Bisections of Lie groupoids as a link between infinite-dimensional and higher geometry

Christoph Wockel (Hamburg)

(with A. Schmeding, NTNU Trondheim)

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Outline

Motivating examples

Reconstructing Lie groupoids from their bisections

Constructing Lie groupoids from candidates for their bisections

Motivating example 1

(M,ω): symplectic manifold.

(M,ω) prequantisable $\rightsquigarrow$ prequantum bundle $P \xrightarrow{U(1)} M$ induces

$$C^\infty(M,U(1)) \to \mathrm{Aut}(P) \to \mathrm{Diff}(M)_0$$

(extension of Lie groups) with Lie algebra extension

$$C^\infty(M) \to \underbrace{\Gamma(TP/U(1))}_{\cong C^\infty(M)\oplus_{\overline{\omega}}\mathcal{V}(M)} \to \mathcal{V}(M)$$

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If (M, ω) is not prequantisable, then

$$C^\infty(M) \to C^\infty(M) \oplus_{\overline{\omega}} \mathcal{V}(M) \to \mathcal{V}(M) \tag{1}$$

(extension of Lie algebras) with

$$[(f, X), (g, Y)] := (X.g - Y.f + \omega(X, Y), [X, Y])$$

still exists.

Question 1: Could it be that (1) integrates to an extension of Lie groups although (M, ω) is not prequantisable?

Motivating example 2

$\mathcal{L} := (A \to M, [\cdot,\cdot], \rho)$: Lie algebroid

If $\mathcal{L}$ is integrable ($\mathcal{L} = \mathbf{L}(\mathcal{G})$ for $\mathcal{G} = (G \rightrightarrows M)$ some Lie groupoid), then

$$\mathrm{Bis}(\mathcal{G}) := \{\sigma \in C^\infty(M, G) \mid s \circ \sigma = \mathrm{id}_M, t \circ \sigma \in \mathrm{Diff}(M)\}$$

is a Lie group with Lie algebra $\mathbf{L}(\mathrm{Bis}(\mathcal{G})) \cong \Gamma(\mathbf{L}(\mathcal{G}))$.

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Question 2: Could it be that $\Gamma(\mathcal{L})$ integrates to a Lie group although $\mathcal{L}$ does not integrate to a Lie groupoid?

Note: Obstructions for integrability of extensions of Lie algebras and Lie algebroids look very similar (e.g. $\pi_2(\mathrm{Diff}(M))$ and $\pi_2(M)$)

$\rightsquigarrow$ suggests a geometric relation b/w them!

Motivating example 3

(M, ω): symplectic manifold

$\Lambda := F(\pi_1(\mathrm{Symp}(\mathrm{M}, \omega))) \subseteq H^1(M, \mathbb{R})$ Flux (sub)group, where

$$F \colon \widetilde{\mathrm{Symp}} \to H^1(M, \mathbb{R}), \quad [\gamma_t] \mapsto \int_0^1 \omega(\gamma(t)^{-1}\gamma'(t), \cdot)dt$$

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Flux Conjecture (Ono '06) implies that Λ is always discrete and that $\mathrm{Ham}(M, \omega) := \langle \exp(\mathfrak{ham}(M, \omega)) \rangle \subseteq \mathrm{Symp}(M, \omega)$ is a closed Lie subgroup (in the C^∞-topology).

(M, ω) Γ-prequantisable (for $\Gamma \subseteq \mathbb{R}$ discrete) $\Rightarrow \Lambda \subseteq H^1(M, \Gamma)$ (trivial case of the Flux conjecture)

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Question 3: Where does the non-discreteness go if (M,ω) is not prequantisable?

Outline

The naïve way

Question: What does $\mathrm{Bis}(\mathcal{G})$ know of $\mathcal{G}$?

$$\mathrm{Bis}(\mathcal{G}) := \{\sigma \in C^\infty(M, G) \mid s \circ \sigma = \mathrm{id}_M, t \circ \sigma \in \mathrm{Diff}(M)\}$$

- $\rightsquigarrow$ obtain smooth action $\mathrm{Bis}(\mathcal{G}) \xrightarrow{t_*} \mathrm{Diff}(M)$
- $\rightsquigarrow$ obtain $\underbrace{\mathrm{Bis}(\mathcal{G}) \rtimes M}_{\text{Lie groupoid}} \xrightarrow{\mathrm{ev}} \mathcal{G}, \quad \underbrace{(\sigma, m) \mapsto \sigma(m)}_{\text{morphism}}$

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Theorem

ev *is a quotient morphism of Lie groupoids if* $\mathcal{G}$ *is source-connected. In particular,*

$$(\mathrm{Bis}(\mathcal{G}) \rtimes M)/\ker(\mathrm{ev}) \cong \mathcal{G}.$$

More generally, ev *is the counit of an adjunction*

$$\mathrm{LieGroups}_{\mathrm{Diff}(M)} \underset{\mathrm{Bis}}{\overset{\rtimes}{\rightleftarrows}} \mathrm{LieGroupoids}_M^{\Sigma}$$

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Upshot: In addition to a smooth action $H \to \mathrm{Diff}(M)$, we need a candidate for $\ker(\mathrm{ev})$ in order to identify H with some $\mathrm{Bis}(\mathcal{G})$.
Problem: Construction of quotients of Lie groupoids by normal subgroupoids only works in the transitive (finite-dimensional) case.

