Canonical Quantization of Poincaré BFCG Theory

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2-group

- A 2-group is a 2-category with one object where all the 1-morphisms and all the 2-morphisms are invertible.
- Equivalent to a pair of groups (G, H) with a group action $\triangleright : G \times H \to H$ and a homomorphism $\partial : H \to G$ such that

$$\partial(g \triangleright h) = g(\partial h)g^{-1}\,, \quad (\partial h) \triangleright h' = hh'h^{-1}\,.$$

- The morphisms are the elements of G, while the 2-morphisms are the elements of the semi-direct product group $G \times_s H$.
- Poincaré 2-group: $G = SO(3,1)$, $H = \mathbf{R}^4$,

$$g \triangleright h = h' \Leftrightarrow \Lambda(g)v(h) = v(h')$$

and $\partial h = Ide$.

2-connection

- If (G, H) is a 2-Lie group and M a manifold, then a 2-connection on M is a pair of forms (A, β) on M such that A is a one-form taking values in **g**, and β is a two-form taking values in **h**,

$$A \to g^{-1}(A + d)\, g\,, \quad \beta \to g^{-1} \triangleright \beta$$

for $g : M \to G$ and

$$A \to A + \partial\epsilon\,, \quad \beta \to \beta + d\epsilon + A \wedge^{\triangleright} \epsilon + \epsilon \wedge \epsilon$$

where ϵ is a one-form from **h** and

$$A \wedge^{\triangleright} \epsilon = A^I \wedge \epsilon^\alpha \, \Delta_{I\alpha}^{\beta} T_\beta\,.$$

- Δ are the structure constants defined by the group action $\triangleright$ for the corresponding Lie algebras. Hence $X_I \triangleright T_\alpha = \Delta_{I\alpha}^{\beta} T_\beta$, where X is a basis for **g** and T is a basis for **h**.

2-connection

- In the Poincaré 2-group case

$$A(x) = \omega^{ab}(x)\, J_{ab}\,, \quad \beta(x) = \beta^a(x)\, P_a\,,$$

where J are the Lorentz group generators and P are the translation generators.

- Infinitesimal gauge transformations

$$\delta_\lambda \omega^{ab} = d\lambda^{ab} + \omega_c^{[a}\, \lambda^{b]c}\,, \quad \delta_\lambda \beta^a = \lambda^a_c\, \beta^c\,.$$

- Infinitesimal 2-morphism gauge transformations

$$\delta_\epsilon \omega = 0\,, \quad \delta_\epsilon \beta^a = d\epsilon^a + \omega^a_c \wedge \epsilon^c\,.$$

2-curvature

- The curvature for a 2-connection (A, β) is a pair of a 2-form $\mathcal{F} \in \mathbf{g}$ and a 3-form $\mathcal{G} \in \mathbf{h}$, given by

$$\mathcal{F} = dA + A \wedge A - \partial\beta\,, \quad \mathcal{G} = d\beta + A \wedge^{\triangleright} \beta\,.$$

- In the Poincaré 2-group case, we have

$$\mathcal{F}^{ab} \equiv R^{ab} = d\omega^{ab} + \omega^{a}{}_{c} \wedge \omega^{cb}$$

$$\mathcal{G}^{a} \equiv G^{a} = \nabla\beta^{a} = d\beta^{a} + \omega^{a}{}_{c} \wedge \beta^{c}\,,$$

so that R^{ab} is the usual spin-connection curvature. The $\partial\beta$ term does not appear in R^{ab} beacuse $\partial\beta = 0$ for the Poincaré 2-group.

BFCG theory

- The dynamics of flat 2-connections for the Poincaré 2-group is given by the BFCG action

$$S = \int_M \left(B_{ab} \wedge R^{ab} + e_a \wedge G^a \right)$$

where B^{ab} is a 2-form and e_a are the tetrads.

- The Lagrange multipliers B and e transform under the usual gauge transformations as

$$B \to g^{-1} Bg \,, \quad e \to g \triangleright e \,,$$

while the 2-morphism transformations are given by

$$B_{ab} \to B_{ab} + e_{[a} \wedge \epsilon_{b]} \,, \quad e_a \to e_a \,.$$

BFCG theory

- If a constraint

$$B_{ab} = \epsilon_{abcd}\, e^c \wedge e^d \,,$$

is imposed in the BFCG action, one obtains a theory which is equivalent to the Einstein-Cartan formulation of General Relativity

$$S_{EC} = \int_M \epsilon^{abcd}\, e_a \wedge e_b \wedge R_{cd} \,.$$

- More precisely, the action

$$S_{2PC} = \int_M \left[B_{ab} \wedge R^{ab} + e_a \wedge G^a - \phi^{ab} \wedge \Big(B_{ab} - \epsilon_{abcd} e^c \wedge e^d \Big) \right] .$$

is dynamically equivalent to S_{EC}.

