# Lie's Third Theorem

Rui Loja Fernandes

Departamento de Matemática
Instituto Superior Técnico

UAB Math Dep Colloquium

# Outline

# Symmetries of Differential Equations

Sophus Lie, influenced by Felix Klein, proposed:

**Definition**

The **group of symmetries** of a differential equation:

$$\Delta(x, y, \ldots, u, v, \ldots, u_x, v_x, u_{xx}, \ldots) = 0,$$

is the set of all transformation of the independent variables $(x, y, \ldots)$ and of the dependent variables $(u, v, \ldots)$ that transform solutions to solutions.

# Symmetries of Differential Equations

Lie aimed (and achieved) a Galois theory for differential equations:

- he proved that if the group of symmetries is *solvable* then the differential equation can be integrated by quadratures.
- he found a method to compute the group of symmetries.

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# Example: The heat equation

The symmetry group of the heat equation:

$$u_t = u_{xx}$$

is generated by the following transformations:

$$(x,t,u) \mapsto (x+\varepsilon, t, u) \qquad (x,t,u) \mapsto (e^{\varepsilon}x, e^{2\varepsilon}t, u)$$

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$$(x,t,u) \mapsto \left(\frac{x}{1-4\varepsilon t}, \frac{t}{1-4\varepsilon t}, u\sqrt{1-4\varepsilon t}\, e^{\frac{-\varepsilon x^2}{1-4\varepsilon t}}\right)$$

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# From global to infinitesimal

**Problem**

*How can one find the symmetry group $G_\Delta$ of a given differential equation $\Delta = 0$?*

Each 1-parameter group of symmetries:

$$\mathbb{R} \ni \varepsilon \mapsto T_\varepsilon \in G_\Delta,$$

determines an infinitesimal symmetry, i.e., a vector field:

$$X(x, y, \dots, u, v \dots) = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} T_\varepsilon(x, y, \dots, u, v, \dots)$$

Lie found that the infinitesimal symmetries of $\Delta$ are the solutions of a system of first order linear p.d.e.

$\Longrightarrow$ systematic method to compute symmetries

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# From global to infinitesimal and back

Lie also noted that:

- The vector space $\mathfrak{g}_\Delta$ of all infinitesimal symmetries is closed under the commutator of vector fields:

$$X_1, X_2 \in \mathfrak{g}_\Delta \implies [X_1, X_2] \in \mathfrak{g}_\Delta.$$

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**Theorem**

*Any space $\mathfrak{g}$ of vector fields closed under the commutator is the set of infinitesimal symmetries of a group of symmetries G.*

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# Lie groups and Lie algebras

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A **Lie group** is a manifold $G$ together with a group structure on $G$ such that the product and inversion are smooth:

$$G \times G \to G, (g,h) \mapsto gh, \quad G \to G, g \mapsto g^{-1}.$$

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A **Lie algebra** is a vector space $\mathfrak{g}$ together with a bilinear, skew-symmetric, bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which satisfies the Jacobi identity:

$$[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0.$$

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# From Lie groups to Lie algebras

Let $G$ be a *finite dimensional* Lie group.

Its Lie algebra $\mathfrak{g} = \mathcal{L}(G)$ is constructed as follows:

- As a vector space, $\mathfrak{g} := T_e G$;
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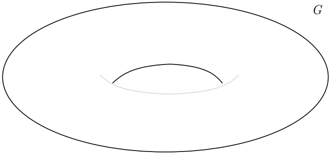

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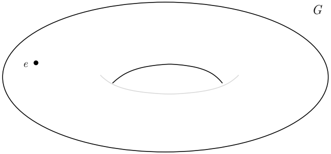

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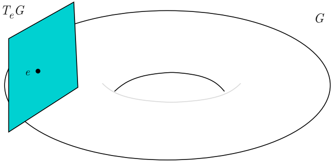

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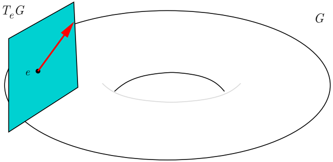

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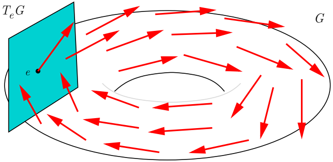

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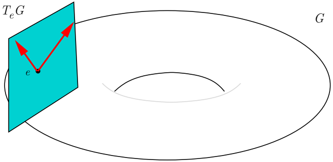

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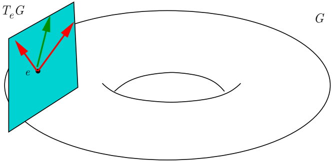

