

INTEGRABILITY OF POISSON-LIE GROUP ACTIONS

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ABSTRACT. We establish a 1:1 correspondence between Poisson-Lie group actions on integrable Poisson manifolds and twisted multiplicative hamiltonian actions on source 1-connected symplectic groupoids. For an action of a Poisson-Lie group G on a Poisson manifold M , we find an explicit description of the lifted hamiltonian action on the symplectic groupoid $\Sigma(M)$. We give applications of these results to the integration of Poisson quotients M/G , Lu-Weinstein quotients $\mu^{-1}(e)/G$ and Poisson homogeneous spaces G/H .

INTRODUCTION

Let (M, π) be a Poisson manifold, G a Lie group and assume that G acts on M by Poisson diffeomorphisms. Such an action usually does not admit a momentum map (e.g., if the action is transverse to the symplectic leaves). However, there is a symplectization functor which turns this action into a hamiltonian action. More precisely, associated with an integrable Poisson manifold (M, π) there is a canonical symplectic groupoid $\Sigma(M) \rightrightarrows M$ and the action lifts to a hamiltonian action on the symplectic groupoid $(\Sigma(M), \Omega)$ with momentum map $J : \Sigma(M) \rightarrow \mathfrak{g}^*$ (see [6, 8, 18] and references there in). One should think of J as a canonical momentum map which is attached to the Poisson action, and which always exists. This map satisfies

$$J(x \cdot y) = J(x) + J(y),$$

and, in fact, it is the Lie groupoid morphism that integrates the Lie algebroid morphism $j : T^*M \rightarrow \mathfrak{g}^*$ canonically associated with the action (here we view \mathfrak{g}^* as an abelian Lie algebra). The momentum map J is relevant, e.g., to understand the reduced space M/G . Namely, $\Sigma(M)//G := J^{-1}(0)/G$ is a symplectic groupoid integrating the Poisson quotient M/G (which, however, does not need to coincide with $\Sigma(M/G)$; see [8]).

Our aim is to understand how this theory extends to actions of Poisson-Lie groups. Suppose $G \times M \rightarrow M$ is a Poisson action of a Poisson-Lie group G , with associated Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$. The map $j : T^*M \rightarrow \mathfrak{g}^*$ is still a Lie algebroid morphism and it integrates to a Poisson-Lie groupoid morphism $J : \Sigma(M) \rightarrow G^*$, where G^* is the

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dual, 1-connected, Poisson-Lie group. We would like to lift the G -action on M to a hamiltonian action on $\Sigma(M)$ with momentum map $J : \Sigma(M) \rightarrow G^*$. As was observed by Lu [11], any Poisson map J into G^* determines a *local* hamiltonian G -action with momentum map J , so there exists a *local* hamiltonian G -action on $\Sigma(M)$.

Recall that a Poisson-Lie group G is said to be complete if the infinitesimal dressing action of \mathfrak{g}^* on G can be integrated to a global action $G \times G^* \rightarrow G$. Our main result is the following result:

Theorem 1. *Let G be a complete Poisson-Lie group, M an integrable Poisson manifold and $G \times M \rightarrow M$ a Poisson action. There exists a lifted Poisson action of G on the symplectic groupoid $\Sigma(M)$, which is hamiltonian with a canonical momentum map $J : \Sigma(M) \rightarrow G^*$.*

Our proof of Theorem 1 takes advantage of the description of $\Sigma(M)$ in terms of cotangent paths (see [4, 5]) to explicitly construct the lifted G -action. In this respect, it is important to observe that in the situation described by Theorem 1, elements of G do not act as groupoid automorphisms. In fact, we have the following result:

Proposition 1. *Let G be a connected, complete Poisson-Lie group, and $\mathcal{G} \rightrightarrows M$ a source-connected symplectic groupoid. For a hamiltonian action $G \times \mathcal{G} \rightarrow \mathcal{G}$ with momentum map $J : \mathcal{G} \rightarrow G^*$ such that $J(M) = e$, the following are equivalent:*

(i) $J : \mathcal{G} \rightarrow G^*$ is a groupoid morphism:

$$J(x \cdot y) = J(x) \cdot J(y), \quad x, y \in \mathcal{G}^{(2)}.$$

(ii) The twisted multiplicativity property holds:

$$(1) \quad g(x \cdot y) = (gx) \cdot (g^{J(x)}y), \quad x, y \in \mathcal{G}^{(2)}, g \in G,$$

where we denote by g^u the right dressing action of $u \in G^*$ on an element $g \in G$.

When the Poisson structure on G vanishes we have $G^* = \mathfrak{g}^*$. In this case, G^* acts trivially on G , so G is always complete. Also, the twisting disappears and the action is by groupoid automorphisms, so we recover the results mentioned before. The twisted multiplicativity property (1) was also noted by Lu in [12].

It is easy to check that, under the assumptions of Proposition 1, the twisted multiplicativity property (1) implies that there is an induced G -action on the units M and that this action is Poisson. Hence, our results establish a 1:1 correspondence:

Poisson actions on integrable Poisson manifolds	\longleftrightarrow	Twisted multiplicative hamiltonian actions on source 1-connected symplectic groupoids
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We illustrate Theorem 1 with a few applications to the problem of integrating various classes of Poisson manifolds. The first simple application is:

Theorem 2. *Let G be a complete Poisson-Lie group, (M, π) an integrable Poisson manifold and $G \times M \rightarrow M$ a proper and free Poisson action. The reduced space*

$$\Sigma(M)//G := J^{-1}(e)/G$$

is a symplectic groupoid integrating the Poisson manifold M/G .

In general, $\Sigma(M)//G \neq \Sigma(M/G)$. We will see that the failure in this equality can be controlled in much the same way as in the case of actions by Poisson diffeomorphisms [8] (see Theorem 3.3 below).

