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Noncommutative Chern-Simons gauge and gravity theories and their geometric Seiberg-Witten map

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I work in the context of formal deformation quantization.

$A = C^\infty(M)[[\hbar]]$ algebra of formal power series in $\hbar$ with coefficients in $C^\infty(M)$.

$A_\star = C^\infty(M)_\star[[\hbar]]$ noncommutative algebra, with the same $\mathbb{C}[[h]]$-module structure as A and with noncommutative and associative product $f \star g$ given by a bidifferential operator.

The bidifferential operator is obtained from a Drinfeld twist.

Example: Moyal-Weyl $\star$-product on $\mathbb{R}^{2n}$ (from now on we absorb $\hbar$ in θ)

$$f \star g = fh + \frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}f\,\frac{\partial}{\partial x^\nu}h \;-\; \frac{1}{8}\theta^{\mu\nu}\theta^{\rho\sigma}\frac{\partial}{\partial x^\rho}\frac{\partial}{\partial x^\mu}f\,\frac{\partial}{\partial x^\sigma}\frac{\partial}{\partial x^\nu}h + \dots$$

$$(f \star h)(x) = \mathrm{e}^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial y^\nu}} f(x)h(y)\Big|_{x=y} \,.$$

Notice that if we set

$$\mathcal{F}^{-1} = \mathrm{e}^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial y^\nu}}$$

then

$$(f \star h)(x) = \mu \circ \mathcal{F}^{-1}(f \otimes h)(x)$$

The element $\mathcal{F} = \mathrm{e}^{-\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial y^\nu}}$ is called a twist. We have $\mathcal{F} \in U(g) \otimes U(g)$ where $U(g)$ is the universal enveloping algebra of the group of tanslations on $\mathbb{R}^{2n}$.

A more general example of a twist associated to a manifold M is given by:

$$\mathcal{F} = e^{-\frac{i}{2}\theta^{IJ}X_I \otimes X_J}$$

where X_I are vector fields, $[X_I, X_J] = 0$, and θ^{IJ} is a constant (antisymmetric) matrix.

$\star$-Noncommutative Manifold $(M, \mathcal{F})$

M smooth manifold

$\mathcal{F} = e^{-\frac{i}{2}\theta^{IJ}X_I \otimes X_J}$

The $\star$-product deformation is obtained via an (abelian) Drinfeld twist

(Prototypical example: an action of an abelian group on M that induces an action of the corresponding abelian Lie algebra g on $C^\infty(M)$).

Usual product of functions:

$$f \otimes h \xrightarrow{\mu} fh$$

$\star$-Product of functions

$$f \otimes h \xrightarrow{\mu_\star} f \star h$$

$$\mu_\star = \mu \circ \mathcal{F}^{-1}$$

A and $A_\star$ are the same as modules over $\mathbb{C}[[\hbar]]$, they have different algebra structure

$\mathcal{F}$ deforms the geometry of M into a noncommutative geometry.

Guiding principle: deform g-invariant bilinear maps via

$$\mu \longrightarrow \mu_\star = \mu \circ \mathcal{F}^{-1}$$

The action of $\mathcal{F}^{-1} = \mathrm{e}^{\frac{i}{2}\theta^{IJ}X_I \otimes X_J}$ will always be via the Lie derivative.

$\star$-Tensor fields

Notation $\mathcal{F}^{-1} = \bar{\mathrm{f}}^\alpha \otimes \bar{\mathrm{f}}_\alpha$

$$\rho \otimes_\star \rho' = \bar{\mathrm{f}}^\alpha(\rho) \otimes \bar{\mathrm{f}}_\alpha(\rho') \ .$$

In particular $h \star v,\ h \star df,\ df \star h$ etc...

Associativity of $\otimes_\star$ follows from twist properties (for abelian twist from $[X_I, X_J] = 0$).

This construction can be also understood as equivalence of symmetric monoidal categories of $U(g)^{\mathcal{F}}$ Hopf algebra modules and $A_\star$-bimodules. (In our case $U(g)^{\mathcal{F}} = U(g)$ since g is abelian).