# Examples

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| General Linear Group: $GL(n) = \{A \in M_n(\mathbb{R}) : \det A \neq 0\}$ | $\mathfrak{gl}(n) = \{A \in M_n(\mathbb{R})\}$ |
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| Group of symplectomorphisms of $(M, \omega)$: $G = \{\phi : M \to M \mid \phi^* \omega = \omega\}$ | $\mathfrak{g} = \{X \in \mathfrak{X}(M) \mid \mathcal{L}_X \omega = 0\}$ |

# From Lie algebras to Lie groups

**Theorem (Lie I)**

*Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. There exists a unique (up to isomorphism) 1-connected Lie group $\widetilde{G}$ with Lie algebra $\mathfrak{g}$.*

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*Let $G$ and $H$ be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, where $G$ is 1-connected. Given a Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{h}$, there exists a unique Lie group homomorphism $\Phi : G \to H$ with $(\Phi)_* = \phi$.*

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# Infinite dimensional Lie groups

Symmetry groups of differential equations can be infinite dimensional (e.g., the heat equation).

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# Example I [Van Est & Korthagen, 1964]

$\mathfrak{g}_0 := \{X : [0,1] \to \mathfrak{su}(2) | \int_0^1 X(t)\mathrm{d}t = 0\}$ with pointwise bracket;

Take the skew-symmetric bilinear form $\tau : \mathfrak{g}_0 \times \mathfrak{g}_0 :\to \mathbb{R}$:

$$\tau(X,Y) := \int_0^1 \mathrm{tr}\left(\int_0^t X(s)\mathrm{d}s \circ Y(t)\right)\mathrm{d}t.$$

and form the central extension $\mathfrak{g} = \mathbb{R} \times \mathfrak{g}_0$:

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}_0 \longrightarrow 0$$

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- The group Diff($M$) is a Fréchet Lie group;
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However, Lie II fails:

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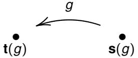


$$\mathcal{G} \underset{\mathbf{s}}{\overset{\mathbf{t}}{\rightrightarrows}} M$$

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$$\mathcal{G}^{(2)} = \{(h, g) \in \mathcal{G} \times \mathcal{G} : \mathbf{s}(h) = \mathbf{t}(g)\}$$

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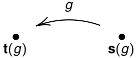


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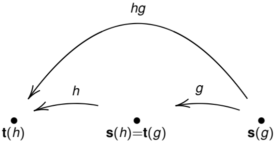


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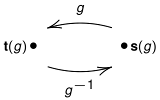

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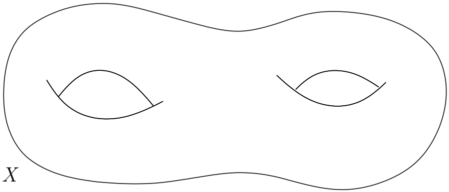

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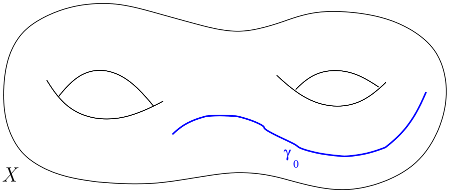

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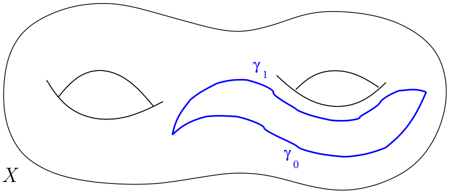

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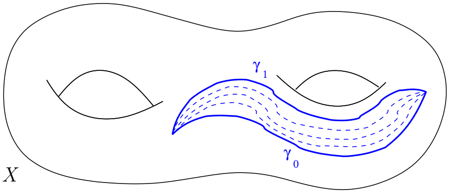

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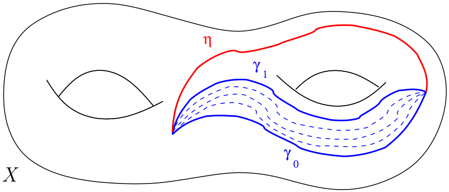

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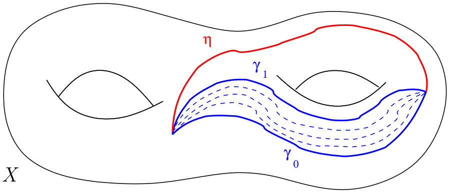


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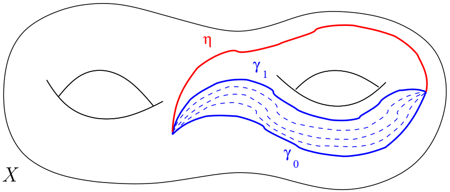


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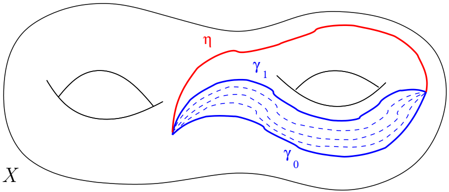


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# Example: Fundamental groupoid of a space