The problem of integrating the Poisson quotient M/G has been discussed by several authors before. The first results in this direction are due to Xu [19]. Later, Lu in [12] and Stefanini in [16] give integrations based on more complicated procedures, using various notions of action algebroids/groupoids and doubles. Our approach gives a clear explanation why the condition of completeness on G always leads to a global action on $\Sigma(M)$, rather than just a local action, a problem circumvented by these authors.

Our second application of Theorem 1 is to the integration of the Poisson manifold obtained by reduction of a hamiltonian action $G \times M \rightarrow M$ with equivariant momentum map $\mu : M \rightarrow G^*$. If the action is proper and free, Lu [11] has shown that the Marsden-Weinstein type quotient $\mu^{-1}(e)/G$ is a Poisson submanifold of M/G . In general, a Poisson submanifold of an integrable Poisson manifold does not need to be integrable (see [5]), and when it is integrable, its symplectic groupoid need not be a subgroupoid of the symplectic groupoid of the ambient Poisson manifold. We will give a simple condition that guarantees the integrability of $\mu^{-1}(e)/G$ and gives a symplectic groupoid integrating $\mu^{-1}(e)/G$ which is a symplectic subgroupoid of $\Sigma(M)//G$. In the case where G is a 1-connected, simple, compact Lie group, a classical result of Alekseev [1] states that one can gauge transform the Poisson structure so that the Poisson action becomes an action by Poisson diffeomorphisms. Our methods allow us to describe what happens to this operation at the level of the lifted action on the symplectic groupoid.

Our last application of Theorem 1 is to the integrability of Poisson homogeneous spaces. If G is any Poisson-Lie group, the action of G on itself by left translations is Poisson. Hence, the theorem shows that it lifts to a hamiltonian G -action on $\Sigma(G)$ with a momentum map $J : \Sigma(G) \rightarrow G^*$. If $H \subset G$ is a closed, connected, coisotropic subgroup, the quotient G/H is a Poisson homogeneous space ([12]). The coisotropy condition is equivalent to the annihilator $\mathfrak{h}^\perp \subset \mathfrak{g}^*$ of the Lie algebra of H being a Lie subalgebra. We will say that the pair (G, H) is *relatively complete* if $\mathfrak{h}^\perp \subset \mathfrak{g}^*$ integrates to a closed subgroup $H^\perp \subset G^*$ and the right dressing action $\mathfrak{g}^* \rightarrow \mathfrak{X}(G)$ restricted to \mathfrak{h}^\perp integrates to an action of H^\perp on H .

Our methods then lead to a simple proof of the following theorem, which improves results of [2] (see, also, [13] for a more general approach to the integration of Poisson homogeneous spaces):

Theorem 3. *Let G be a Poisson-Lie group and let $H \subset G$ be a closed, coisotropic subgroup, such that (G, H) is relatively complete. Then $J^{-1}(H^\perp)/H \rightrightarrows G/H$ is a symplectic groupoid integrating the Poisson homogeneous space G/H .*

The results presented in this paper are part of a wider picture: since Poisson-Lie groups are the group-like objects in the category of Poisson groupoids, one should expect them to appear as the *group of symmetries* of such objects. A systematic

study of symmetries of Poisson groupoids and their infinitesimal counterparts, Lie bialgebroids, will be the subject of a separate publication [7].

The rest of this paper is organized as follows. In Section 1 we review a few notions and facts we will need to state and prove our results. Section 2 contains a proof of Theorem 1 and other results concerning Poisson actions. Section 3 contains the three applications of Theorem 1 mentioned before.

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1. BASIC NOTIONS

In this section we give a short review of all the basic notions we will need: Poisson and symplectic groupoids, Lie bialgebra(oid)s and Poisson actions.

1.1. Poisson and symplectic groupoids. Let \mathcal{G} be a Lie groupoid over M . We denote by \mathbf{s} and \mathbf{t} the source and target maps, by $\mathbf{m} : \mathcal{G}^2 \rightarrow \mathcal{G}$ the multiplication (defined on the space \mathcal{G}^2 of pairs of composable arrows), by $\mathbf{i} : \mathcal{G} \rightarrow \mathcal{G}$ the inverse map, and by $\varepsilon : M \rightarrow \mathcal{G}$ the identity section. Our convention for the groupoid multiplication is such that, given two arrows $x, y \in \mathcal{G}$, the product $x \cdot y := \mathbf{m}(x, y)$ is defined provided $\mathbf{s}(x) = \mathbf{t}(y)$. Also, if $m \in M$ we write $1_m := \varepsilon(m)$ for the unit arrow over m , and if $x \in \mathcal{G}$ we write $x^{-1} := \mathbf{i}(x)$ for the inverse arrow. We denote the groupoid by $\mathcal{G} \rightrightarrows M$.

We will be interested in Lie groupoids $\mathcal{G} \rightrightarrows M$ carrying Poisson structures on the space of arrows and on the space of units. A *Poisson groupoid* is a pair (\mathcal{G}, Π) , where \mathcal{G} is a Lie groupoid and $\Pi \in \mathfrak{X}^2(\mathcal{G})$ is a multiplicative Poisson structure. This means that the graph of the groupoid multiplication

$$\text{graph}(\mathbf{m}) := \{(x, y, x \cdot y) \mid \mathbf{s}(x) = \mathbf{t}(y)\}$$

is a coisotropic submanifold of $\mathcal{G} \times \mathcal{G} \times \bar{\mathcal{G}}$ ([20]). When Π is non-degenerate, so $\Omega = \Pi^{-1}$ is a symplectic form, the multiplicativity condition amounts to:

$$(2) \quad \mathbf{m}^* \Omega = \pi_1^* \Omega + \pi_2^* \Omega,$$

where $\pi_i : \mathcal{G}^2 \rightarrow \mathcal{G}$ are the projections on each factor. In this case, we say that the pair (\mathcal{G}, Ω) is a *symplectic groupoid*. A *morphism of Poisson groupoids* is a Lie groupoid morphism $\Phi : (\mathcal{G}_1, \Pi_1) \rightarrow (\mathcal{G}_2, \Pi_2)$ which is also a Poisson map.

