Perturbative BV-BFV theories on manifolds with boundary

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Outline

Introduction

- Lift Atiyah–Segal's axioms to the perturbative QFTs
  boundaries $\rightsquigarrow$ vector spaces
  manifolds (with boundaries) $\rightsquigarrow$ states/operators
- Do it for general Lagrangian theories (including gauge theories)
- First understand classical pictureÎ
- then the perturbative quantum BV picture:

Lift Atiyah–Segal to the cochain level

Lagrangian Mechanics

- In Lagrangian mechanics $S = \int_{t_0}^{t_1} L\, dt$ as a functional on the path space $N^{[t_0,t_1]}$.
- Usual example: $L = \frac{1}{2}m||v||^2 - V(q)$.
- Newton's equation are recovered as Euler–Lagrange equations (EL), i.e., critical points: $\delta S = 0$.
- A solution is uniquely specified by its initial conditions. Set $C := TN$, the space of **Cauchy data**. Î
- For this, one sets conditions at t_0 and t_1 (usually by fixing the path endpoints). Otherwise

$$\delta S = \mathrm{EL} + \alpha|_{t_0}^{t_1}$$

$$\alpha = \sum_i \frac{\partial L}{\partial v^i} dq^i \in \Omega^1(C) \qquad \text{Noether's one-form}$$

Here EL denotes the term containing the EL equations. By *EL* we will denote the space of solutions to EL.

Symplectic formulation

$\omega := \mathrm{d}\alpha$ is symplectic iff L is regular. In this case:

- ω is the pullback on $C = TN$ of the canonical symplectic form on T^*N by the Legendre mapping.
- Time evolution is given by a Hamiltonian flow ϕ. In particular,

$$L := \mathrm{graph}\, \phi_{t_0}^{t_1} \in \overline{TN} \times TN$$

is Lagrangian (**canonical relation**).

Î

Remark

L may also be defined directly as $L = \pi(EL)$ with

$$\begin{array}{cccc} \pi: & N^{[t_0,t_1]} & \to & TN \times TN \\ & \{x(t)\} & \mapsto & ((x(t_0), \dot{x}(t_0)), (x(t_1), \dot{x}(t_1))) \end{array}$$

This picture has to be generalized

Example1: Geodesics

We discuss geodesics on $\mathbb{E}^2$ Î(Minkowski would be more realistic).

$$L = ||v||$$

S is defined on $\mathcal{F} := N_0^{[t_0,t_1]} := \{\text{immersed paths}\}$.Î

- $EL =$ straight lines
- Initial data:
  $\mathcal{F}|_{((t_0))} = \mathbb{R}^2 \times \mathbb{R}^2_* \times \mathbb{R}^\infty = \mathbb{R}^2 \times S^1 \times \mathbb{R}_{>0} \times \mathbb{R}^\infty \ni (\mathbf{q}, \mathbf{v}, \rho, \mathbf{q}_2, \mathbf{q}_3, \dots)$.
- $\alpha = \mathbf{v} \cdot \mathrm{d}\mathbf{q}$
- ω degenerate
- $\tilde{L} := \pi(EL) = \{(\mathbf{q}_1, \mathbf{v}, \rho_1, \dots), (\mathbf{q}_2, \mathbf{v}, \rho_2, \dots)) : \mathbf{q}_1 - \mathbf{q}_2 \,||\, \mathbf{v}\}$
  Not a graph!

Geodesics (continued)

However:

- $\omega|_{\tilde{L}} = 0$, so $\tilde{L}$ is isotropic (actually Lagrangian).Î
- $\ker\omega = \text{span}\left(\mathbf{v}\cdot\frac{\partial}{\partial\mathbf{q}}, \frac{\partial}{\partial\rho}, \frac{\partial}{\partial\mathbf{q}_2}, \dots\right) =$
directions parallel to **v**, rescalings of velocity, higher jets;
so
$$\varpi\colon \mathcal{F}|_{((t_0))} \to \mathcal{F}^\partial := \mathcal{F}|_{((t_0))}/\ker\omega = TS^1$$
with canonical symplectic form (identify T and T^* using the metric).
- $L := \varpi(\tilde{L}) = \text{graph}\,\text{Id}$, so a graph and Lagrangian.Î
- Actually, no time evolution after reduction (an example of topological theory).Î
- With target $\mathbb{R}^{n+1}$ and Minkowski metric, one gets $\mathcal{F}^\partial = T\mathcal{H}^n$, with $\mathcal{H}^n$ the n-dimensional hyperboloid with induced hyperbolic metric.

