Poisson 2000

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Anton Alekseev Lie group valued moment maps.

We review the theory of group valued moment maps. As in the standard moment map theory, there is a notion of reduction, and there is an analogue to the nonabelian convexity theorem for group valued moment maps. Recently, counterparts to the Duistermaat-Heckman formula and to the "quantization commutes with reduction" principle have been proved for group valued moment maps. We end with a list of open problems.

Sam Evens On the variety of Lagrangian subalgebras.

This talk is based on joint work with Jiang-Hua Lu. We reinterpret a result of Drinfeld classifying Poisson homogeneous spaces for a Poisson Lie group U as an assertion that for any Poisson homogeneous space M, there is a Drinfeld map from M to the variety L of Lagrangian subalgebras of the double $\mathfrak{d}$ of the Lie algebra $\mathfrak{u}$ of U. We show further that the space L has the natural structure of a Poisson variety, and that the map from M to L is a Poisson mapping. As Alan Weinstein has pointed out, in the special case when U has zero Poisson Lie group structure and M is symplectic, the Drinfeld map may be interpreted as the moment map. In the special case where U is a compact semisimple Lie group with Poisson bracket induced from a classical r-matrix, we determine the geometry of L using results of Karolinsky. More precisely, we show that the irreducible components of L are smooth, determine the irreducible components, and show that their geometry is related to that of a De Concini-Procesi compactification. We further observe that L can be used to produce examples of Poisson homogeneous spaces. In particular, we verify that open orbits of equal rank real semisimple Lie groups on the flag variety can be realized inside of L.

Boris Fedosov Pseudo-differential operators and deformation quantization.

Using a Riemannian connection on a manifold X, we show that the algebra of pseudo-differential operators generates a canonical deformation quantization on the cotangent bundle T^*X. The corresponding Abelian connection is calculated explicitly in terms of the exponential mapping. We also prove that the Atiyah-Singer index theorem for elliptic operators may be obtained as a consequence of the index theorem for deformation quantization.

Rui Loja Fernandes Connections and characteristic classes in Poisson geometry.

The theory of connections is a classical topic in differential geometry. Connections provide an extremely important tool for the study of geometric structures on manifolds and, as such, have been used with great success in many different settings. However, the use of connections has been very limited whenever singular behavior is present. The reason is that, if a geometric structure admits a compatible connection, then the parallel transport will preserve any algebraic invariant of the structure, and that excludes the presence of singular behavior. In this talk, we explain how one can extend the notion of connection in order to include geometric structures that may exhibit singular behavior: we consider a notion of connection on a Lie algebroid which is a natural extension of the usual concept of connection. Using connections, we are able to define the holonomy of the orbit foliation of a Lie algebroid and to prove a stability theorem. We also introduce secondary or exotic characteristic classes that generalize the modular class of a Lie algebroid, and are analogues to the secondary characteristic classes of foliation theory. These classes provide information on the topology of the singular foliation associated with a Lie algebroid. These results extend corresponding results for Poisson manifolds (see [1]), and are explained in detail in [2].

References:

[1] Connections in Poisson geometry I: holonomy and invariants, to appear in J. Diff. Geometry, preprint math.DG/0001129.

[2] R. Loja Fernandes, Lie algebroids, holonomy and characteristic classes, preprint math.DG/0007132.

Vladimir Fock Geometry of the moduli space of flat Kac-Moody connections on Riemann surfaces.

According to an observation by Hitchin, the complexification of the moduli space of flat connections on a Riemann surface is isomorphic to the space of flat Kac-Moody connections on it. This isomorphism provides the former space with a hyperkähler structure. The aim of the talk is to describe a hyperkähler structure on an analogue to the space of flat connections - the space of quasi-fuchsian groups. It turns out that such a construction does exist and, moreover, the space of quasi-fuchsian groups turns out to be isomorphic to the space of global solutions of the cosh-Gordon equation, thus providing this equation with a clear geometric meaning.

Viktor L. Ginzburg Poisson topology: Morita equivalence, characteristic classes, and all that.

We introduce a version of Poisson K-theory which takes an intermediate place between commutative and non-commutative K-theories. These K-theory rings satisfy the following three conditions:

(1) for a manifold P with zero Poisson structure, the K-theory is equal to the ordinary ring $K(P)$,

(2) for a symplectic manifold, the Poisson K-theory is determined by the space of representations of the fundamental group of the manifold,

(3) for the dual space of a Lie algebra, the Poisson K-theory is determined by the space of representations of the Lie algebra.

The rings are defined using a suitable class of Poisson vector bundles, *i.e.*, vector bundles with an action of the Poisson algebra of functions on the manifold. The notion of Poisson K-theory fits well in the context of a newly emerging direction in Poisson Geometry which can be called Poisson topology.