For this paper, the two most important examples are the following:

Example 1.1. Lie groups are precisely the Lie groupoids for which the space of units reduces to a single object. For a Lie group G , a Poisson structure π_G is multiplicative iff the multiplication $m : G \times G \rightarrow G$ is a Poisson map (where we consider the Poisson structure $\pi_G \oplus \pi_G$ on $G \times G$). In this case, one calls (G, π_G) a *Poisson-Lie group*.

Example 1.2. Let (M, π) be a Poisson manifold. Its *Weinstein groupoid* $\Sigma(M) \rightrightarrows M$ is defined as (see [5]):

$$\Sigma(M) := \frac{\{\text{cotangent paths}\}}{\{\text{cotangent homotopies}\}},$$

where multiplication is concatenation of cotangent paths. If $p : T^*M \rightarrow M$ denotes the projection, the source and target maps are given by:

$$\mathbf{s}([a]) = p(a(0)), \quad \mathbf{t}([a]) = p(a(1)).$$

A Poisson manifold (M, π) is called *integrable* if its Weinstein groupoid $\Sigma(M)$ is smooth (in which case, one has $\dim \Sigma(M) = 2 \dim M$). The obstructions to integrability were determined in [4, 5]. When (M, π) is integrable, $\Sigma(M)$ carries a natural multiplicative symplectic form Ω . Moreover, the source (respectively, target map) is a Poisson (resp. anti-Poisson) map.

Alan Weinstein in [20] observed that the properties of the source/target maps in the last example are by no means exceptional: given a Poisson groupoid (\mathcal{G}, Π) with base M there exists a unique Poisson structure π on M , such that $\mathbf{s} : \mathcal{G} \rightarrow M$ is a Poisson map and $\mathbf{t} : \mathcal{G} \rightarrow M$ is anti-Poisson.

1.2. Lie bialgebras and Lie bialgebroids. Now let us go to the infinitesimal level. We will denote by A a Lie algebroid with bundle projection $p : A \rightarrow M$, anchor $\# : A \rightarrow TM$, and Lie bracket $[\cdot, \cdot]_A$ on its space of sections. The A -differential forms are $\Omega^\bullet(A) := \Gamma(\wedge^\bullet A^*)$ and they form a complex with the A -differential d_A (see, e.g., [14]). Our conventions are such that if $\mathcal{G} \rightrightarrows M$ is a Lie groupoid, then its Lie algebroid $A = A(\mathcal{G})$ has $A_x := \text{Ker } d_{1_x} \mathbf{s}$ and $\#_x := d_{1_x} \mathbf{t}$. Moreover, $\Gamma(A) = \mathfrak{X}(A)$ is identified with the space $\mathfrak{X}_r(\mathcal{G})$ of right invariant vector fields on \mathcal{G} . Similarly, one obtains identifications of $\mathfrak{X}^\bullet(A)$ and $\Omega^\bullet(A)$ with the spaces $\mathfrak{X}_r^\bullet(\mathcal{G})$ and $\Omega_r^\bullet(\mathcal{G})$ of right invariant multivector fields and differential forms on \mathcal{G} (note that a right invariant differential form is a \mathbf{s} -foliated differential form on \mathcal{G}). Under these identifications, the bracket $[\cdot, \cdot]_A$ and the differential d_A is identified with the Schouten bracket on right invariant multivector fields and the de Rham differential on right invariant differential forms.

We recall the following basic proposition, due to Weinstein [20]:

Proposition 1.3. *If (\mathcal{G}, Π) is a Poisson groupoid then it induces a Lie algebroid structure on $A(\mathcal{G})^*$, the dual of the Lie algebroid, whose $A(\mathcal{G})^*$ -differential is characterized by*

$$(3) \quad \overrightarrow{d_{A(\mathcal{G})^*} X} = -[\overrightarrow{X}, \Pi], \quad (X \in \mathfrak{X}(A)).$$

This leads to the infinitesimal counterpart of a Poisson groupoid, i.e., the notion of a *Lie bialgebroid*. This is pair of Lie algebroid structures (A, A^*) , on a bundle $A \rightarrow M$ and on its dual bundle $A^* \rightarrow M$, such that for any $X, Y \in \mathfrak{X}(A)$,

$$d_{A^*}[X, Y]_A = \mathcal{L}_X d_{A^*} Y - \mathcal{L}_Y d_{A^*} X.$$

Just like for a Poisson groupoid, if (A, A^*) is a Lie bialgebroid over M , there exists a Poisson structure π_M on M which is characterized by

$$\pi_M(df, dg) = \#(d_{A^*}f)(g) = \langle d_{A^*}f, d_Ag \rangle, \quad (f, g \in C^\infty(M)).$$

A *morphism* of Lie bialgebroids $\phi : (A_1, A_1^*) \rightarrow (A_2, A_2^*)$ is a Lie algebroid morphism $\phi : A_1 \rightarrow A_2$ which is also a Poisson map (note that A_i has a fiberwise linear Poisson structure, being the dual of the Lie algebroid A_i^*).

If (\mathcal{G}, Π) is a Poisson groupoid, it follows from Proposition 1.3 that $(A(\mathcal{G}), A(\mathcal{G})^*)$ is a Lie bialgebroid. Conversely, it is proved in [15] that any Lie bialgebroid structure (A, A^*) , where A can be integrated to a Lie groupoid, can actually be integrated to a Poisson groupoid $(\mathcal{G}(A), \Pi)$. Here $\mathcal{G}(A)$ is just the source 1-connected Lie groupoid integrating A . In this situation, the Poisson structures on M induced by $(\mathcal{G}(A), \Pi)$ and (A, A^*) coincide. Similarly, Poisson groupoid morphisms $\Phi : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ are in 1:1 correspondence with Lie bialgebroid morphisms $\phi : (A_1, A_1^*) \rightarrow (A_2, A_2^*)$, provided \mathcal{G}_1 is source 1-connected.