General case

Following ideas by Tulczjiew, Gawedzki, Schwarz, Fock,...

- Let $S_M = \int_M L$ be a class of local actions determined by a Lagrangian L. Here M is a d-manifold.
- S_M is defined on a space of fields F_M (e.g., maps from M to another manifold, connections on M, sections of a fiber bundle,. . . .)Î

To a $(d-1)$-manifold Σ we associate the space $\tilde{F}_\Sigma$ of jets of fields at $\Sigma \times \{0\}$ on $\Sigma \times [0, \epsilon]$ ("normal derivatives").

The boundary term in the variational calculus defines a one-form $\tilde{\alpha}_\Sigma$ on $\tilde{F}_\Sigma$, for every Σ, with the property

$$\delta S_M = \mathrm{EL}_M + \tilde{\pi}_M^* \tilde{\alpha}_{\partial M}$$

with $\tilde{\pi}_M \colon F_M \to \tilde{F}_{\partial M}$ the natural surjective submersion and EL_M the "EL one-form."Î

Define $\tilde{\omega}_\Sigma := \mathrm{d}\tilde{\alpha}_\Sigma$.

Assumption

We assume that $\tilde{\omega}_\Sigma$ is presymplectic for every Σ.

Boundary structure

- Denote by $(F^\partial_\Sigma, \omega^\partial_\Sigma)$ the reduction of $\tilde{F}_\Sigma$ by the kernel of $\tilde{\omega}_\Sigma$.
- For simplicity, we assume that $\tilde{\alpha}_\Sigma$ also descends to a one-form α_Σ on F^∂_Σ.

ThenÎ

1. $\omega_\Sigma = \mathrm{d}\alpha_\Sigma$.
2. For every M, we get a projection $\pi_M\colon F_M \to F^\partial_{\partial M}$ and the equation

$$\delta S_M = \mathrm{EL}_M + \pi_M^* \alpha^\partial_{\partial M}$$

Now define $L_M := \pi_M(EL_M)$, which by the previous equation is automatically isotropic.Î

Boundary structure (continued)

Assumption

We assume that L_M is Lagrangian for every M.Î

Remark

This is a requirement for a well-defined theory. It requires, e.g., that YM, CS and *BF* theories should be defined in terms of Lie algebras or the PSM in terms of a Poisson tensor (not just any bivector field).Î

Definition

For every Σ we define C_Σ as the space of points of F^∂_Σ that can be completed to a pair belonging to $L_{\Sigma\times[0,\epsilon]}$ for some ϵ.

By the assumption, C_Σ is coisotropic. It represents the space of Cauchy data. Its reduction is called the reduced phase space.
Its symplectic reduction is usually singular!

Boundary structure: composition

Remark (Composition)

If $M = M_1 \cup_\Sigma M_2$, where Σ is (part of) the boundary of M_1 and of M_2,

$$L_M = L_{M_1} \circ L_{M_2} \subset F^\partial_{(\partial M_1 \setminus \Sigma) \coprod (\partial M_2 \setminus \Sigma)},$$

where $\circ$ denotes the composition of relations. Î

Definition

We call L_M the **evolution relation**. (More precisely, we split $\partial M = \partial_{\text{in}} M \coprod \partial_{\text{out}} M$ and regard L_M as a relation in $\overline{F^\partial_{(\partial_{\text{in}} M)^{\text{opp}}}} \times F^\partial_{\partial_{\text{out}} M}$.)