$\star$-Forms

$$\tau \wedge_\star \tau' = \bar{\mathsf{f}}^\alpha(\tau) \wedge \bar{\mathsf{f}}_\alpha(\tau') \ .$$

- Covariance: $\wedge_\star$ contains only Lie derivatives of the vectorfields X_I
- Compatibility with the undeformed exterior differential:

$$d(\tau \wedge_\star \tau') = d(\tau) \wedge_\star \tau' + (-1)^{deg(\tau)} \tau \wedge_\star d\tau' \tag{1}$$

- Compatibility with the undeformed integral (graded cyclicity property):

$$\int \tau \wedge_\star \tau' = (-1)^{deg(\tau)deg(\tau')} \int \tau' \wedge_\star \tau \tag{2}$$

(up to boundary terms). Indeed we can write

$$\begin{aligned} \tau \wedge_\star \tau' &= \tau \wedge \tau' + \ell_{A_1} \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{i}{2}\right)^n \theta^{A_1B_1} \cdots \theta^{A_nB_n} (\ell_{A_2} \cdots \ell_{A_n} \tau)(\ell_{B_1} \ell_{B_2} \cdots \ell_{B_n} \tau') \\ &= \tau \wedge \tau' + \ell_{A_1} Q^{A_1} \end{aligned} \tag{3}$$

where $\ell_A = \ell_{X_A}$ is the Lie derivative along X_A, and $[\ell_A, \ell_B] = 0$

- Compatibility with the undeformed complex conj.: $(\tau \wedge_\star \tau')^* = (-1)^{deg(\tau)deg(\tau')} \tau'^* \wedge_\star \tau^*$

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Along these lines deformation of connections on commutative vector bundles over M to noncommutative vector bundles ($A_\star$-bimodules) [Aschieri, Schenkel 2014].

$\star$-Riemannian Geometry and $\star$-gravity [WESS GROUP '06]
[Aschieri, Blohmann, Dimitrijevic, Mayer, Schupp, Wess]

Another route to the construction of gauge theories on noncommutative manifolds

Describe (locally) a connection as a 1-form $\widehat{\Omega}$ with values in a Lie algebra: $\widehat{\Omega} = \widehat{\Omega}^a T^a$.

Consider the Lie algebra of $U(n)$.

Define the curvature of $\widehat{\Omega}$

$$\widehat{R} = d\widehat{\Omega} - \widehat{\Omega} \wedge_\star \widehat{\Omega}$$

where $\widehat{\Omega} \wedge_\star \widehat{\Omega} = \widehat{\Omega}^a \wedge_\star \widehat{\Omega}^b \, T^a T^b$.

Noncommutative infinitesimal $U(N)$ gauge transformations

$$\widehat{\delta}_{\widehat{\varepsilon}} \widehat{\Omega} = d\widehat{\varepsilon} - \widehat{\Omega} \star \widehat{\varepsilon} + \widehat{\varepsilon} \star \widehat{\Omega}, \quad \Rightarrow \quad \widehat{\delta}_{\widehat{\varepsilon}} \widehat{R} = -\widehat{R} \star \widehat{\varepsilon} + \widehat{\varepsilon} \star \widehat{R} \, . \tag{4}$$

$\widehat{\varepsilon} = \widehat{\varepsilon}^a T^a$ Lie algebra valued function (gauge parameter).

Consider an action functional invariant under gauge transformations.

Example: Yang-Mills on $\mathbb{R}^4$ with Moyal-Weyl noncommutativity and Minkowski metric:

$$\int Tr\big(\widehat{R} \wedge_\star {}^{*H}(\widehat{R})\big) = \int Tr(\widehat{R}_{\mu\nu} \star \widehat{R}^{\mu\nu}) d^4x \, .$$

These NC theories are useful to describe the low energy dynamics of D-branes in the presence of a constant magnetic field B. They arise studying T-duality of open strings, T-duality acts within NC Yang-Mills theories.

Also these theories can be understood as deformations of commutative theories: Seiberg-Witten map (SW map) determines $\widehat{\Omega}$ as a field redefinition of a commutative gauge potential Ω; with usual gauge transf. $\delta_\varepsilon \Omega = d\varepsilon - \Omega\varepsilon + \varepsilon\Omega$.

Seiberg Witten Map

SW map determines

$$\widehat{\Omega} = \widehat{\Omega}[\Omega, \theta] \quad , \quad \widehat{\epsilon} = \widehat{\epsilon}[\epsilon, \Omega, \theta]$$

such that

$$\widehat{\Omega}[\Omega, \theta] + \widehat{\delta}_{\widehat{\epsilon}} \widehat{\Omega}[\Omega, \theta] = \widehat{\Omega}[\Omega + \delta_\epsilon \Omega, \theta] \tag{5}$$

In this way gauge equivalent classes of noncommutative gauge transformations are in one-to-one correspondence with gauge equivalence classes of commutative gauge transformations. This implies that the physical degrees of freedom of the noncommutative and commutative theories are the same. [Cond. (5) can be formulated for $U(1)$ gauge fields also in the context for Kontsevich def. quant. [Jurco, Schupp]].