Note that the notion of Lie bialgebroid is symmetric: if (A, A^*) is a Lie bialgebroid so is (A^*, A) . On the other hand, at the level of groupoids things are more subtle: for example, in a Lie bialgebroid (A, A^*) it is possible that A is integrable while A^* is not.

Example 1.4. It is well known that if (M, π) is a Poisson manifold, then $A = T^*M$ is Lie algebroid with anchor $\pi^\sharp : T^*M \rightarrow TM$ and Lie bracket:

$$[\alpha, \beta] = \mathcal{L}_{\pi^\sharp(\alpha)}\beta - \mathcal{L}_{\pi^\sharp(\beta)}\alpha - d(\pi(\alpha, \beta)).$$

When one consider $A^* = TM$ with its canonical Lie algebroid structure, the pair (T^*M, TM) becomes a Lie bialgebroid. While $A^* = TM$ is always integrable, $A = T^*M$ does not have to be integrable. Its integrability is equivalent to the integrability of (M, π) . When (M, π) is integrable, $(\Sigma(M), \Omega)$ is the source 1-connected symplectic groupoid integrating the Lie bialgebroid (T^*M, TM) .

Example 1.5. If (G, π_G) is a Poisson-Lie group, the corresponding Lie algebroid is just a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$. According to our conventions, \mathfrak{g} is the space of right invariant vector fields on G . We can also identify \mathfrak{g}^* with the space of right invariant 1-forms on G . The bracket on 1-forms induced by π_G (see the previous example) preserves the right invariant forms, and it induces the Lie bracket $[\cdot, \cdot]_{\mathfrak{g}^*}$ on \mathfrak{g}^* .

The 1-connected Lie group integrating \mathfrak{g}^* , denoted G^* , is called the dual Poisson-Lie group: its Lie bialgebra is $(\mathfrak{g}^*, \mathfrak{g})$.

1.3. Poisson actions. Let (G, π_G) be a Poisson-Lie group and let (M, π) be a Poisson manifold. Recall that a smooth action $\Psi : G \times M \rightarrow M$ is called a *Poisson action* if Ψ is a Poisson map. Here the product $G \times M$ is furnished with the direct sum Poisson structure $\pi_G \oplus \pi$.

For a smooth action $\Psi : G \times M \rightarrow M$ of a Lie group on a manifold M , we will denote by $\psi : \mathfrak{g} \rightarrow \mathfrak{X}(M)$ the corresponding infinitesimal Lie algebra action defined by:

$$\psi(\xi)_a = \left. \frac{d}{dt} \exp(t\xi)a \right|_{t=0} \quad (\xi \in \mathfrak{g}).$$

According to our conventions, \mathfrak{g} is identified with the space of right invariant vector fields on G and it follows that $\psi : \mathfrak{g} \rightarrow \mathfrak{X}(M)$ is a Lie algebra homomorphism. The following characterization of Poisson actions is due to Lu [10]:

Proposition 1.6. *Let (G, π_G) be a connected Poisson-Lie group and let (M, π) be a Poisson manifold. For a smooth action $\Psi : G \times M \rightarrow M$ the following two properties are equivalent:*

- (i) *The action Ψ is Poisson;*
- (ii) *Setting $\delta := d_e \pi : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$, the infinitesimal action satisfies:*

$$\mathcal{L}_{\psi(\xi)} \pi = (\psi \wedge \psi) \delta(\xi), \quad (\xi \in \mathfrak{g}).$$

The map δ is just (the dual of) the Lie bracket on \mathfrak{g}^* . Hence, the proposition leads to the notion of an infinitesimal action of a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ on a Poisson manifold (M, π) .

Example 1.7. Let (G, π_G) be a Poisson-Lie group with Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$. According to our conventions, we can identify \mathfrak{g}^* with the space of right invariant 1-forms on G . The map $\lambda : \mathfrak{g}^* \rightarrow \mathfrak{X}(G)$ which to a right invariant 1-form $\eta \in \mathfrak{g}^*$ associates the vector field $\pi_G^\#(\eta)$ is a Lie algebra morphism and so defines an (left) infinitesimal action of \mathfrak{g}^* on G . Using Proposition 1.6, one checks that this is an infinitesimal action of $(\mathfrak{g}^*, \mathfrak{g})$ on the Poisson manifold (G, π_G) , called the *left dressing action*. Similarly, the identification of \mathfrak{g}^* with the left invariant 1-forms on G , leads to Lie algebra anti-morphism $\rho : \mathfrak{g}^* \rightarrow \mathfrak{X}(G)$ and hence to a *right dressing action*. Switching the role of G and G^* we also obtain left/right dressing actions of \mathfrak{g} on G^* . If one of the infinitesimal dressing actions is complete so is the other. We say that (G, π_G) is a *complete Poisson-Lie group* if the right dressing action $\rho : \mathfrak{g} \rightarrow \mathfrak{X}(G^*)$ integrates to a (Poisson) right action of (G^*, π_{G^*}) on (G, π_G) .

There is another useful characterization of Poisson actions, due to Xu [18]:

Proposition 1.8. *Let (G, π_G) be a connected Poisson-Lie group and let (M, π) be a Poisson manifold. For a smooth action $\Psi : G \times M \rightarrow M$ define $j : T^*M \rightarrow \mathfrak{g}^*$ by*

$$\langle j(\alpha), \xi \rangle = \langle \alpha, \psi(\xi) \rangle, \quad (\xi \in \mathfrak{g}).$$

Then the following two properties are equivalent:

- (i) *The action Ψ is Poisson;*
- (ii) *The map $j : T^*M \rightarrow \mathfrak{g}^*$ is a Lie bialgebroid morphism.*

1.4. Hamiltonian actions. Let $\Psi : G \times M \rightarrow M$ be a Poisson action. A smooth map $\mu : M \rightarrow G^*$ is called a *momentum map* for the action if:

$$(4) \quad \psi(\xi) = \pi^\#(\mu^* \xi^R) \quad (\xi \in \mathfrak{g}).$$

Here, $\xi^R \in \Omega^1(G^*)$ is the right invariant 1-form on G^* with value $\xi \in \mathfrak{g}$ at the identity $e \in G^*$. Notice that when $\pi_G = 0$, so that $G^* = \mathfrak{g}^*$, the momentum map condition (4) reduces to the usual condition. These generalized momentum maps were first studied by Lu in [10, 11].