Remark (EL)

By definition the fiber of EL_M over L_M is just one point if M is a short cylinder, but in general it may be much bigger.
So it makes sense to remember it and think of $EL_M \to F^\partial_{\partial_M}$ as a correspondence, the **evolution correspondence**.

Axiomatics

We may then think of a classical Lagrangian field theory in d dimensions as the following data:

- A space of field F_M for every d-manifold M
- A symplectic space F^{∂}_{Σ} for every $(d-1)$-manifold Σ
- A Lagrangian correspondence $\pi\colon EL_M \to F^{\partial}_{\partial M}$ for every M.
- $(F_\bullet, C_\bullet)$ should be thought as a functor.Î

Quantization of regular Lagrangian field theories

- In a regular theory, $\underline{C_\Sigma} = C_\Sigma = F^\partial_\Sigma$ is symplectic; geometric quantization: vector space H_Σ.Î
- For simplicity, assume that the symplectic manifold $C_{\partial M}$ is endowed with a Lagrangian foliation along which $\alpha_{\partial M}$ vanishes and with a smooth leaf space $B_{\partial M}$. (One may change $\alpha_{\partial M}$ to this goal.) Then $H_{\partial M}$ is a space of functions on $B_{\partial M}$.ÎDenote by $p_{\partial M}$ the projection $C_{\partial M} \to B_{\partial M}$.
- The canonical relation $L_M \subset C_{\partial M}$ is quantized to a state $\psi_M \in H_{\partial M}$. Asymptotically,

$$\psi_M(\varphi) = \int_{\Phi \in \pi_M^{-1}(p_{\partial M}^{-1}(\varphi))} \mathrm{e}^{\frac{i}{\hbar} S_M(\Phi)}\, [D\Phi], \qquad \varphi \in B_{\partial M}$$

- If $\partial M = \partial_{\text{in}} M \coprod \partial_{\text{out}} M$, then $\psi_M \in H^*_{\partial_{\text{in}} M} \otimes H_{\partial_{\text{out}} M}$.
  Hence, operator $H_{\partial_{\text{in}} M} \to H_{\partial_{\text{out}} M}$.
  Composition of relations goes to composition of operators.
- Cfr. Segal's axiomatization of CFT and Atiyah's axiomatization of TFT.Î

The local, finite-dimensional BV formalism

The Batalin–Vilkovisky (BV) formalism is used to gauge fix gauge theories and check gauge-fixing independence.Î
We start with a local, finite-dimensional version.

- Consider super coordinates q^i, p_i and the symplectic form $\omega = \sum_i \mathrm{d}p_i \mathrm{d}q^i$.
- Functions are ordinary smooth functions of the even coordinates tensor the Grassmann algebra generated by the odd coordinates. Here p_i has parity opposite to q^i. Î
- The BV Laplacian is defined as

$$\Delta = \sum_i (-1)^{|q_i|} \frac{\partial^2}{\partial q^i \partial p_i}.$$

Equivalently, $\Delta f = -\frac{1}{2} \operatorname{div} X_f$.

Lemma

$$\Delta^2 = 0, \qquad \Delta(fg) = \Delta f\, g \pm f \Delta g \pm (f, g).$$

Here (,) denotes the BV bracket (odd Poisson bracket given by ω).

Let f be a function of the p, qs and ψ a function of the qs only. One defines the BV integral

$$\int_{\mathcal{L}_\psi} f := \int f(q, p_i = \partial_i \psi)\, dq^1 \dots dq^n$$

to be intended as the integral of f on the Lagrangian submanifold

$$\mathcal{L}_\psi = \text{graph}\, \mathrm{d}\psi.$$

Remark

$dq^1 \dots dq^n$ denotes Berezinian integration:
In the even coordinates it is the standard integration; in the odd coordinates it is just the selection of the top coefficient in the Grassmann algebra (with a choice of orientation).