In the present context condition (5) holds if

$$\delta_\theta \widehat{\Omega} \equiv \delta\theta^{AB} \frac{\partial}{\partial \theta^{AB}} \widehat{\Omega} = \frac{i}{4} \delta\theta^{AB} \{\widehat{\Omega}_A, \ell_B \widehat{\Omega} + \widehat{R}_B\}_\star \ , \tag{6}$$

$$\delta_\theta \widehat{\varepsilon} \equiv \delta\theta^{AB} \frac{\partial}{\partial \theta^{AB}} \widehat{\varepsilon} = \frac{i}{4} \delta\theta^{AB} \{\widehat{\Omega}_A, \ell_B \widehat{\varepsilon}\}_\star \ , \tag{7}$$

where:

- $\widehat{\Omega}_A$, $\widehat{R}_A$ are defined as the contraction i_A along the tangent vector X_A of the exterior forms $\widehat{\Omega}$, $\widehat{R}$, i.e. $\widehat{\Omega}_A \equiv i_A \widehat{\Omega}$, $\widehat{R}_A \equiv i_A \widehat{R}$; $\ell_B = i_B d + d i_B$ Lie derivative along X_B).
- The bracket $\{\ ,\ \}_\star$ is the usual $\star$-anticommutator, e.g. $\{\widehat{\Omega}_A, \widehat{R}_B\}_\star = \widehat{\Omega}_A \star \widehat{R}_B + \widehat{R}_B \star \widehat{\Omega}_A$.

The differential equations (6)-(7) hold for any abelian twist defined by arbitrary commuting vector fields X_A [Aschieri, Castellani, Dimitrijevic].

Chern-Simons forms

$$dL_{CS}^{(2n-1)} = Tr(R^n)\ .$$

For example:

$$L_{CS}^{(1)} = Tr[\Omega] \tag{8}$$

$$L_{CS}^{(3)} = Tr[R\Omega + \frac{1}{3}\Omega^3] \tag{9}$$

$$L_{CS}^{(5)} = Tr[R^2\Omega + R\Omega^3 + \frac{1}{10}\Omega^5] \tag{10}$$

$$L_{CS}^{(7)} = Tr[R^3\Omega + \frac{2}{5}R^2\Omega^3 + \frac{1}{5}R\Omega^2R\Omega + \frac{1}{5}R\Omega^5 + \frac{1}{35}\Omega^7] \tag{11}$$

NC Chern-Simons forms obtained by replacing $\wedge \to \widehat{\wedge}_\star$, $\Omega \to \widehat{\Omega}$.

Since $\tau \wedge_\star \tau' = \tau \wedge \tau' + \ell_{A_1} Q^{A_1}$, we have

$$Tr(\mathcal{T} \wedge_\star \mathcal{T}') = (-1)^{deg(\mathcal{T})\,\mathrm{deg}(\mathcal{T}')}\, Tr(\mathcal{T}' \wedge_\star \mathcal{T}) + \ell_A Q^A\ ;$$

$\mathcal{T}, \mathcal{T}'$ homogeneous forms that are Lie algebra valued. Hence

$$dL_{CS^*}^{(2n-1)} = Tr(R^{\wedge_\star n}) + \ell_C Q^{(2n)\,C}$$

The SW variation of NC Chern-Simons forms Notation: we omit the hat denoting noncommutative fields, the $\star$ and $\wedge_\star$ products, and simply write $\{\ ,\ \}$, $[\ ,\ ]$ for the $\star$-anticommutator and the $\star$-commutator $\{\ ,\ \}_\star$, $[\ ,\ ]_\star$.

SW variation of $Tr(R^n)$. The SW variation of the connection implies the following variation for the curvature 2-form $R = d\Omega - \Omega\Omega$

$$\delta_\theta R = \frac{i}{4}\delta\theta^{AB}(\{\Omega_{A,B}\, R\} - [R_A, R_B]) \ .$$

From this formula and using induction:

$$\delta_\theta Tr(R^n) = \frac{i}{2}\delta\theta^{AB} Tr\Big(\frac{1}{n+1} i_B i_A R^{n+1}\Big) + \frac{i}{2}\delta\theta^{AB}\Big(dU_{AB} + \ell_C Q^C_{AB}\Big) \tag{12}$$

where the $(2n-1)$-form U_{AB} is given by

$$U_{AB} = Tr\Big(\sum_{i=2}^{n-1} R^{i-1} D R_{[A}(R^{n-i})_{B]}\Big) \tag{13}$$

with $(R^{n-i})_B \equiv i_B(R^{n-i})$. Antisymmetrization in the indices ${}_{A\ B}$