The elaborate way

Choose once and for all a base-point $m \in M$ (and restrict from now on to the transitive case).

Definition

A transitive pair is a smoothly transitive action $H \xrightarrow{\theta} \mathrm{Diff}(M)$, together with $N_m \trianglelefteq H_m$ (regular, co-Banach).

Note: Transitive pairs are Klein geometries for principal bundles!

Example

$P \to M$: principal K-bundle $\Rightarrow \mathrm{Aut}(P) \to \mathrm{Diff}(M)$, together with $\mathrm{Aut}_p(P)$ is a transitive pair.

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(θ, N) trans. pair $\Rightarrow H/N_m \to M$ principal H_m/N_m-bundle.

Proof: Use ∞-dim. implicit function theorems for co-Banach Lie subgroups.

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$P \cong \mathrm{Aut}(P)/\mathrm{Aut}_p(P) \to M$ w.r.t. $K \cong \mathrm{Aut}(P)_m/\mathrm{Aut}_p(P)$

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Transitive pairs from geometric Lie group extensions

Setting: M: 1-connected, $K \leq \mathrm{Diff}(M)$ connected, $K.m = M$,
$\omega \in \Omega^2(M)$ closed

$\rightsquigarrow \overline{\omega}\colon \mathfrak{k} \times \mathfrak{k} \to C^\infty(M)$, $(X, Y) \mapsto \omega(X, Y)$ abelian cocycle

$\rightsquigarrow \mathrm{per}_{[\overline{\omega}]}\colon \pi_2(K) \to \mathbb{R}$, $[\sigma] \mapsto \int_{\mathbb{S}^2} \sigma^* \overline{\omega}^{\mathrm{eq}}$ period homomorphism

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Theorem (Neeb)

If $\mathrm{per}_{[\overline{\omega}]}(\pi_2(K)) \subseteq \Gamma$ *with* $\Gamma \leq \mathbb{R}$ *discrete, then*

$$C^\infty(M) \to C^\infty(M) \oplus_{\overline{\omega}} \mathfrak{k} \to \mathfrak{k}$$

integrates (with $T_\Gamma := \mathbb{R}/\Gamma$*) to*

$$C^\infty(M, T_\Gamma) \to \widehat{K} \to \widetilde{K}.$$

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$\rightsquigarrow$ obtain: action $\theta\colon \widehat{K} \to \widetilde{K} \to K \to \mathrm{Diff}(M)$

$\rightsquigarrow$ need: normal subgroup $N_m \trianglelefteq \widehat{K}_m$

Candidate: Integration of $C^\infty_m(M) \oplus_{\overline{\omega}} \mathfrak{k}_m$ (if possible)

$(C^\infty_m(M) \oplus_{\overline{\omega}} \mathfrak{k}_m \cong \mathbf{L}(\mathrm{Aut}_p(P))$ if $[\omega] \in H^2(M, \Gamma))$

The main theorem

The fibration $K \xrightarrow{\mathrm{ev}_m} M$ and $\mathrm{per}_{[\omega]} \colon \pi_2(M) \to \mathbb{R}$ induce

$$\begin{array}{ccccc} \mathrm{im}(\pi_2(K) \to \pi_2(M)) & \longrightarrow & \pi_2(M) & \longrightarrow & \Delta := \ker(\pi_1(K_m) \to \pi_1(K)) \\ \downarrow{\scriptstyle \mathrm{per}_{[\overline{\omega}]}} & & \downarrow{\scriptstyle \mathrm{per}_{[\omega]}} & & \downarrow{\scriptstyle \mathrm{per}^\flat_{[\omega]}} \\ \Gamma & \longrightarrow & \mathbb{R} & \longrightarrow & T_\Gamma \end{array}$$

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Theorem

The following are equivalent:

1. (M, ω) *is prequantisable.*
2. $\mathrm{per}_{[\overline{\omega}]}(\pi_2(K))$ *and* $\mathrm{per}^{\flat}_{[\omega]}(\Delta)$ *are discrete.*
3. *The Lie algebroid* $C^\infty(M) \oplus_{\overline{\omega}} \mathfrak{k}$ *integrates to a Lie groupoid.*
4. $C^\infty(M) \oplus_{\overline{\omega}} \mathfrak{k}$ *integrates to an extension of Lie groups and* $C^\infty_m(M) \oplus_{\overline{\omega}} \mathfrak{k}_m$ *to a closed Lie subgroup thereof.*

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Upshot: Discreteness of $\mathrm{per}^{\flat}_{[\omega]}(\Delta)$ is *not* relevant for abstract integrability, but for the integrability of a subalgebra to a closed Lie subgroup (cf. $\langle \exp(\mathfrak{ham}(M, \omega)) \rangle$).