Lemma

Assume that integrals converge. Then:

- *If $f = \Delta g$, then $\int_{\mathcal{L}_\psi} f = 0$.*
- *If $\Delta f = 0$, then $\int_{\mathcal{L}_\psi} f$ is invariant under deformations of ψ.*

The main application

- Suppose $\int_{\mathcal{L}_0} f$ is ill defined but $\Delta f = 0$. Then we can replace the ill-defined integral by a well-defined one $\int_{\mathcal{L}_\psi} f$ and the above Lemma says that it does not matter which ψ we choose (as long as the integral converges). This procedure is called gauge fixing.Î
- In view of applications to path integrals, we write $f = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}$. Then $\Delta f = 0$ corresponds to the Quantum Master Equation (QME)

$$\frac{1}{2}(S,S) - \mathrm{i}\hbar\Delta S = 0$$

The central idea is to allow S to depend on the parameter $\hbar$ and solve the QME order by order (if possible). The lowest order term is the Classical Master Equation (CME)

$$(S,S) = 0$$

- The main point here is that the CME may be defined on infinite dimensional manifolds (needed in field theory). Integration together with the actual definition of Δ are deferred to a second step (e.g., perturbative path integral quantization).

Further remarks

- It is convenient to introduce a $\mathbb{Z}$-grading. One assigns degrees so that ω has degree -1, so the Hamiltonian vector field Q of a degree 0 function S has degree $+1$. The CME for S is equivalent to $[Q, Q] = 0$. One says that Q is a cohomological vector field.
- One may generalize the BV integral to a partial integration. Assume a splitting of coordinates $(p, q) = (p', p'', q', q'')$ with $\omega = \omega' + \omega''$ and $\Delta = \Delta' + \Delta''$. If f is a function of all coordinates and ψ a function of the q''s, one defines the BV pushforward

$$\int_{\mathcal{L}_\psi} f := \int f|_{p_i''=\partial_i \psi}\, dq''$$

One can then prove that

$$\Delta' \int_{\mathcal{L}_\psi} f = \int_{\mathcal{L}_\psi} \Delta f$$

and that, if $\Delta f = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{L}_\psi(t)} f = \Delta'(\cdots)$$

- There is a global description of the BV formalism due to A. Schwarz formulated on odd symplectic manifolds. The BV Laplacian is canonically defined on half densities, which in turn can be integrated on Lagrangian submanifolds. BV integration of Δ-closed half densities turns out to be invariant under deformations of Lagrangian submanifolds (and under some further transformations). The BV pushforward may be defined on appropriate fiber bundles. One usually prefers to choose a reference half density and to work with functions again.
- The BV action S satisfying the CME arises in field theory as follows. One starts with an action functional S_0, defined on a space of fields, and its symmetries. One then look for an odd symplectic manifold that contains the space of fields and for an extension of S_0 that satisfies the CME and whose Hamiltonian vector field "restricted" to the original space of fields yields the symmetries. Under certain weak assumptions existence and uniqueness (up to...) is guaranteed.

BV theories

Let us go back to field theory.

- We start with a field theory on a space of fields F_M and an action S^0, plus symmetries.
- If M has no boundary, the BV construction yields a BV manifold $(\mathcal{F}_M, \omega_M, S_M)$, where
  1. $\mathcal{F}_M$ is a supermanifold with additional $\mathbb{Z}$-grading (containing the original F_M as its degree zero component).
  2. ω_M is an odd symplectic form of degree -1 on $\mathcal{F}_M$.
  3. S_M is an even function of degree zero on $\mathcal{F}_M$ which extends the classical action and satisfies the CME

$$(S_M, S_M) = 0.$$

One defines Q_M as the Hamiltonian vector field of S_M

$$\iota_{Q_M}\omega_M = \mathrm{d}S_M$$

Q_M has degree one and $[Q_M, Q_M] = 0$ (cohomological vector field).

