Canonical Quantization of Poincaré BFCG Theory

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A 2-group is a 2-category with one object where all the 1-morphisms and all the 2-morphisms are invertible.

Equivalent to a pair of groups \((G, H)\) with a group action \(\triangleright: G \times H \to H\) and a homomorphism \(\partial: H \to G\) such that

\[
\partial(g \triangleright h) = g(\partial h)g^{-1}, \quad (\partial h) \triangleright h' = hh'h^{-1}.
\]

The morphisms are the elements of \(G\), while the 2-morphisms are the elements of the semi-direct product group \(G \rtimes_s H\).

Poincaré 2-group: \(G = SO(3, 1), \, H = \mathbb{R}^4\),

\[
g \triangleright h = h' \iff \Lambda(g)v(h) = v(h')
\]

and \(\partial h = \text{Id}e\).
If \((G, H)\) is a 2-Lie group and \(M\) a manifold, then a 2-connection on \(M\) is a pair of forms \((A, \beta)\) on \(M\) such that \(A\) is a one-form taking values in \(g\), and \(\beta\) is a two-form taking values in \(h\),

\[
A \rightarrow g^{-1}(A + d)g, \quad \beta \rightarrow g^{-1} \triangleright \beta
\]

for \(g : M \rightarrow G\) and

\[
A \rightarrow A + \partial \epsilon, \quad \beta \rightarrow \beta + d \epsilon + A \wedge^\triangleright \epsilon + \epsilon \wedge \epsilon
\]

where \(\epsilon\) is a one-form from \(h\) and

\[
A \wedge^\triangleright \epsilon = A^I \wedge \epsilon^\alpha \Delta^\beta_{I\alpha} T_{\beta}.
\]

\(\Delta\) are the structure constants defined by the group action \(\triangleright\) for the corresponding Lie algebras. Hence \(X_I \triangleright T_\alpha = \Delta^\beta_{I\alpha} T_\beta\), where \(X\) is a basis for \(g\) and \(T\) is a basis for \(h\).
In the Poincaré 2-group case

\[ A(x) = \omega^{ab}(x) J_{ab}, \quad \beta(x) = \beta^a(x) P_a, \]

where \( J \) are the Lorentz group generators and \( P \) are the translation generators.

Infinitesimal gauge transformations

\[ \delta_\lambda \omega^{ab} = d\lambda^{ab} + \omega_c^{[a} \lambda^{b]} c, \quad \delta_\lambda \beta^a = \lambda^a_c \beta^c. \]

Infinitesimal 2-morphism gauge transformations

\[ \delta_\epsilon \omega = 0, \quad \delta_\epsilon \beta^a = d\epsilon^a + \omega^a_c \wedge \epsilon^c. \]
The curvature for a 2-connection \((A, \beta)\) is a pair of a 2-form \(F \in g\) and a 3-form \(G \in h\), given by

\[
F = dA + A \wedge A - \partial \beta, \quad G = d\beta + A \wedge^\triangledown \beta.
\]

In the Poincaré 2-group case, we have

\[
F^{ab} \equiv R^{ab} = d\omega^{ab} + \omega^a_c \wedge \omega^{cb}
\]

\[
G^a \equiv G^a = \nabla \beta^a = d\beta^a + \omega^a_c \wedge \beta^c,
\]

so that \(R^{ab}\) is the usual spin-connection curvature. The \(\partial \beta\) term does not appear in \(R^{ab}\) because \(\partial \beta = 0\) for the Poincaré 2-group.
The dynamics of flat 2-connections for the Poincaré 2-group is given by the BFCG action

\[
S = \int_M \left( B_{ab} \wedge R^{ab} + e_a \wedge G^a \right)
\]

where \( B^{ab} \) is a 2-form and \( e_a \) are the tetrads.