Canonical formulation

- Dirac: Given an action for variables $Q : \mathbf{R} \to \mathbf{R}^n$

$$S = \int_a^b L(Q, \dot{Q})\, dt\,,$$

where $\dot{Q} = dQ/dt$, then S is dynamically equivalent to

$$S_D = \int_a^b dt \left[P_k \dot{Q}^k - H_0(P,Q) - \lambda^a\, G_a(P,Q) - \mu^\alpha\, \theta_\alpha(P,Q) \right] .$$

- The first-class constraintsl satisfy

$$\{G_a, G_b\}_D = f_{ab}{}^c(P,Q)\, G_c\,, \quad \{G_a, H_0\}_D = h_a^b(P,Q)\, G_b\,,$$

where

$$\{X, Y\}_D = \{X, Y\} - \{X, \theta_\alpha\}\Delta^{\alpha\beta}\{\theta_\beta, Y\}\,,$$

$\Delta = \{\theta, \theta\}^{-1}$ and

$$\{X, Y\} = \frac{\partial X}{\partial Q^k}\frac{\partial Y}{\partial P_k} - \frac{\partial X}{\partial P_k}\frac{\partial Y}{\partial Q^k}\,.$$

Canonical formulation

- If $L(Q,\dot{Q}) = p_i \dot{q}^i - \lambda^c G_c(p,q)$ and
$$\{G_c, G_d\}^* = f_{cd}{}^e(p,q)\, G_e \,,$$
where $\{,\}^*$ is the (p,q) Poisson bracket, then S is a gauge-fixed form of S_D where the second-class constraints have been eliminated and some of the phase-space coordinates have been set to zero.
- In the Poincare BFCG case, let $M = \Sigma \times \mathbf{R}$, and
$$X_{\mu\cdots}\, Y^{\mu\cdots} = X_{0\cdots}\, Y^{0\cdots} + X_{i\cdots}\, Y^{i\cdots} \,,$$
so that
$$\mathcal{L} = \pi^i{}_{ab}\dot{\omega}_i^{ab} + \Pi^{ij}_a \dot{\beta}^a_{ij} - \lambda_1 \mathcal{C}_1 - \lambda_2 \mathcal{C}_2 - \Lambda_1 \mathcal{G}_1 - \Lambda_2 \mathcal{G}_2 \,,$$
where
$$\mathcal{C}^i_{1ab} = \frac{1}{2}\epsilon^{ijk} R_{ab\,jk} \,, \quad \mathcal{C}^a_2 = \frac{1}{2}\epsilon^{ijk}\nabla_i \beta^a_{jk} \,,$$
$$\mathcal{G}_{1ab} = \nabla_i \pi^i{}_{ab} - \beta_{[a|ij}\, \Pi^{ij}_{b]} \,, \quad \mathcal{G}^{ai}_2 = \nabla_j \Pi^{aji} \,,$$

Canonical formulation

and

$$\pi^{ab\,i} = \frac{1}{2}\,\epsilon^{ijk}\,B^{ab}_{jk}\,, \quad \Pi^{a\,ij} = -\frac{1}{2}\,\epsilon^{ijk}\,e^a_k\,.$$

- Constraint algebra

$$\{C_I(x), C_J(y)\} = f_{IJ}^{K}\,C_K(x)\,\delta(x-y)\,.$$

- Poincare BFCG is dynamically the same as the Poincare BF theory

$$\int_M \left(B^{ab}\wedge R_{ab} + e^a \wedge \nabla\beta_a\right) = \int_M \left(B^{ab}\wedge R_{ab} + \nabla e^a \wedge \beta_a\right)$$

$$= \int_M \left(B^{ab}\wedge R_{ab} + \beta^a \wedge T_a\right)\,.$$

Canonical formulation

- Canonical formulation for BF Poincare:

$$A_i^I = (\omega_i^{ab}, e_i^a)\,, \quad E_I^i = (\pi^i{}_{ab}, p_a^i)\,,$$

  and the constraint algebra is the same as in the 2-Poincare case.
- Relation to the CF for 2-Poincare: $(e,p) \to (\beta, \Pi)$ such that

$$\beta_{ij}^a = \varepsilon_{ijk}\, p^{ak}\,, \quad \Pi_a^{ij} = -\epsilon^{ijk}\, e_{ak}\,.$$