Mark Gotay On quantizing nilpotent and solvable basic algebras.

We prove an algebraic "no-go theorem" to the effect that a nontrivial Poisson algebra cannot be realized as an associative algebra with the commutator bracket. Using it, we show that there is an obstruction to quantizing the Poisson algebra of polynomials generated by a nilpotent basic algebra on a symplectic manifold. This result generalizes Groenwald's famous theorem on the impossibility of quantizing the Poisson algebra of polynomials on $\mathbb{R}^{2n}$. Finally, we explicitly construct a polynomial quantization of a symplectic manifold with a solvable basic algebra, thereby showing that the obstruction in the nilpotent case does not extend to the solvable case.

Janusz Grabowski Graded brackets associated with Lie algebroids and Poisson structures.

Generalized Schouten, Frölicher-Nijenhuis, etc. graded brackets are defined for any Lie algebroid. We present certain relations between these brackets. In particular, it turns out that the classical Frölicher-Nijenhuis bracket on a manifold M can be embedded in the Schouten bracket on the cotangent bundle of M. The derived graded Loday brackets are discussed together with a construction of skew-symmetric "derived" brackets for triangular bi-algebroids. They can be viewed as natural, *i.e.*, respecting Poisson maps, graded extensions of Poisson brackets to differential forms.

Johannes Huebschmann Stratified Kähler spaces and quantization.

An appropriate notion of Kähler space with singularities implies the result that, in the world of "singular Poisson-Kähler geometry", reduction after quantization coincides with quantization after reduction: For a stratified symplectic space, the concept of stratified polarization, which is defined in terms of an appropriate Lie-Rinehart algebra, encapsulates polarizations on the strata, as well as the behaviour of the polarizations across the strata. Symplectic reduction carries a Kähler manifold to a suitably defined (normal complex analytic) stratified Kähler space in such a way that the sheaf of germs of polarized functions thereupon coincides with the ordinary sheaf of germs of holomorphic functions. When the notion of prequantum module is defined as a generalization of prequantum bundles, for a Kähler manifold, reduction after quantization coincides with quantization after reduction in the sense that not only the reduced and unreduced quantum phase spaces correspond, but the unreduced and reduced quantum observables correspond as well. Explicit examples for stratified Kähler spaces are provided by certain moduli spaces, by the closure of holomorphic nilpotent orbits in real semisimple Lie algebras, and by reduced spaces arising from angular momentum zero.

Patrick Iglésias Parenthèses de Lagrange, crochets de Poisson : la naissance d'une nouvelle mécanique.

Dans les cours de mécanique actuels, il est plus fréquent de voir cités les crochets de Poisson que les crochets ou les parenthèses de Lagrange ; or, il ne fait aucun doute que l'introduction des divers types de crochets/parenthèses en mécanique analytique est due à J.-L. Lagrange. Quel a donc été le rôle de Poisson dans cette affaire?

Mikhail Karasev Quantized restriction of Lie-Poisson algebras.

In a quantized Poisson manifold with a noncommutative product in the space of functions, let us consider a symplectic leaf and ask whether there exists a noncommutative product on the space of functions over the leaf, such that the operation of restriction of functions to the leaf is a homomorphism of associative algebras. Actually, it was recently proved that even in the case of linear Lie-Poisson brackets, when the symplectic leaves are just coadjoint orbits, such a quantum product over the leaves does not exist, in general. In this talk, we discuss a way to obtain a positive answer to this question, replacing the usual operation of restriction of functions to the leaf by a new quantum restriction operation, which "knows" the properties of functions not only along the leaf but also in the transversal directions. In the simple example of the orbits of maximal dimension in the space dual to a compact Lie algebra, we show how to define such a quantum restriction operation, using the irreducible representation of the Lie algebra in the space of anti-holomorphic sections over the orbit. In the semiclassical approximation, when there exists a natural deformation parameter, or several parameters, say, highest weights of the representation, the quantum restriction operation can be expanded in a formal power series whose leading term is just the classical restriction to the orbit, plus additional terms which are quantum corrections. We explicitly derive the first quantum correction, and show that it is a first-order differential operator acting in a direction transversal to the orbit, so, it is a normal vector field over the orbit. (After the talk, A.Weinstein mentioned that this vector may be the mean curvature of the orbit; and several days later the author proved that it is indeed half the mean curvature vector defined by the affine connection on the Lie coalgebra and by the Levi-Civita connection on the orbit.)

Boris Khesin Holomorphic Poisson structures and gauge groups on complex manifolds.