Example 1.9. From each very definitions, the left/right dressing actions of G on G^* have momentum map $G^* \rightarrow G^*$ the identity map. Similarly, the left/right dressing actions of G^* on G also have momentum map $G \rightarrow G$ the identity. These actions satisfy versions of the twisted multiplicativity property (1). For example, the (left) dressing $G \times G^* \rightarrow G^*$ satisfies:

$$(5) \quad g(u_1 \cdot u_2) = (gu_1) \cdot (g^{u_1}u_2), \quad (g \in G, u_1, u_2 \in G^*).$$

A proof of the following basic fact can be found in [11]:

Proposition 1.10. *Let (G, π_G) be a connected and complete Poisson-Lie group. A momentum map $\mu : M \rightarrow G^*$ for a Poisson action $G \times M \rightarrow M$ is G -equivariant (relative to the left dressing action of G on G^*) if and only if it is a Poisson map.*

We will say that a Poisson action $G \times M \rightarrow M$ is a *hamiltonian action* if admits an equivariant momentum map $\mu : M \rightarrow G^*$. Lu has also shown that the usual Marsden-Weinstein symplectic reduction extends to these hamiltonian actions.

In order to explain this, let $G \times M \rightarrow M$ be a hamiltonian action on a Poisson manifold, with momentum map $\mu : M \rightarrow G^*$. If $u \in G^*$, denote by G_u the isotropy group of u for the the left dressing action of G on G^* . Then we have the following result (see Lu [11]):

Theorem 1.11. *Let $G \times M \rightarrow M$ be a proper and free hamiltonian action, with momentum map $\mu : M \rightarrow G^*$. For each $u \in G^*$, the level set $\mu^{-1}(u)$ carries a natural Dirac structure L_u , the space $\mu^{-1}(u)/G_u$ carries a natural Poisson structure and we have a commutative diagram:*

$$\begin{array}{ccc} & M & \\ \nearrow & & \searrow \\ \mu^{-1}(u) & & M/G \\ \searrow & & \nearrow \\ & \mu^{-1}(u)/G_u & \end{array}$$

where the inclusions are backward Dirac maps and the projections are forward Dirac maps.

Remark 1.12. If one starts with a hamiltonian action on a symplectic manifold (S, ω) the reduced spaces $\mu^{-1}(u)/G_u$ are also symplectic. In fact, their connected components are symplectic leaves of the quotient Poisson manifold M/G .

Finally, if we start with a Poisson-Lie group (G, π_G) and a Poisson manifold (M, π) , any Poisson map $\mu : M \rightarrow G^*$ determines an infinitesimal Poisson action $\psi : \mathfrak{g} \rightarrow \mathfrak{X}(M)$ by setting:

$$\psi(\xi) := \pi^\sharp(\mu^* \xi^R) \quad (\xi \in \mathfrak{g}).$$

Integration gives a *local* Poisson action with equivariant momentum map μ .

2. INTEGRATION OF POISSON ACTIONS

In this section, we will prove Theorem 1 and other results concerning the integration of Poisson actions.

2.1. Poisson actions on symplectic groupoids. Before considering the problem of lifting a Poisson action on M to a Poisson action on the symplectic groupoid $\Sigma(M)$, we discuss actions on symplectic groupoids and how the twisted multiplicativity property arises.

Proposition 2.1. *Let (G, π_G) be a connected, complete Poisson-Lie group, and let $\mathcal{G} \rightrightarrows M$ be a source-connected symplectic groupoid. If $G \times \mathcal{G} \rightarrow \mathcal{G}$ is a hamiltonian action with momentum map $J : \mathcal{G} \rightarrow G^*$ such that $J(M) = e$, the following are equivalent:*

(i) $J : \mathcal{G} \rightarrow G^*$ is a groupoid morphism:

$$J(x \cdot y) = J(x) \cdot J(y), \quad x, y \in \mathcal{G}^{(2)}.$$

(ii) The twisted multiplicativity property holds:

$$(6) \quad g(x \cdot y) = (gx) \cdot (g^{J(x)}y), \quad x, y \in \mathcal{G}^{(2)}, g \in G,$$

where we denote by g^u the right dressing action of $u \in G^*$ on an element $g \in G$.

Proof. Denote by $\psi : \mathfrak{g} \rightarrow \mathfrak{X}(\mathcal{G})$ the infinitesimal \mathfrak{g} -action. For the proof, we remark that the multiplicativity property (2) of the symplectic form Ω , when evaluated at (x, y) on the pair $(\psi(\xi)_x, \psi(\text{Ad}^* J(x) \cdot \xi)_y), (v, 0) \in T_{(x,y)}\mathcal{G}^{(2)}$ yields:

$$(7) \quad \Omega_{x,y}(\mathbf{d}_{(x,y)}\mathbf{m}(\psi(\xi)_x, \psi(\text{Ad}^* J(x) \cdot \xi)_y), \mathbf{d}_x R_y v) = \Omega_x(\psi(\xi)_x, v).$$

where $\mathbf{m} : \mathcal{G}^{(2)} \rightarrow \mathcal{G}$ is the groupoid multiplication and R_y denotes right translation by the element $y \in \mathcal{G}$ (here v is any vector tangent to the source fiber at x).