The case with boundary

- The equation
$$\iota_{Q_M}\omega_M = \mathrm{d}S_M$$
no longer holds if M has boundary. We have to deal with the boundary terms in computing $\mathrm{d}S_M$ as in the first part of this talk.
- Define the space $\tilde{\mathcal{F}}_\Sigma$ of preboundary fields on a $(d-1)$-manifold Σ as the jets at $\Sigma \times \{0\}$ of $\mathcal{F}_{\Sigma\times[0,\epsilon]}$. Integration by parts in the computation of $\mathrm{d}S_{\Sigma\times[0,\epsilon]}$ yields a one-form $\tilde{\alpha}_\Sigma$ of degree zero on $\tilde{\mathcal{F}}_\Sigma$. We denote by $\tilde{\omega}_\Sigma$ its differential.

Assumption

We assume that $\tilde{\omega}_\Sigma$ is presymplectic.

- Denote by $(\mathcal{F}^\partial_\Sigma, \omega^\partial_\Sigma)$ the reduction of $(\tilde{\mathcal{F}}_\Sigma, \tilde{\omega}_\Sigma)$.
- For simplicity, we assume that $\tilde{\alpha}_\Sigma$ also descends to a one-form α^∂_Σ on $\mathcal{F}^\partial_\Sigma$.

The case with boundary (continued)

Let $\pi_M\colon \mathcal{F}_M \to \mathcal{F}^\partial_{\partial M}$ be the induced surjective submersion.
One can then prove that

1. Q_M descends to a cohomological vector field $Q^\partial_{\partial M}$ which is Hamiltonian w.r.t. $\omega^\partial_{\partial M}$.Î

Remark

One then says that the triple $(\mathcal{F}^\partial_{\partial M}, \omega^\partial_{\partial M}, Q^\partial_{\partial M})$ is a BFV manifold. Notice that the degree of $\omega^\partial_{\partial M}$ is now zero. The zero locus of $Q^\partial_{\partial M}$ is coisotropic. Its degree zero component $C_{\partial M}$ is also coisotropic. If its reduction is smooth, its Poisson algebra of functions is the same as the cohomology of $Q^\partial_{\partial M}$ in degree zero. The BFV construction has to be thought of as a resolution of this quotient.Î

2. We have the fundamental equation of the BV theory for manifolds with boundary [C, Mnëv, Reshetikhin]:

$$\iota_{Q_M}\omega_M = \mathrm{d}S_M + \pi_M^*\alpha^\partial_{\partial M}$$

Example: Electromagnetism

- Maxwell's equations: $\mathrm{d}^*\mathrm{d}A = 0$, A connection 1-form.Î
- First-order formalism: $S^{\mathrm{cl}}_M = \int_M B\,\mathrm{d}A + \frac{1}{2} B * B$
  B a $(d-2)$-form. Then $EL = \{*B = \mathrm{d}A,\ \mathrm{d}B = 0\}$.Î
- BV: $S_M = \int_M B\,\mathrm{d}A + \frac{1}{2} B * B + A^+\,\mathrm{d}c$
  A^+: $(d-1)$-form, ghost number -1; c: 0-form, ghost number 1.
  $\omega_M = \int_M \delta A\,\delta A^+ + \delta B\,\delta B^+ + \delta c\,\delta c^+$,
  B^+ and c^+ do not show up in the action.
  $QA = \mathrm{d}c,\ QA^+ = \mathrm{d}B,\ QB^+ = *B + \mathrm{d}A,\ Qc^+ = \mathrm{d}A^+$.Î
- Boundary fields: A, B, A^+, c,
  $S^\partial_\Sigma = \int_\Sigma c\,\mathrm{d}B$,
  $\alpha^\partial_\Sigma = \int_\Sigma B\,\delta A + A^+\,\delta c$,
  $Q^\partial A^+ = \mathrm{d}B$, $Q^\partial A = \mathrm{d}c$.Î
  Interpretation:
  $A =$ vector potential, up to gauge transformations $A \mapsto A + \mathrm{d}c$
  $B =$ electric field constrained by Gauss law $\mathrm{d}B = 0$.