This is joint work with A. Rosly (ITEP, Moscow). We give a comparative description of the Poisson structures on the moduli spaces of flat connections on real surfaces and of the holomorphic Poisson structures on the moduli spaces of holomorphic bundles on complex surfaces. The symplectic leaves of the latter are classified by restrictions of the bundles to certain divisors. This can be regarded as fixing a "complex analogue to the holonomy" of a connection along a "complex analogue to the boundary", in analogy with the real case.

Alexander A. Kirillov Family algebras.

A new class of associative algebras, called family algebras, is introduced and studied. These algebras are related to an irreducible representation of a simple complex Lie algebra. A family algebra is a sort of finite approximation to the enveloping algebra $U(\mathfrak{g})$ viewed as a module over its center. It turns out that several important questions about semi-simple algebras and their representations can be formulated, studied and sometimes solved in terms of family algebras. Here we only start this program and hope that it will be continued and developed.

Zhang-Ju Liu The local structure of Lie bialgebroids.

This is joint work with Ping Xu (Penn. State Univ.). We study the Lie bialgebroid structures on a given transitive Lie algebroid. For the local aspect, it is enough to deal with the gauge Lie algebroids $\mathcal{A} = TM \times \mathfrak{g}$, where $\mathfrak{g}$ is a Lie algebra and the anchor is the projection onto TM. Under the assumption that $H^1(M, \mathbb{R}) = 0$, we show that any Lie bialgebroid structure on $\mathcal{A} = TM \times \mathfrak{g}$ is determined by a pair (Λ, δ_0), where $\Lambda \in \Gamma(\bigwedge^2 \mathcal{A})$ and $\delta_0 : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ is a 1-cocycle, such that $[\Lambda, \cdot] + \delta_0$ is the differential on $\Gamma(\bigwedge \mathcal{A})$ coming from the Lie algebroid structure on $\mathcal{A}^*$. We then derive the compatibility conditions among δ_0, π, θ and τ, where Λ is split as $\pi + \theta + \tau$. Here π is a bivector field on M, $\theta \in \Gamma(TM \wedge \mathfrak{g})$ is the mixed term, and τ is a map from M to $\bigwedge^2 \mathfrak{g}$, a generalized dynamical r-matrix. One of these compatibility conditions generalizes the dynamical Yang-Baxter equation.

Kirill Mackenzie On symplectic double groupoids and the duality of Poisson groupoids.

The cotangent bundle of a Lie group(oid) has a well-known symplectic groupoid structure which realizes the standard Poisson structure on the Lie algebr(oid) dual. In this talk we show that given a double Lie groupoid S, the cotangent bundle T^*S has a symplectic double groupoid structure, the sides of which are the Lie algebroid duals of the side groupoids of S, and the base of which is the Lie algebroid dual of the core groupoid of S. This proves a result outlined by Weinstein in 1988: the side groupoids of a symplectic double groupoid are Poisson groupoids in duality. Further, it provides a construction of dual pairs of Poisson groupoids (and thus of Lie bialgebroids), associated to each double Lie groupoid. The constructions use an extension of the dualities and canonical isomorphisms introduced by the author and Ping Xu.

Reference: On symplectic double groupoids and the duality of Poisson groupoids, Int. J. Math. 10(4), 1999, 435–456.

Charles-Michel Marle Symmetries of Hamiltonian systems and Poisson manifolds.

Let (M, Λ) be a Poisson manifold and $H : M \to \mathbf{R}$ a smooth Hamiltonian on M. A Lie group G is said to be a *group of symmetries of the Poisson manifold* (M, Λ) if G acts on M by a left action $\Phi : G \times M \to M$ in such a way that for every $g \in G$, the map $\Phi_g : M \to M$ defined by $\Phi_g(x) = \Phi(g, x)$ is a Poisson diffeomorphism of (M, Λ). The group G is said to be a *group of symmetries of the Hamiltonian system* (M, Λ, H) if in addition, for every $g \in G$, $\Phi_g^* H = H$. We discuss the properties of the symmetry groups of a Poisson manifold (M, Λ). Then we consider two symmetry groups G_1 and G_2 of the Poisson manifold (M, Λ); denoting by Φ_1 and Φ_2 their respective actions on M, we say that these two actions are *orthogonal* (resp., *exactly orthogonal*) if for any $x \in M$, $T_x\big(\Phi_1(G_1, x)\big) \cap C_x$ is contained in (resp., is equal to) $\mathrm{orth}\Big(T_x\big(\Phi_2(G_2, x)\big) \cap C_x\Big)$. We have denoted by C_x the tangent space at x to the symplectic leaf through that point and by orth the symplectic orthogonality in that symplectic vector space. We prove a reduction theorem for a pair (Φ_1, Φ_2) of commuting orthogonal (resp., exactly orthogonal) actions such that the set P of orbits of the restriction of Φ_1 to a normal subgroup H of G_1 has a smooth manifold structure: the actions on P of G_1/H and G_2 obtained after quotient constitute a pair of orthogonal (resp., exactly orthogonal) actions.