Now, since the Lie group G is connected, the twisted multiplicativity property (6) is equivalent to its infinitesimal version, which reads:

$$(8) \quad \psi(\xi)_{x,y} = \mathbf{d}_{(x,y)}\mathbf{m}(\psi(\xi)_x, \psi(\text{Ad}^* J(x) \cdot \xi)_y), \quad x, y \in \mathcal{G}^{(2)}, \xi \in \mathfrak{g}.$$

So if this condition holds, we conclude from (7) that

$$\Omega_{x,y}(\psi(\xi)_{x,y}, \mathbf{d}_x R_y v) = \Omega_x(\psi(\xi)_x, v),$$

for any vector v tangent to the source fiber at x . In other words, $i_{\psi(\xi)}\Omega = J^*\xi^R$ is a right invariant 1-form, for all $\xi \in \mathfrak{g}$. But if $J^*\xi^R$ is a right invariant 1-form, for all $\xi \in \mathfrak{g}$ and $J(M) = e$, then $J : \mathcal{G} \rightarrow G^*$ must be a groupoid homomorphism.

Conversely, assume that $J : \mathcal{G} \rightarrow G^*$ is a groupoid homomorphism. Then:

$$(9) \quad \psi(\xi) = \pi^\sharp(J^*\xi^R),$$

where $\pi = \Omega^{-1}$ is a multiplicative Poisson structure. Let $\xi \in \mathfrak{g}$ and $(x, y) \in \mathcal{G}^{(2)}$. Since J is a groupoid morphism, we have that

$$J^*(\theta_1 \star_{G^*} \theta_2) = (J^*\theta_1) \star_{\mathcal{G}} (J^*\theta_2), \quad (\theta_1 \star \theta_2) \in (T^*(G^*))^{(2)},$$

where $\star_{\mathcal{G}}$ (respectively, \star_{G^*}) denotes the groupoid multiplication in $T^*\mathcal{G}$ (respectively, $T^*(G^*)$). Now, using (9), we deduce

$$\begin{aligned} d_{(x,y)}\mathbf{m}(\psi(\xi)_x, \psi(\mathrm{Ad}^* J(x) \cdot \xi)_y) &= d_{(x,y)}\mathbf{m}(\pi^\sharp(J^* \xi^R)_x, \pi^\sharp(J^*(\mathrm{Ad}^* J(x) \cdot \xi)^R)_y) \\ &= \pi^\sharp((J^* \xi^R)_x \star_{\mathcal{G}} (J^*(\mathrm{Ad}^* J(x) \cdot \xi)^R)_y) \\ &= \pi^\sharp(J^* \xi^R)_{x,y} = \psi(\xi)_{x,y}. \end{aligned}$$

Here, we have also used that $\pi^\sharp : T^*\mathcal{G} \rightarrow T\mathcal{G}$ is a groupoid morphism, i.e.,

$$d\mathbf{m}(\pi^\sharp(\eta_1), \pi^\sharp(\eta_2)) = \pi^\sharp(\eta_1 \star_{\mathcal{G}} \eta_2), \quad (\eta_1, \eta_2) \in (T^*\mathcal{G})^{(2)}.$$

Therefore, the infinitesimal condition (8) is satisfied and, as a consequence, the twisted multiplicativity condition holds. \square

Our next remark is even more general.

Proposition 2.2. *Let (G, π_G) be a complete Poisson-Lie group, $G \times \mathcal{G} \rightarrow \mathcal{G}$ a smooth action on a Lie groupoid, and $J : \mathcal{G} \rightarrow G^*$ a groupoid morphism. If the action satisfies the twisted multiplicativity property (6), then there is an induced action on the Lie algebroid A of \mathcal{G} . Moreover, if \mathcal{G} is source 1-connected the G -action on \mathcal{G} is completely determined by J and the induced G -action on A .*

Remark 2.3. Note that in this proposition there is no assumption about a symplectic or Poisson structure on \mathcal{G} . Also, the induced action on A , in general, is not by Lie algebroid automorphisms.

Proof. First, we remark that the twisted multiplicativity property (6) and the fact that J is a homomorphism implies that, for any $x \in \mathcal{G}$, we have:

$$\begin{cases} gx = g(1_{\mathfrak{t}(x)} \cdot x) = (g 1_{\mathfrak{t}(x)}) \cdot (gx) \\ gx = g(x \cdot 1_{\mathfrak{s}(x)}) = (gx) \cdot (g^{J(x)} 1_{\mathfrak{s}(x)}) \end{cases} \Rightarrow \begin{cases} g 1_{\mathfrak{t}(x)} = 1_{\mathfrak{t}(gx)} \\ g^{J(x)} 1_{\mathfrak{s}(x)} = 1_{\mathfrak{s}(gx)} \end{cases}$$

Therefore, we have an induced G -action on M such that:

$$g 1_m = 1_{gm}.$$

Moreover, we also find that:

$$\mathfrak{t}(gx) = g \mathfrak{t}(x), \quad \mathfrak{s}(gx) = g^{J(x)} \mathfrak{s}(x).$$

We will also need the identity:

$$(gx)^{-1} = g^{J(x)} x^{-1},$$

whose proof is straightforward from (6) and the fact that J is a homomorphism.

The previous identities show that the G -action sends \mathfrak{t} -fibers to \mathfrak{t} -fibers, but does not preserve source fibers. However, we can consider a new G -action on \mathcal{G} defined by:

$$g \odot x := (gx^{-1})^{-1},$$

which does preserve \mathbf{s} -fibers, and induces the same action on the identity section. Hence, we have an induced G -action on the Lie algebroid A of \mathcal{G} , by vector bundle automorphisms (but, in general, not Lie algebroid automorphisms), defined by:

$$ga := \left. \frac{d}{dt} g \odot \gamma(t) \right|_{t=0}, \quad (g \in G, a \in A_m)$$

where $\gamma(t)$ is any curve lying in the source fiber $\mathbf{s}^{-1}(m)$ with $\gamma(0) = 1_m$ and $\dot{\gamma}(0) = a$.