Canonical quantization

- $(p, q) \in \mathbf{R}^n \times Q_n \to (\hat{p}, \hat{q})$ acting on $L_2(Q_n)$ such that

$$\hat{p}_k \, \Psi(q) = i \, \frac{\partial \Psi(q)}{\partial q^k} \, , \quad \hat{q}_k \, \Psi(q) = q_k \, \Psi(q) \, .$$

- Constrained CQ: solve $\hat{C}_I \Phi(q) = 0$ such that $\Phi(q) \in L_2(Q^*)$ and $Q^* \subset Q_n$.
- In the BF case Q^* is the space of flat connections on Σ modulo gauge transformations, i.e. $Q^* = \mathcal{M}(\Sigma)$. Hence

$$Q^*_{PBF} = \mathcal{M}_{ISO(3,1)}(\Sigma) = VB(\mathcal{M}_{SO(3,1)}) \, ,$$

where the fibers are solutions of $de + \omega \wedge e = 0$ and $\omega \in \mathcal{M}_{SO(3,1)}$.
- Because of the dynamical equivalence

$$Q^*_{2PF} \simeq Q^*_{PBF} \, .$$

Loop quantization

- Instead of (A^I_i, E^i_I) use $(Hol_\gamma(A), \Phi_\sigma(B))$ to quantize, where $\gamma = \partial\sigma$ and $B_{ij} = \epsilon_{ijk}E^k$.
- Represent the flux-holonomy algebra in the spin-network basis

$$W_{\hat{\gamma}}(A) = Tr\left(\prod_{v\in\gamma} C^{(\iota_v)} \prod_{l\in\gamma} D^{(\Lambda_l)}(A)\right) \equiv \langle A|\hat{\gamma}\rangle\,,$$

  where $\hat{\gamma} = (\gamma, \Lambda, \iota)$ denotes a spin network associated to a closed graph γ.
- When A is a flat connection, than W is invariant under a homotopy of the graph γ, so that we can label the spin-network wavefunctions by combinatorial (abstract) graphs γ.

Loop quantization

- By requiring that $W_{\hat{\gamma}}(A)$ form a basis in $\mathcal{H}$, we obtain

$$|\Psi\rangle = \int DA\,|A\rangle\langle A|\Psi\rangle = \sum_{\hat{\gamma}} |\hat{\gamma}\rangle\langle\hat{\gamma}|\Psi\rangle$$

where

$$\langle\hat{\gamma}|\Psi\rangle = \int DA\,\langle\hat{\gamma}|A\rangle\langle A|\Psi\rangle = \int DA\,W^*_{\hat{\gamma}}(A)\,\Psi(A)$$

is the loop transform.

2-holonomy quantization

- How to generalize the spin-network basis $W_{\hat{\gamma}}(A)$ to a spin-foam basis $W_{\hat{\Gamma}}(A,\beta)$ where $\hat{\Gamma} = (\Gamma, L, \Lambda, \iota)$?
- Conjecture

$$W_{\hat{\Gamma}}(\omega_l, \beta_f) = Tr\left(\prod_{v\in\Gamma} C^{(\iota_v)} \prod_{l\in\Gamma} D^{(\Lambda_l)}(\omega) \prod_{f\in\Gamma} D^{(L_f)}(\omega,\beta)\right)$$

where

$$D^{(L_f)}(\omega,\beta) = D^{(L_f)}(g_{l(f)} \triangleright h_f)\,,$$

and $f \in \partial p$ such that

$$h_p = \prod_{f\in\partial p} g_{l(f)} \triangleright h_f \,,$$

is a 2-holonomy.

Conclusions

- In the 2-Poincare case one can also use

$$W_{\hat{\Gamma}}(\omega_l, \beta_f) \sim W_{\hat{\gamma}}(\omega_l, e_l) \sim \tilde{W}_{\hat{\gamma}}(\omega_l, \beta_\Delta)\,,$$

  where $\tilde{W}_{\hat{\gamma}}$ is the Fourier transform of $W_{\hat{\gamma}}$. This may give clues about a Peter-Weil theorem for 2-groups.
- Quantum Gravity implications: $SU(2)$ spin-network basis can be generalized to a spin-foam basis (edge lengths and face areas) for the 3d Euclidean 2-group.

References

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