Properties

The fundamental equation

$$\iota_{Q_M}\omega_M = \mathrm{d}S_M + \pi_M^*\alpha^\partial_{\partial M} \tag{1}$$

has several consequences. Among them

1. Q_M is not symplectic

$$L_{Q_M}\omega_M = \pi_M^*\omega^\partial_{\partial M}$$

2. Modified CME (mCME)

$$\text{“}(S_M, S_M)\text{”} := \iota_{Q_M}\iota_{Q_M}\omega_M = \pi_M^*(2S^\partial_{\partial M})$$

Boundaries of boundaries

- On every boundary component Σ, we now have a BFV manifold $(\mathcal{F}^{\partial}_{\Sigma}, \omega^{\partial}_{\Sigma}, Q^{\partial}_{\Sigma})$. Assume it is given by local data. Let S^{∂}_{Σ} be the Hamiltonian function of Q^{∂}_{Σ}: $\iota_{Q^{\partial}_{\Sigma}}\omega^{\partial}_{\Sigma} = \mathrm{d}S^{\partial}_{\Sigma}$.
- If Σ has a boundary γ, we may repeat the previous construction verbatim. We get
  1. A triple $(\mathcal{F}^{\partial\partial}_{\gamma}, \omega^{\partial\partial}_{\gamma} = \mathrm{d}\alpha^{\partial\partial}_{\gamma}, Q^{\partial\partial}_{\gamma})$ with $\omega^{\partial\partial}_{\gamma}$ symplectic of degree one and $Q^{\partial\partial}_{\gamma}$ cohomological and Hamiltonian.Î
  2. The fundamental equation
  $$\iota_{Q^{\partial}_{\Sigma}}\omega^{\partial}_{\Sigma} = \mathrm{d}S^{\partial}_{\Sigma} + \pi^{*}_{\Sigma}\alpha^{\partial\partial}_{\partial\Sigma}$$
  3. and so on.Î

Remark

It makes sense however to stop if the fibers of the correpondences become infinite dimensional. In TFTs and in 2d YM one can go down up to dimension zero (fully extended field theories).

Example: EM

- Boundary fields: A, B, A^+, c, $\quad S^\partial_\Sigma = \int_\Sigma c\,\mathrm{d}B$,
  $\alpha^\partial_\Sigma = \int_\Sigma B\,\delta A + A^+\,\delta c$, $\quad Q^\partial A^+ = \mathrm{d}B$, $Q^\partial A = \mathrm{d}c$.Î
- Boundary of boundary: $\gamma = (d-2)$-manifold
  BB fields: B, c, $\quad \alpha^{\partial\partial}_\gamma = \int_\gamma B\,\delta c$, of degree $+1$
  $S^{\partial\partial}_\gamma = 0$, $\quad Q^{\partial\partial}_\gamma = 0$.

Quantization

1. The first step is to fix a polarization $\mathcal{P}$ on $\mathcal{F}^{\partial}_{\partial M}$. We assume that the leaf space $\mathcal{B}^{\mathcal{P}}_{\partial M}$ is smooth. We set

$$\mathcal{H}^{\mathcal{P}}_{\partial M} = \text{functions on } \mathcal{B}^{\mathcal{P}}_{\partial M}$$

We also assume for simplicity that the 1-form $\alpha^{\partial}_{\partial M}$ vanishes on fibers. (We also allow shifting it by an exact 1-form if necessary.)Î

2. We assume a splitting of the fibration $\mathcal{F}_M \to \mathcal{B}^{\mathcal{P}}_{\partial M}$

$$\mathcal{F}_M = \mathcal{B}^{\mathcal{P}}_{\partial M} \times \mathcal{Y}$$

such that ω_M is constant on the base $\mathcal{B}^{\mathcal{P}}_{\partial M}$.Î

3. Such a splitting leads to a fiberwise version of the mCME. As a result the exponential of the action is Δ-closed only up to boundary terms that can be summarized as the action of a differential operator $\Omega^{\mathcal{P}}_{\partial M}$ on $\mathcal{B}^{\mathcal{P}}_{\partial M}$ that quantizes $\mathcal{S}^{\partial}_{\partial M}$

$$(\hbar^2\Delta + \Omega^{\mathcal{P}}_{\partial M})e^{\frac{i}{\hbar}\mathcal{S}_M} = 0$$

We assume $(\Omega^{\mathcal{P}}_{\partial M})^2 = 0$. (No anomaly condition.)