SW variation of $Tr(R^n)$ on a $2n$-dimensional manifold M (recall $\ell_C = i_C d + d i_C$)

$$\delta_\theta Tr(R^n) = \frac{i}{2}\delta\theta^{AB} d\big(U_{AB} + i_C Q^C_{AB}\big) \tag{14}$$

The nontrivial information in (14) is that $U_{AB} + i_C Q^C_{AB}$ where the second term is a contraction of a $2n$-form (in the NC connection), and the first term is expressed only in terms of products of curvatures, their contractions and covariant derivatives, i.e., in terms of only gauge covariant fields.

SW variation of $L^{(2n-1)}_{CS^\star}$

The SW variation of $L^{(2n-1)}_{CS^\star}$ can be inferred from $dL^{(2n-1)}_{CS^\star} = Tr(R^n) + \ell_C Q^{(2n)\,C}$, For forms living in a $2n$-dimensional manifold M,

$$\delta_\theta d\, L^{(2n-1)}_{CS^\star} = \frac{i}{2}\delta\theta^{AB} d\big(U_{AB} + i_C Q^C_{AB}\big) + d i_C(\delta_\theta Q^{(2n)\,C})$$

where we used $\ell_C = i_C d + d\, i_C$ and the vanishing of forms of degree higher than $2n$. Equivalently on M we have

$$\delta_\theta L^{(2n-1)}_{CS^\star} = \frac{i}{2}\delta\theta^{AB}\Big(U_{AB} + i_C Q^C_{AB}\Big) + i_C(\delta_\theta Q^{(2n)\,C}) + d\varphi$$

for some $(2n-1)$-form φ written in terms of the connection, of exterior derivatives and of contraction operators along the noncommutative directions.

We now consider a $(2n-1)$-dimensional submanifold N of M and choose commuting vector fields $\{X_A\}$ on M that restrict to vector fields on N. In this case $L^{(2n-1)}_{CS^\star}$ is a top form on N, and $Q^C_{AB} = \delta_\theta Q^{(2n)\,C} = 0$ being $2n$-forms on the $(2n-1)$-dimensional manifold N. The SW variation of the CS action on a manifold N with no boundary or with fields that have appropriate boundary conditions is therefore

$$\begin{aligned}
\delta_\theta \int L^{(2n-1)}_{CS^\star} &= \frac{i}{2}\delta\theta^{AB}\int U_{AB} \\
&= \frac{i}{2}\delta\theta^{AB}\int Tr\Big(\sum_{i=2}^{n-1} R^{i-1} DR_A (R^{n-i})_B\Big) \\
&= \frac{i}{2}\delta\theta^{AB}\int Tr\Big(RDR_A \sum_{k=0}^{n-3}(k+1)R^{n-3-k}R_B R^k\Big)
\end{aligned} \tag{15}$$

This variation is zero for $n = 1, 2$. The first nonvanishing SW variation of a Chern-Simons action occurs for $n = 3$. In particular in three dimensions the SW expansion of the noncommutative Chern-Simons action equals the commutative Chern-Simons action; this result, for Moyal-Groenewold noncommutativity, was obtained in [Grandi Silva].

In higher dimensions the variation is nonvanishing, and is expressed in terms of the gauge covariant quantities R, R_A and their covariant derivatives.

Slowly varying fields and invariance of NC CS action under SW map
The gauge covariant formulation of the slowly varying field strength condition is $L_A R \sim 0$, where $L_A = i_A D + D i_A$. In this case the noncommutative and commutative CS actions coincide. Indeed $DR_A = i_A DR + D i_A R = L_A R \sim 0$.

This result nicely matches that for Born-Infeld lagrangians. [Seiberg, Wittten].

Extended CS actions from NC CS actions

Example: the $D = 5$ noncommutative CS action is given by

$$\int L^{(5)}_{CS^\star} = \int L^{(5)}_{CS} + \frac{i}{2}\theta^{AB} \int Tr(RDR_A R_B) + \mathcal{O}(\theta^2) \tag{16}$$

Notice that this action is well defined for any gauge group G, and that it has the same (off shell) degrees of freedom as the usual CS action. Like in modified gravity theories the θ correction is just a further interaction term among the fields. For $G = SU(2, 2)$ we have Chern-Simons gravity theory on 5-dim. AdS with higer derivative corrections coming from noncommutativity.