For the fundamental groupoid

$$\Pi(X) = \{[\gamma] \mid \gamma : [0,1] \to X\}$$

the structure maps are:

- *source*/*target* give initial/final points: $\mathbf{s}([\gamma]) = \gamma(0)$, $\mathbf{t}([\gamma]) = \gamma(1)$;
- *product* is concatenation of curves: $[\gamma] \cdot [\eta] = [\gamma \cdot \eta]$;
- *units* are the constant curves: $1_x = [\gamma]$, where $\gamma(t) = x$;
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# Lie groupoids

**Definition (Charles Ehresmann, 1950's)**

A **Lie groupoid** is a groupoid where $\mathcal{G}$ and $M$ are manifolds and all structure maps are smooth.

**Examples**

- A Lie group $G$ is a Lie groupoid: $\mathcal{G} := G \rightrightarrows \{*\}$;
- For a manifold $M$, $\Pi(M) \rightrightarrows M$ and $M \times M \rightrightarrows M$ are Lie groupoids;
- For a foliation $\mathcal{F}$, $\Pi(\mathcal{F}) \rightrightarrows M$ is a Lie groupoid;
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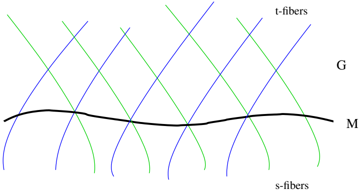

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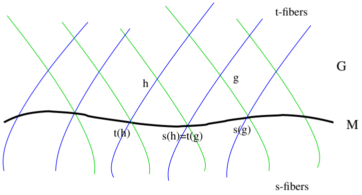

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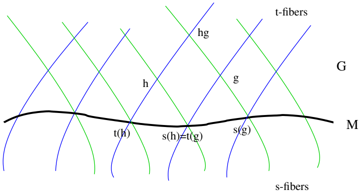

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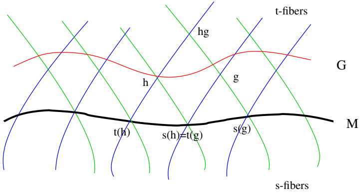

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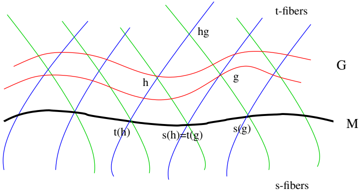

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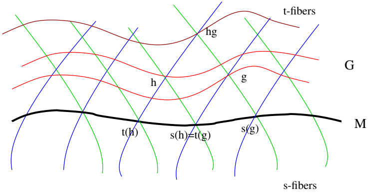

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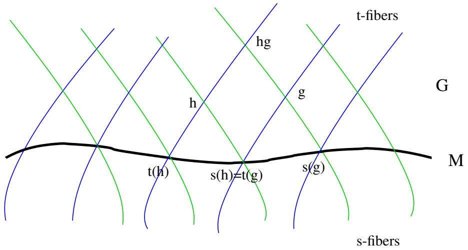

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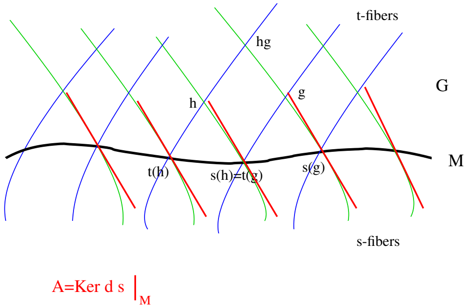

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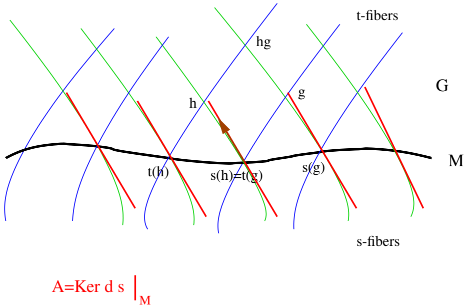

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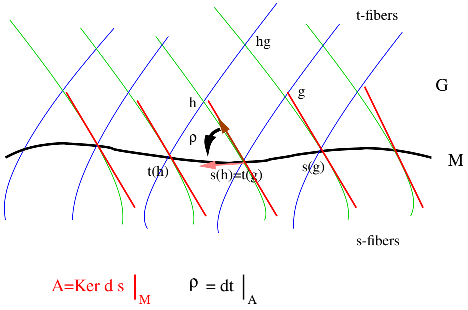

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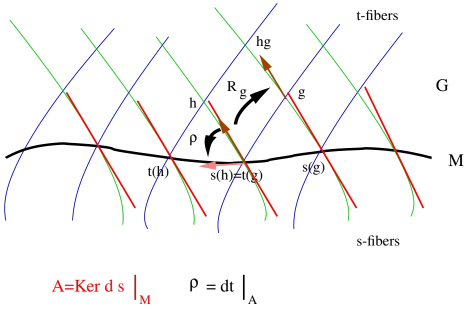