Perturbative quantization

1. Define
$$\psi_M = \int_{\mathcal{L}} e^{\frac{i}{\hbar} S_M} \in \mathcal{H}^{\mathcal{P}}_{\partial M}$$
where $\mathcal{L}$ is a Lagrangian submanifold of $\mathcal{Y}$.
2. By the standard techniques in BV, one gets
$$\Omega^{\mathcal{P}}_{\partial M} \psi_M = 0.$$
Moreover, changing gauge fixing modifies ψ_M by an $\Omega_{\partial M}$-exact term. Thus,
ψ_M defines a class in the physical space $H^0_{\Omega_{\partial M}}(\mathcal{H}^{\mathcal{P}}_{\partial M})$.

Residual fields

Usually, the only way of computing the functional integral is to perturb around a quadratic theory.

Let S^0 be the quadratic theory. Denote by $\mathcal{V}_M^{\mathcal{P}}$ the space of critical points of S^0 relative to the boundary polarization $\mathcal{P}$ modulo symmetries.

1. We assume a symplectic splitting

$$\mathcal{Y} = \mathcal{V}_M^{\mathcal{P}} \times \mathcal{Y}'$$

2. We now define ψ_M as a BV-pushforward:

$$\psi_M = \int_{\mathcal{L}'} e^{\frac{i}{\hbar} S_M} \in \mathcal{H}_{\partial M}^{\mathcal{P}} \otimes \mathcal{Z}_M^{\mathcal{P}}$$

where $\mathcal{L}'$ is a Lagrangian submanifold of $\mathcal{Y}'$ and

$$\mathcal{Z}_M^{\mathcal{P}} = \text{functions on } \mathcal{V}_M^{\mathcal{P}}$$

3. We finally get the modified quantum master equation (mQME)

$$(\hbar^2 \Delta_{\mathcal{V}_M^{\mathcal{P}}} \psi_M + \Omega_{\partial M}^{\mathcal{P}})\, \psi_M = 0$$

Axiomatics

- To each $(d-1)$-manifold Σ we associate a complex $(\mathcal{H}_\Sigma, \Omega_\Sigma)$.
- To each d-manifold M we as associate a state ψ_M satisfying the mQME.
- Plus functorial properties.Î
  In particular, gluing is given by pairing states and doing a BV-pushforward

$$\mathcal{V}_{M_1} \times \mathcal{V}_{M_1} \to \mathcal{V}_{M_1 \cup_\Sigma M_2}$$

Remark

The full power of this approach is that we may cut the original manifold M into simple, or tiny, pieces; do the perturbative quantization there; and eventually glue and reduce.
This could provide some new insight for physical theories.
In TFTs it yields a perturbative version of Atiyah's axioms. We expect to be able to compute, e.g., perturbative CS invariants.

BF theory

- *BF* theory

$$S_M = \int_M \left\langle B, \mathrm{d}A + \frac{1}{2}[A, A] \right\rangle, \qquad A \in \Omega(M, \mathfrak{g}),\ B \in \Omega(M, \mathfrak{g}^*)$$

- Here

$$S^0_M = \int_M \langle B, \mathrm{d}A \rangle$$

It turns out that $\mathcal{V}_M$ is the odd cotangent bundle of the (relative) cohomology of M with values in $\mathfrak{g}$.Î

- The gluing in the quadratic theory consists of
  - Gluing of torsions
  - Mayer–Vietoris

Nonabelian *BF* theory

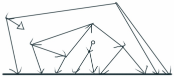

Figure: $\frac{\delta}{\delta B}$-polarizationÎ

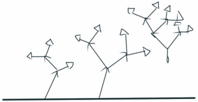

Figure: $\frac{\delta}{\delta A}$-polarization