasons as before, is different from the evolution operator of a disk). Calling g_1, g_2 and g_3 the fixed edge colourings, and h the 2-cell, we get the following colouring condition:

$$\partial(h) = g_1 g_2 g_3$$

Now, if the product $g_1 g_2 g_3$ is not in A, there are no possible colourings. If it is, then we can choose h amongst $|ker(\partial)|$ possibilities. Therefore, in this case, the evolution operator $Z_{Tr.}$ is:

$$\langle \varnothing | Z_{Tr.} | g_1, g_2, g_3 \rangle = \frac{|H|^3}{|G|^3 |H|^3} \frac{|H|}{|A|}$$
$$\langle \varnothing | Z_{Tr.} | g_1, g_2, g_3 \rangle = \frac{|H|}{|G|^3 |A|}$$

As before, even though they are topologically equivalent, the evolution operator of the triangle is the evolution operator of the disk multiplied by the factor $\frac{1}{|G|^2}$, since the triangle has two more vertices than the disk.

Let us now take a look at the cylinder. Decomposing it like in section 2.5.2, we call the fixed boundary edges colourings g_1 and g_2, and we call k the third edge colouring. We also call the 2-cell colouring h. Cut the cylinder the same way as before, and we get a rectangle. The colouring condition becomes:

$$\partial(h) = g_1 k g_2^{-1} k^{-} 1$$

Therefore, the evolution operator Z_C is given by:

$$\langle g_1 | Z_C | g_2 \rangle = \frac{|H|^2}{|G|^2 |H|^3} \sum_{col.}$$
$$\langle g_1 | Z_C | g_2 \rangle = \frac{\#\{(h, k) \in H \times G : \partial(h) = g_1 k g_2^{-1} k^{-1}\}}{|G|^2 |H|}$$

Our last example is the pair of pants, decomposed into CW cells the same way as before. The upper edge is coloured with g_1 and the two lower edges are coloured with g_2 and g_3. The two inner edges are coloured with j_1 and j_2 and the single 2-cell is coloured with h. Cutting and opening the pair of pants, we get a coloured heptagon. The colouring condition is:

$$\partial(h) = j_1 g_2^{-1} j_1^{-1} j_2 g_3^{-1} j_2^{-1} g_1$$

Therefore, the evolution operator is:

$$\langle g_1 | Z_{Pants} | g_2, g_3 \rangle = \frac{|H|^3}{|G|^3 |H|^5} \sum_{col.}$$
$$\langle g_1 | Z_{Pants} | g_2, g_3 \rangle =$$
$$\frac{\#\{(h, j_1, j_2) \in H \times G^2 : \partial(h) = j_1 g_2^{-1} j_1^{-1} j_2 g_3^{-1} j_2^{-1} g_1\}}{|G|^3 |H|^2}$$

Now we just need to present the formula for combination of surfaces to start combining surfaces to try and get some 2-group identities.

3.7 Composition of surfaces

In section 2.6, when we talked about surface composition, we came to the conclusion that we had to multiply the ordinary composition formula by $|G|^n$, where n was the number of glued edges, in order to get the right results. This happened because when two edges are glued, the surface obtained has one vertex and one edge less than the sum of the edges and vertices of the two surfaces taken individually, and, as the evolution operator depended on $\frac{1}{|G|^v}$ where v is the number of vertices, we had to insert the factor $|G|^n$ to take that loss into account.

We are using the same procedure for 2-groups. As the evolution operator depends on $\frac{|H|^v}{|G|^v |H|^e}$, where e is the number of edges, in order to translate mathematically the loss of one vertex and one edge in the evolution operator, we have to multiply it by $\frac{|G|^n |H|^n}{|H|^n} = |G|^n$, where n is the number of glued edges (that is equal to the number of "glued" vertices). Therefore, the factor that appears in the composition formula using 2-groups should be the same as the one used in section 2.6.

As an example, we study the gluing of two disks in order to get a sphere. We colour the glued edge with g. If we call Z_S the evolution operator of the resulting sphere and Z_D the evolution operator for a disk, we get:

$$\begin{aligned} Z_S &= |G| \cdot \sum_{g \in G} \langle \varnothing \,| Z_D |\, g \rangle \langle g \,| Z_D |\, \varnothing \rangle \\ &= |G| \cdot \sum_{g \in A} \frac{|H|}{|G||A|} \frac{|H|}{|G||A|} \\ &= \frac{|H|^2}{|G||A|^2} \cdot \sum_{g \in A} 1 \\ &= \frac{|H|^2}{|G||A|} \end{aligned}$$

We recover the invariant for a sphere, confirming our gluing formula.

Let's study some other gluing examples to see if we find some 2-group identities.