Tudor Ratiu Hamiltonian dynamics around critical elements.

This talk presents some of the results jointly obtained with J.-P. Ortega regarding the lower bound for the number of relative equilibria and relative periodic orbits around stable and formally unstable equilibria and relative equilibria. These results are extensions of the Weinstein theorem to the symmetric case. The techniques used in the proofs are a combination of bifurcation theoretical methods, singular reduction results, Lyapunov-Schmidt reduction approaches and normal form methods, both from the point of view of bifurcation theory and that of symplectic geometry.

Dmitri Roytenberg Equivariant cohomology via symplectic supermanifolds.

We show that a Lie bialgebroid structure is equivalent to a pair of commuting Hamiltonian vector fields on an even symplectic supermanifold. It turns out that the equivariant de Rham complexes, in both the Weil and Cartan models, arise naturally as special cases of this construction. This point of view offers the hope of generalizing the notion of equivariant cohomology to arbitrary Lie groupoids and groupoid actions.

Pavol Severa Courant algebroids and variational problems.

There is an infinite sequence of notions, starting with symplectic manifolds, Poisson manifolds, and Courant algebroids. We shall call them Σ_n-manifolds, where $n \in \mathbb{N}$. The basic idea comes from Sullivan's rational homotopy theory: there one can see the groupoid corresponding to a Lie algebroid as the fundamental groupoid. Moreover one can consider appropriate generalizations of Lie algebroids (supermanifolds with additional structure) with nontrivial higher homotopies. These higher homotopies are the basic topic of the talk. Putting a symplectic structure on our supermanifolds we arrive at Σ_n-manifolds. Their homotopies (in the case of Courant algebroids, *i.e.*, $n = 2$) give double symplectic groupoids, symplectic groupoids over Poisson homogeneous spaces, etc. Finally, Σ_n-manifolds play much the same role in n-dimensional variational problems as Poisson manifolds do in classical mechanics.

Izu Vaisman Locally Lagrangian symplectic and Poisson manifolds.

This is a survey of those symplectic manifolds whose local structure is the one encountered in Lagrangian dynamics. Examples and characteristic properties are given. Then, we discuss the computation of the Maslov classes of a Lagrangian submanifold. Finally, we indicate the generalization of this type of symplectic structures to Poisson manifolds.

Yurii Vorobjev On linearized transverse Poisson structures and coupling tensors

A contravariant version of Sternberg's minimal coupling is discussed in the context of Poisson geometry. Using the techniques of Poisson-Ehresmann connections on fiber bundles, we introduce and study a class of Poisson structures called coupling tensors. This geometric approach gives us an effective tool for the investigation of some problems in the classification theory of Poisson manifolds near a single symplectic leaf. For example, in view of the results on the linearization problem, we show that the notion of a linearized Poisson structure at a given closed symplectic leaf can be defined as an equivalence class of isomorphic coupling tensors which live naturally on the normal bundle to the leaf. The key observation here is that various couplings of the symplectic form on the leaf with the linearized transverse Poisson structure give the same structure up to isomorphism. Several other applications and examples are given to illustrate the approach.

Alan Weinstein Doubles of Leibniz algebras and Courant algebroids.

In this joint work with Michael Kinyon, we show that the skew-symmetrized product on every Leibniz algebra $\mathcal{E}$ can be realized on a reductive complement to a subalgebra in a Lie algebra. As a consequence, we construct a nonassociative multiplication on $\mathcal{E}$ which, when $\mathcal{E}$ is a Lie algebra, is derived from the integrated adjoint representation. We apply this construction to realize the bracket operations on the sections of Courant algebroids and on the "omni-Lie algebras" recently introduced by the speaker. Finally, we discuss the extension of a path-group construction of Duistermaat and Kolk to the case of Leibniz algebras.

Ping Xu Quantization of dynamical r-matrices.

We provide a general study of triangular dynamical r-matrices using Poisson geometry. We show that a triangular dynamical r-matrix always gives rise to a regular Poisson manifold. Using the Fedosov method, we prove that the triangular dynamical r-matrices $r : \mathfrak{h}^* \to \wedge^2 \mathfrak{g}$ which are non-degenerate, *i.e.*, such that the corresponding Poisson manifolds are symplectic, are quantizable, and that the quantizations are classified by the relative Lie algebra cohomology $H^2(\mathfrak{g}, \mathfrak{h})[[\hbar]]$.