The Lagrange multipliers \( B \) and \( e \) transform under the usual gauge transformations as

\[
B \rightarrow g^{-1} B g, \quad e \rightarrow g \triangleright e,
\]

while the 2-morphism transformations are given by

\[
B_{ab} \rightarrow B_{ab} + e_{[a} \wedge \epsilon_{b]}, \quad e_a \rightarrow e_a.
\]
If a constraint

\[ B_{ab} = \epsilon_{abcd} e^c \wedge e^d, \]

is imposed in the BFCG action, one obtains a theory which is equivalent to the Einstein-Cartan formulation of General Relativity

\[ S_{EC} = \int_M \epsilon^{abcd} e_a \wedge e_b \wedge R_{cd}. \]

More precisely, the action

\[ S_{2PC} = \int_M \left[ B_{ab} \wedge R^{ab} + e_a \wedge G^a - \phi^{ab} \wedge \left( B_{ab} - \epsilon_{abcd} e^c \wedge e^d \right) \right]. \]

is dynamically equivalent to \( S_{EC} \).
Canonical formulation

- Dirac: Given an action for variables $Q : \mathbb{R} \to \mathbb{R}^n$

$$S = \int_a^b L(Q, \dot{Q}) \, dt ,$$

where $\dot{Q} = dQ/dt$, then $S$ is dynamically equivalent to

$$S_D = \int_a^b dt \left[ P_k \dot{Q}^k - H_0(P, Q) - \lambda^a G_a(P, Q) - \mu^\alpha \theta_\alpha(P, Q) \right] .$$

- The first-class constraints satisfy

$$\{ G_a, G_b \}_D = f_{ab}^c(P, Q) \, G_c , \quad \{ G_a, H_0 \}_D = h_a^b(P, Q) \, G_b ,$$

where

$$\{ X, Y \}_D = \{ X, Y \} - \{ X, \theta_\alpha \} \Delta^{\alpha\beta} \{ \theta_\beta, Y \} ,$$

$$\Delta = \{ \theta, \theta \}^{-1} \quad \text{and} \quad \{ X, Y \} = \frac{\partial X}{\partial Q^k} \frac{\partial Y}{\partial P_k} - \frac{\partial X}{\partial P_k} \frac{\partial Y}{\partial Q^k} .$$
Canonical formulation

If \( L(Q, \dot{Q}) = p_i \dot{q}^i - \lambda^c G_c(p, q) \) and

\[
\{ G_c, G_d \}^* = f_{cd}^e(p, q) G_e,
\]

where \( \{ , \}^* \) is the \((p, q)\) Poisson bracket, then \( S \) is a gauge-fixed form of \( S_D \) where the second-class constraints have been eliminated and some of the phase-space coordinates have been set to zero.

In the Poincare BFCG case, let \( M = \Sigma \times \mathbf{R} \), and

\[
X_{\mu\ldots} Y^{\mu\ldots} = X_{0\ldots} Y^{0\ldots} + X_{i\ldots} Y^{i\ldots},
\]

so that

\[
\mathcal{L} = \pi_{ab}^i \dot{\omega}^i_{ab} + \Pi_a^i \dot{\beta}_a^i - \lambda_1 C_1 - \lambda_2 C_2 - \Lambda_1 G_1 - \Lambda_2 G_2,
\]

where

\[
C_{1ab}^i = \frac{1}{2} \epsilon^{ijk} R_{abjk}, \quad C_a^i = \frac{1}{2} \epsilon^{ijk} \nabla_i \beta_{jk}^a,
\]

\[
G_{1ab} = \nabla_i \pi_{ab}^i - \beta_{[a|ij} \Pi_{b]}^{ij}, \quad G_{2}^{ai} = \nabla_j \Pi^{aji},
\]
and
\[ \pi^{ab i} = \frac{1}{2} \epsilon^{ijk} B^{ab}_{jk}, \quad \Pi^{a ij} = -\frac{1}{2} \epsilon^{ijk} e^a_k. \]

- Constraint algebra

\[ \{ C_I(x), C_J(y) \} = f^K_{IJ} C_K(x) \delta(x - y). \]

- Poincare BFCG is dynamically the same as the Poincare BF theory

\[
\int_M \left( B^{ab} \wedge R_{ab} + e^a \wedge \nabla \beta_a \right) = \int_M \left( B^{ab} \wedge R_{ab} + \nabla e^a \wedge \beta_a \right) \\
= \int_M \left( B^{ab} \wedge R_{ab} + \beta^a \wedge T_a \right). 
\]
Canonical formulation

Canonical formulation for BF Poincare:

\[ A_i^I = (\omega_i^{ab}, e_i^a), \quad E_i^I = (\pi_i^a, p_i^a), \]

and the constraint algebra is the same as in the 2-Poincare case.

