On the string Lie algebra

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Crossed modules of Lie algebras

Definition

A crossed module of Lie algebras is a homomorphism $\mu : \mathfrak{m} \to \mathfrak{n}$ *together with an action of* $\mathfrak{n}$ *on* $\mathfrak{m}$ *by derivations such that*

(a) *for all* $n \in N$, $m \in M$: $\mu(n \cdot m) = [n, \mu(m)]$, *and*

(b) *for all* $m, m' \in M$: $\mu(m) \cdot m' = [m, m']$.

Main features of crossed modules include the realization of 3-cohomology classes and the interpretation of $\mathfrak{m} \rtimes \mathfrak{n} \to \mathfrak{n}$ as a strict Lie 2-algebras with target map $\beta(m, n) = \mu(m) + n$.

Construction of abelian representatives

Given a short exact sequence of $\mathfrak{g}$-modules

$$0 \to V_1 \xrightarrow{i} V_2 \xrightarrow{\pi} V_3 \to 0$$

and an abelian extension

$$0 \to V_3 \to V_3 \times_\alpha \mathfrak{g} \to \mathfrak{g} \to 0,$$

their Yoneda product is a crossed module

$$V_2 \to V_3 \times_\alpha \mathfrak{g}$$

with 3-cohomology class $\partial[\alpha] \in H^3(\mathfrak{g}, V_1)$. It is called an *abelian representative* of the equivalence class of crossed modules associated to the 3-cohomology class $\partial[\alpha] = [\theta]$.

Construction of the connecting homomorphism

Let $\sigma : V_3 \to V_2$ be a linear section of π.

$$\begin{array}{ccccc}
 & & \tilde{\alpha} = \sigma\alpha & \overset{\pi}{\longmapsto} & \alpha \\
 & & \big| & & \big| \\
C^2(\mathfrak{g}, V_1) & \overset{i_*}{\rightarrowtail} & C^2(\mathfrak{g}, V_2) & \overset{\pi_*}{\twoheadrightarrow} & C^2(\mathfrak{g}, V_3) \\
\Big\downarrow d & & \Big| d & & \Big| d \\
C^3(\mathfrak{g}, V_1) & \overset{i_*}{\rightarrowtail} & C^3(\mathfrak{g}, V_2) & \overset{\pi_*}{\twoheadrightarrow} & C^3(\mathfrak{g}, V_3) \\
 & & \Big\downarrow & & \Big\downarrow \\
\partial\alpha = \theta & \overset{i}{\longmapsto} & d\tilde{\alpha} = \Phi & \overset{\pi}{\longmapsto} & 0 = d\alpha
\end{array}$$

Example of an abelian representative

Let $W_1 = \prod_{n \geq -1} \mathbb{C}\, z^{n+1} \frac{d}{dz}$ Lie algebra of formal vector fields in one variable, F_i the W_1-module of formal i-forms $(i = 0, 1)$, then

$$F_0 \to F_1 \times_\alpha W_1$$

is a crossed module representing the Godbillon-Vey 3-class $[\theta] \in H^3(W_1, \mathbb{C})$ with

$$\alpha(f,g) = \begin{vmatrix} f' & g' \\ f'' & g'' \end{vmatrix} (dz)^1, \quad \theta(f,g,h) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix}.$$

For the subalgebra $\mathfrak{sl}_2(\mathbb{C}) \subset W_1$, we get a crossed module representing the Cartan cocycle $\langle [,], \rangle \in Z^3(\mathfrak{sl}_2(\mathbb{C}), \mathbb{C})$.

Explicit formula for the cochain α

Let $\mathfrak{g}$ be a Lie algebra and $U\mathfrak{g}$ its universal enveloping algebra. Furthermore

$$\mathrm{pr}(x) := \sum_{k\geq 0} \frac{(-1)^k}{k+1}(\mathrm{Id} - \eta\epsilon)^{\star(k+1)}(x), \quad \phi_t(x) := \sum_{n\geq 0} \frac{t^n}{n!}\mathrm{pr}^{\star n}(x),$$

$$A_t(x, g) := \sum_{(x)} \phi_{-t}(x^{(1)})\phi_t(x^{(2)}g).$$

Define α for $g_1, g_2 \in \mathfrak{g}$ and $x \in U\mathfrak{g}$ by

$$\tilde{\alpha}(x, g_1, g_2) := \sum_{(x)} \int_0^1 dt\, \theta\big(\mathrm{pr}(x^{(1)}), A_t(x^{(2)}, g_1), A_t(x^{(3)}, g_2)\big).$$

Proposition

If θ is a 3-cocycle with values in $\mathbb{C}$, then α is a 2-cocycle in $C^2(\mathfrak{g}; U\mathfrak{g}^+)$ and $[\theta] = \partial[\alpha]$ in $H^3(\mathfrak{g}; \mathbb{C})$.

Definition

A quasi-invariant tensor for $\mu : \mathfrak{m} \to \mathfrak{n}$ *is a triple* (r, c, ξ) *where:*

(a) $r = \sum_q s_q \otimes t_q$ *is a symmetric tensor;*

(b) $\xi : \mathfrak{n} \to (\mathfrak{n} \otimes \mathfrak{m}) \oplus (\mathfrak{m} \otimes \mathfrak{n})$ *is a linear map whose image is symmetric,*

(c) c *is an* $\mathfrak{n}$*-invariant element in* $\ker(\mu) \subset \mathfrak{m}$.

This data is supposed to satisfy the following conditions:

(a) $X \cdot r = \beta(\xi(X))$ *for all* $X \in \mathfrak{n}$,

(b) $u \cdot r = \xi(\mu(u))$ *for all* $u \in \mathfrak{m}$,

(c) $\xi([X, Y]) = X \cdot \xi(Y) - Y \cdot \xi(X)$ *for all* $X, Y \in \mathfrak{n}$.

Construction of quasi-invariant tensors

(a) Let $\overline{r}$ be a lift of r to $V_3 \times_\alpha \mathfrak{g} =: E$,

(b) let $\xi = -\xi_0 - C : E \to (E \otimes V_2) \oplus (V_2 \otimes E)$ where
   (i) $\xi_0(\overline{X}) = \sum_i \widetilde{\alpha}(s_i, X) \otimes \overline{t}_i + \overline{s}_i \otimes \widetilde{\alpha}(t_i, X)$ and
   (ii) $\xi_0(\overline{h}) = \sum_i (s_i \cdot \sigma(h)) \otimes \overline{t}_i + \overline{s}_i \otimes (t_i \cdot \sigma(h))$, and
   (iii) $C(h, X) = 1_{V_2} \otimes \overline{X} + \overline{X} \otimes 1_{V_2}$.

(c) and let $c \in \ker(\mu) = \mathbb{C} \subset V_2$.

Compatibility condition: We have for all $X, Y \in \mathfrak{g}$

$$\sum_i \Phi(s_i, X, Y) \otimes t_i = 1_{V_2} \otimes [X, Y], \sum_i s_i \otimes \Phi(t_i, X, Y) = [X, Y] \otimes 1_{V_2}.$$

Theorem (RW 2015)

Suppose that in the above situation the compatibility condition is satisfied. Then the triplet $(\overline{r}, \xi, c)$ is a quasi-invariant tensor for the crossed module $\mu : V_2 \to V_3 \times_\alpha \mathfrak{g}$.