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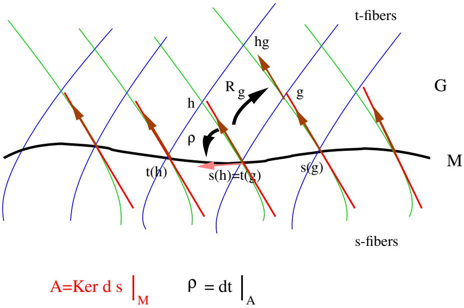

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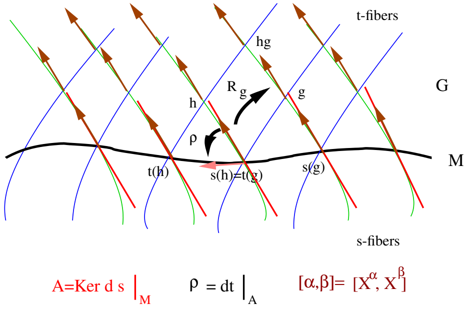

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**Theorem (Lie I)**

*Let $\mathcal{G}$ be a Lie groupoid with Lie algebroid $A$. There exists a unique (up to isomorphism) source 1-connected Lie groupoid $\widetilde{\mathcal{G}}$ with Lie algebroid $A$.*

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*Let $\mathcal{G}$ and $\mathcal{H}$ be Lie groupoids with Lie algebroids $A$ and $B$, where $\mathcal{G}$ is source 1-connected. Given a Lie algebroid homomorphism $\phi : A \to B$, there exists a unique Lie groupoid homomorphism $\Phi : \mathcal{G} \to \mathcal{H}$ with $(\Phi)_* = \phi$.*

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- Fix $\omega \in \Omega^2(M)$, closed, and take the associated Lie algebroid $A = TM \oplus \mathbb{R}$.

**Theorem**

*The Lie algebroid A integrates to a Lie groupoid $\mathcal{G}$ iff the group of spherical periods of $\omega$:*

$$N_x := \{\int_\gamma \omega \mid \gamma \in \pi_2(M,x)\} \subset \mathbb{R}$$

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The *obstructions to integrability* are completely described by:

**Theorem (Crainic & RLF, 2003)**

*For a Lie algebroid $A$, there exist monodromy groups $N_x \subset A_x$ such that $A$ is integrable iff the groups $N_x$ are uniformly discrete for $x \in M$.*

Each $N_x$ is the image of a monodromy map:

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with $L$ the leaf through $x$ and $\mathfrak{g}_x := \operatorname{Ker} \rho_x$ the isotropy Lie algebra.

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# Proof: The Weinstein groupoid

## Notations

- An *A-path* is a Lie algebroid map $TI \to A$;
- An *A-homotopy* is a Lie algebroid map $T(I \times I) \to A$;

### Definition

For a Lie algebroid $\pi : A \to M$, the **Weinstein Groupoid** of $A$ is:

$$\mathcal{G}(A) = P(A)/\sim \quad \text{where} \quad \begin{array}{|l} \mathbf{s} : \mathcal{G}(A) \to M, \quad [a] \mapsto \pi(a(0)) \\ \mathbf{t} : \mathcal{G}(A) \to M, \quad [a] \mapsto \pi(a(1)) \\ M \hookrightarrow \mathcal{G}(A), \quad x \mapsto [0_x] \end{array}$$

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- *$\mathcal{G}(A)$ is a topological groupoid with source 1-connected fibers;*
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Fix leaf $L \subset M$ and $x \in L$:

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$$\Downarrow$$

$$\cdots \to \pi_2(L,x) \stackrel{\partial}{\longrightarrow} \mathcal{G}(\mathfrak{g}_L)_x \to \mathcal{G}(A)_x \longrightarrow \pi_1(L,x) \longrightarrow 1$$

The **monodromy group** at $x$ is: $N_x(A) := \operatorname{Im}\partial \subset Z(\mathfrak{g}_L)$.

# Proof: The obstructions

To measure the discreteness of $N_x(A)$ we set:

$$r(x) := d(N_x - \{0\}, \{0\}) \qquad \text{(with } d(\emptyset, \{0\}) = +\infty\text{).}$$

**Theorem (Crainic & RLF, 2003)**

*A Lie algebroid is integrable iff both the following conditions hold:*

- (i) *Each monodromy group is discrete, i.e.,* $r(x) > 0$,
- (ii) *The monodromy groups are uniformly discrete, i.e.,* $\liminf_{y\to x} r(y) > 0$,

*for all* $x \in M$.

...in many examples it is possible to compute the monodromy...

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