If \mathcal{G} has source 1-connected fibers, then we can identify an element $x \in \mathcal{G}$ with the homotopy class $[x(t)]$, where $x(t)$ is any \mathbf{s} -path, i.e., a path lying in the source fiber through x and such that $x(0) = 1_{\mathbf{s}(x)}$ and $x(1) = x$ (see [4]). Then we can identify \mathcal{G} with the Weinstein groupoid $\mathcal{G}(A)$ consisting of A -paths modulo A -homotopy. This identification can be done at the level of paths by setting:

$$x(t) \longmapsto a(t) := \left. \frac{d}{ds} x(s) \cdot x(t)^{-1} \right|_{s=t}.$$

Using this identification, we transport the G -action on \mathcal{G} to an action on $\mathcal{G}(A)$: if x is represented by the \mathbf{s} -path $x(t)$ then gx is represented by the \mathbf{s} -path:

$$\bar{x}(t) := g^{J(x)J(x(t))^{-1}} x(t).$$

In fact, we find $\mathbf{s}(\bar{x}(t)) = g^{J(x)} \mathbf{s}(x(t)) = g^{J(x)} \mathbf{s}(x) = \mathbf{s}(gx)$ and $\bar{x}(1) = gx$. Then we compute its associated A -path:

$$\begin{aligned} \bar{a}(t) &:= \left. \frac{d}{ds} \bar{x}(s) \cdot \bar{x}(t)^{-1} \right|_{s=t} \\ &= \left. \frac{d}{ds} \left(g^{J(x)J(x(s))^{-1}} x(s) \right) \cdot \left(g^{J(x)J(x(t))^{-1}} x(t) \right)^{-1} \right|_{s=t} \\ &= \left. \frac{d}{ds} \left(g^{J(x)J(x(t))^{-1}} (x(s) \cdot x(t)^{-1})^{-1} \right) \right|_{s=t} \\ &= \left. \frac{d}{ds} \left(g^{J(x)J(x(t))^{-1}} (x(s) \cdot x(t)^{-1}) \right) \right|_{s=t} = g^{J(x)J(x(t))^{-1}} a(t). \end{aligned}$$

This last expression shows that the action of G on \mathcal{G} is completely determined by J and the action of G on A , as claimed. \square

2.2. Lifting of local Poisson actions. Let us now consider the problem of lifting a Poisson action on M to a Poisson action on the symplectic groupoid $\Sigma(M)$.

Given a Poisson action $\Psi : G \times M \rightarrow M$, it follows from Proposition 1.8 that the induced map $j : T^*M \rightarrow \mathfrak{g}^*$ is a Lie bialgebroid morphism from (T^*M, TM) to $(\mathfrak{g}^*, \mathfrak{g})$. Integrating this morphism (see [18, Theorem 5.5]), we conclude that:

Corollary 2.4. *Let $\Psi : G \times M \rightarrow M$ be a Poisson action of a Poisson-Lie group (G, π_G) on a Poisson manifold (M, π) . The Lie bialgebroid morphism $j : T^*M \rightarrow \mathfrak{g}^*$ integrates to a morphism of Poisson groupoids $J : \Sigma(M) \rightarrow G^*$.*

At the level of cotangent paths, the map J is simply given by the formula:

$$J([a]) = [j \circ a]$$

(see [4], where it is explained how to integrate morphisms of Lie algebroids to morphisms of Lie groupoids in terms of cotangent paths).

Since $J : \Sigma(M) \rightarrow G^*$ is a Poisson map and a groupoid morphism, we conclude from Proposition 2.1 that:

Proposition 2.5. *Let $\Psi : G \times M \rightarrow M$ be a Poisson action of a Poisson-Lie group (G, π_G) on a Poisson manifold (M, π) . There exists a local hamiltonian action of G on $\Sigma(M)$ with momentum map $J : \Sigma(M) \rightarrow G^*$ which satisfies the infinitesimal twisted multiplicativity property (8).*

Later, we will give an explicit expression for this local action (see Remark 2.8). The following example shows that, in general, the lifted action will not be a *global action*.

Example 2.6. Let G be any Poisson-Lie group which is not complete. The action of G on itself by left translations $G \times G \rightarrow G$ is a Poisson action. The lifted (local) action on $\Sigma(G)$ is not a global action. In fact, note that the identity $e \in G$ is a fixed point for the Poisson structure where the isotropy Lie algebra is \mathfrak{g}^* . Hence, the corresponding isotropy group is:

$$\Sigma(G)_e = \mathfrak{s}^{-1}(e) = \mathfrak{t}^{-1}(e) \simeq G^*.$$

The restriction of $J : \Sigma(G) \rightarrow G^*$ to this isotropy group is an isomorphism, so if the lifted action was a global action, the dressing action would have to be complete.

2.3. Lifting to global Poisson actions. Our main results states that if (G, π_G) is a complete Poisson-Lie group, then the lifted action is a global action. In the sequel, we will assume that G is complete and will denote by g^u the right dressing action of an element $u \in G^*$ on a element $g \in G$.

Theorem 2.7. *Let (G, π_G) be a complete Poisson-Lie group, (M, π) an integrable Poisson manifold and $G \times M \rightarrow M$ a Poisson action. Let $J : \Sigma(M) \rightarrow G^*$ be the integration of $j : T^*M \rightarrow \mathfrak{g}^*$. Then there exists a lifted hamiltonian action of (G, π_G) on the symplectic groupoid $\Sigma(M)$ with momentum map J , such that:*

(1) *J is equivariant:*

$$J(gx) = gJ(x), \quad (g \in G, x \in \Sigma(M)).$$

(2) *The action is twisted multiplicative:*

$$g(x \cdot y) = (gx) \cdot (g^{J(x)}y), \quad (g \in G, (x, y) \in \Sigma(M)^{(2)}).$$

Proof. Let $a : I \rightarrow T^*M$ be a cotangent path and define a new path $\bar{a} : I \rightarrow T^*M$ by:

$$(10) \quad \bar{a}(t) := g^{J(x)J(x(t))^{-1}} a(t).$$

In this formula, we use the lifted cotangent action of G on T^*M and $x(t)$ denotes the element in $\Sigma(M)$ which, for a fixed $t \in I$, is defined by the cotangent path $s \mapsto t a(st)$. The motivation for this definition can be found in the proof of Proposition 2.2.