3.7.1 Other examples of gluing surfaces

To start, we glue a disk (Z_D) and a cylinder (Z_C), and this way we should get a disk. Applying the gluing formula, colouring the glued edge with g_1 and the remaining cylinder edge g_2, and calling Z'_D the evolution operator of the remaining surface, we get:

$$\begin{aligned} \langle \varnothing \,| Z'_D |\, g_2 \rangle &= |G| \cdot \sum_{g_1 \in G} \langle \varnothing \,| Z'_D |\, g_1 \rangle \langle g_1 \,| Z'_D |\, g_2 \rangle \\ &= |G| \cdot \sum_{g_1 \in A} \frac{|H|}{|G||A|} \frac{\#\{(h,k) \in H \times G : \partial(h) = g_1 k g_2^{-1} k^{-1}\}}{|G|^2|H|} \\ &= \frac{1}{|G|^2|A|} \sum_{g_1 \in A} \#\{(h,k) \in H \times G : \partial(h) = g_1 k g_2^{-1} k^{-1}\} \end{aligned}$$

For simplification, we will adopt the notation:

$$\tilde{C}(g_1, g_2) := \#\{(h,k) \in H \times G : \partial(h) = g_1 k g_2^{-1} k^{-1}\}$$

However, our expression for the evolution operator Z'_D should be equal to:

$$\langle \varnothing \,| Z'_D |\, g_2 \rangle = \begin{cases} \frac{|H|}{|G||A|}, \ g_2 \in A \\ 0, \ g_2 \notin A \end{cases}$$

And the only way for this to be correct is that:

$$\sum_{g_1 \in A} \tilde{C}(g_1, g_2) = \begin{cases} |H| \cdot |G|, \ g_2 \in A \\ 0, \ g_2 \notin A \end{cases}$$

In fact, it is possible to derive this result using group theory, but it requires more time. Nevertheless, this TQFT construction allowed us to find this 2-group identity in a very elegant way.

In order to try to get more of those 2-group identities, we will perform the gluing of two cylinders along one edge, coloured by g_2. The remaining edge of the first cylinder is coloured with g_1, and the remaining edge of the second cylinder is coloured with g_3. Applying our gluing formula again, calling Z'_C the evolution operator of the surface obtained, we get:

$$\begin{aligned} \langle g_1 \,| Z'_C |\, g_3 \rangle &= |G| \cdot \sum_{g_2 \in G} \frac{\tilde{C}(g_1, g_2)\tilde{C}(g_2, g_3)}{|G|^4|H|^2} \\ &= \frac{1}{|G|^3|H|^2} \sum_{g_2 \in G} \tilde{C}(g_1, g_2)\tilde{C}(g_2, g_3) \end{aligned}$$

However, we know that the evolution operator for the obtained cylinder is equal to:

$$\langle g_1 \,| Z'_C |\, g_3 \rangle = \frac{\tilde{C}(g_1, g_3)}{|G|^2|H|}$$

For these last results to be coherent, we must have:

$$\sum_{g_2 \in G} \tilde{C}(g_1, g_2)\tilde{C}(g_2, g_3) = |G||H| \cdot \tilde{C}(g_1, g_3)$$

This last equation is very similar to the one found in section 2.6.1. However, this one requires much more work to prove using group identities.

We have derived some 2-group identities using this TQFT formalism. Certainly, there are many more identities that can be found with this method. This is therefore a great utility for a 2-group based TQFT.

IV. Conclusion

In this article, we first used a finite group based TQFT to get topological invariants of closed surfaces and, defining a formula for gluing surfaces, we also found a quite simple group theory identity. Moreover, we used a finite 2-group based TQFT also to get topological invariants of surfaces and to find more complex 2-group identities. We sensed that many other identities can be found using this useful formalism, but we did not have time to work on more than a few identities.

The physical meaning, in terms of gauge theory, of the evolution operator formula for finite groups could be that (the number of G-connections on the surface) divided by (the number of gauge transformations) is constant, and for 2-groups it would be that (the number of 2-group connections) divided by (the number of 1-gauge transformations, divided by the number of 2-gauge transformations between 1-gauge transformations) is constant.

I. Appendix

Every group identity displayed in the article can be derived using group theory. Here, we are going to derive equation (36), that is:

$$\sum_{h\in G} C(g,h)C(h,i) = |G| \cdot C(g,i)$$

where $g, i \in G$.

First, we are going to derive a lemma:

Lemma 1:

If g and h are in the same conjugacy class, then:

$$C(g,h) = C(g,g)$$

We will reason by contradiction:

Suppose $C(g,h) < C(g,g)$. Let's choose a $j \in G$ such that:

$$g = jhj^{-1}$$

Furthermore, there exist k_i, $1 \leq i \leq C(g,g)$, such that:

$$g = k_i g k_i^{-1}$$

and, for $i \neq j$, $k_i \neq k_j$.

Hence, replacing the last equation in the first one, we get that:

$$g = k_i^{-1} jhj^{-1} k_i$$

So, we conclude that $C(g,h) \geq C(g,g)$, and that contradicts our supposition.

Now, suppose $C(g,g) < C(g,h)$. Arguing in a similar way, we conclude that $C(g,g) \geq C(g,h)$ and that contradicts the supposition.

Therefore, the only possibility left is:

$$C(g,g) = C(g,h) \tag{60}$$

This lemma is closely related to the Orbit-stabilizer theorem in group theory, since it implies that the number of elements of each conjugacy class is a divisor of $|G|$.

An immediate corollary of the lemma is that, if g, h and i are in the same conjugacy class, then:

$$C(g,h) = C(h,i) = C(i,g)$$

Now, we will just point out that, if g, h and i are not in the same conjugacy class, $C(g,h)C(h,i) = 0$. Moreover, it is trivial that $\sum_{h\in G} C(g,h) = |G|$, since all the elements of the conjugacy class of g are included in the sum. Therefore, we have:

$$\begin{aligned}\sum_{h\in G} C(g,h)C(h,i) &= \sum_{h\in G} C(g,h)C(g,i) \\ &= C(g,i)\sum_{h\in G} C(g,h) \\ &= C(g,i)|G|.\end{aligned}$$

That concludes the proof.

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