Relation to the CF for 2-Poincare: \((e, p) \rightarrow (\beta, \Pi)\) such that

\[ \beta^a_{ij} = \varepsilon_{ijk} p^{ak}, \quad \Pi^{ij}_a = -\varepsilon^{ijk} e_{ak}. \]
Canonical quantization

- \((p, q) \in \mathbb{R}^n \times Q_n \rightarrow (\hat{p}, \hat{q})\) acting on \(L_2(Q_n)\) such that

\[
\hat{p}_k \psi(q) = i \frac{\partial \psi(q)}{\partial q^k}, \quad \hat{q}_k \psi(q) = q_k \psi(q).
\]

- Constrained CQ: solve \(\hat{C}_I \Phi(q) = 0\) such that \(\Phi(q) \in L_2(Q^*)\) and \(Q^* \subset Q_n\).

- In the BF case \(Q^*\) is the space of flat connections on \(\Sigma\) modulo gauge transformations, i.e. \(Q^* = \mathcal{M}(\Sigma)\). Hence

\[
Q_{PBF}^* = \mathcal{M}_{ISO(3,1)}(\Sigma) = VB(\mathcal{M}_{SO(3,1)}),
\]

where the fibers are solutions of \(de + \omega \wedge e = 0\) and \(\omega \in \mathcal{M}_{SO(3,1)}\).

- Because of the dynamical equivalence

\[
Q_{2PF}^* \simeq Q_{PBF}^*.
\]
Loop quantization

- Instead of \((A_i^I, E_j^I)\) use \((\text{Hol}_\gamma(A), \Phi_\sigma(B))\) to quantize, where \(\gamma = \partial \sigma\) and \(B_{ij} = \epsilon_{ijk}E^k\).

- Represent the flux-holonomy algebra in the spin-network basis

\[
\mathcal{W}_\gamma(A) = \text{Tr} \left( \prod_{\nu \in \gamma} C^{(\nu)} \prod_{l \in \gamma} D^{(\Lambda_l)}(A) \right) \equiv \langle A|\hat{\gamma}\rangle,
\]

where \(\hat{\gamma} = (\gamma, \Lambda, \nu)\) denotes a spin network associated to a closed graph \(\gamma\).

- When \(A\) is a flat connection, then \(\mathcal{W}\) is invariant under a homotopy of the graph \(\gamma\), so that we can label the spin-network wavefunctions by combinatorial (abstract) graphs \(\gamma\).
By requiring that $W_{\hat{\gamma}}(A)$ form a basis in $\mathcal{H}$, we obtain

$$|\psi\rangle = \int DA |A\rangle \langle A|\psi\rangle = \sum_{\hat{\gamma}} |\hat{\gamma}\rangle \langle \hat{\gamma}|\psi\rangle$$

where

$$\langle \hat{\gamma}|\psi\rangle = \int DA \langle \hat{\gamma}|A\rangle \langle A|\psi\rangle = \int DA W_{\hat{\gamma}}^*(A) \psi(A)$$

is the loop transform.
How to generalize the spin-network basis \( W_{\hat{\gamma}}(A) \) to a spin-foam basis \( W_{\hat{\Gamma}}(A, \beta) \) where \( \hat{\Gamma} = (\Gamma, L, \Lambda, \iota) \) ?

Conjecture

\[
W_{\hat{\Gamma}}(\omega_I, \beta_f) = Tr \left( \prod_{\nu \in \Gamma} C^{(\nu)} \prod_{l \in \Gamma} D^{(\Lambda_l)}(\omega) \prod_{f \in \Gamma} D^{(L_f)}(\omega, \beta) \right)
\]

where

\[
D^{(L_f)}(\omega, \beta) = D^{(L_f)}(g_l(f) \triangleright h_f),
\]

and \( f \in \partial p \) such that

\[
h_p = \prod_{f \in \partial p} g_l(f) \triangleright h_f,
\]

is a 2-holonomy.
Conclusions

- In the 2-Poincare case one can also use

\[ W_{\tilde{\Gamma}}(\omega_I, \beta_f) \sim W_{\tilde{\gamma}}(\omega_I, e_I) \sim \tilde{W}_{\tilde{\gamma}}(\omega_I, \beta_{\Delta}), \]

where \( \tilde{W}_{\tilde{\gamma}} \) is the Fourier transform of \( W_{\tilde{\gamma}} \). This may give clues about a Peter-Weil theorem for 2-groups.

- Quantum Gravity implications: \( SU(2) \) spin-network basis can be generalized to a spin-foam basis (edge lengths and face areas) for the 3d Euclidean 2-group.
References