One now checks that:

- (a) For any cotangent path $a(t)$, the path $\bar{a}(t)$ defined by (10) is also a cotangent path.
- (b) If a_ε is a cotangent homotopy, then the corresponding family \bar{a}_ε defined by (10) is also a cotangent homotopy.

This means that formula (10) leads to a map $G \times \Sigma(M) \rightarrow \Sigma(M)$ by setting at the level of cotangent homotopy classes:

$$(11) \quad g[a(t)] = [g^{J(x)J(x(t))^{-1}} a(t)].$$

Using this formula, we will show the G -equivariance of $J : \Sigma(M) \rightarrow G^*$, i.e., that:

$$(12) \quad J(gx) = gJ(x), \quad (x \in \Sigma(M), g \in G),$$

where g acts on the left by (11) and on the right by the (left) dressing action on G^* .

In order to check (12) one starts by remarking that, since G^* is simply-connected, one can identify G^* with paths $\xi : I \rightarrow \mathfrak{g}^*$ up to \mathfrak{g}^* -homotopy. This is a special instance of the general construction mentioned in the proof of Proposition 2.2: given an element $u \in G^*$ we first identify it with (the homotopy class of) a path $u(t) \in G^*$ starting at the identity $u(0) = e$ and ending at $u(1) = u$. Then we associate to it (the \mathfrak{g}^* -homotopy class of) a path in the Lie algebra $\xi : I \rightarrow \mathfrak{g}^*$ by setting:

$$\xi(t) = \left. \frac{d}{ds} u(s)u^{-1}(t) \right|_{s=t}.$$

By the formula proved at the end of Proposition 2.2, under this identification, the dressing action $G \times G^* \rightarrow G^*$ is given at the level of \mathfrak{g}^* -paths by:

$$g[\xi(t)] = [\text{Ad}^* g^{u(1)u(t)^{-1}} \xi(t)].$$

Using this relation, we see that if $x = [a(t)] \in \Sigma(M)$ and $g \in G$, then:

$$\begin{aligned} J(gx) &= [j(g^{J(x)J(x(t))^{-1}} a(t))] \\ &= [\text{Ad}^* g^{J(x)J(x(t))^{-1}} j(a(t))] \\ &= g[j(a(t))] = gJ(x), \end{aligned}$$

so the equivariance follows.

We now check that (11) defines a G -action, i.e., that:

- (a) If $e \in G$ is the identity element, then $ex = x$, for all $x \in \Sigma(M)$;
- (b) If $g, h \in G$, then $g(hx) = (gh)x$, for all $x \in \Sigma(M)$;

In order to check that (a) holds, one observes that the identity element $e \in G$ is fixed by the right dressing action of G^* , so if $x = [a(t)] \in \Sigma(M)$ we find:

$$e[a(t)] = [e^{J(x)J(x(t))^{-1}} a(t)] = [a(t)],$$

and (a) follows.

To prove (b), we first observe that if $x = [a(t)] \in \Sigma(M)$ so that $hx = [h^{J(x)J(x(t))^{-1}} a(t)]$, then:

$$J(hx) = hJ(x), \quad J((hx)(t)) = h^{J(x)J(x(t))^{-1}} J(x(t)).$$

It then follows from the twisted multiplicativity of the left dressing action of G on G^* that:

$$\begin{aligned}
g(hx) &= g(h[a(t)]) \\
&= [g^{J(hx)J((hx)(t))^{-1}} h^{J(x)J(x(t))^{-1}} a(t)] \\
&= [g^{(hJ(x))(h^{J(x)}J(x(t))^{-1})} h^{J(x)J(x(t))^{-1}} a(t)] \\
&= [g^{h(J(x)J(x(t))^{-1})} h^{J(x)J(x(t))^{-1}} a(t)].
\end{aligned}$$

Using the twisted multiplicativity of the right dressing action of G^* on G we obtain:

$$\begin{aligned}
g(hx) &= g(h[a(t)]) \\
&= [g^{h(J(x)J(x(t))^{-1})} h^{J(x)J(x(t))^{-1}} a(t)] \\
&= [(gh)^{J(x)J(x(t))^{-1}} a(t)] \\
&= (gh)[a(t)] = (gh)x,
\end{aligned}$$

so (b) holds.

Finally, we need to check that the G -action (11) is hamiltonian with momentum map J . By Proposition 2.5 we have a hamiltonian G -action on $\Sigma(M)$ with momentum map J . This local action satisfies the twisted multiplicativity property and the induced action on the Lie algebroid $A(\Sigma(M)) = T^*M$ is the cotangent lifted action. On the other hand, the G -action (11) also satisfies the twisted multiplicativity property and induces the same action on T^*M . By the uniqueness in Proposition 2.2, the two actions must coincide, so we conclude that our action is hamiltonian with momentum map J . \square

Remark 2.8. Assume that G is not a complete Poisson-Lie group. The identity $e \in G$ is a fixed point for the infinitesimal right dressing action of G^* on G . It follows that, for any element $u \in G^*$, the (local) dressing action g^u is defined for g in a small enough neighborhood of $e \in G$ (which depends on u). Similarly, if $u : I \rightarrow G^*$ is any path starting at $u(0) = e \in G^*$, a compactness argument shows that $g^{u(1)u(t)^{-1}}$ is defined provided g is close enough to $e \in G^*$. It follows that, when G is not complete, formula (11) still defines a *local action* of G on $\Sigma(M)$. The proof of the Theorem shows that this will be twisted multiplicative and hamiltonian, with momentum map $J : \Sigma(M) \rightarrow G^*$. In other words, it is the local action given by Proposition 2.5.

3. APPLICATIONS

In this section we will consider three applications of Theorem 2.7. The first application is to the integration of a Poisson quotient M/G . The second application concerns hamiltonian actions and the integration of Lu-Weinstein quotients. The third application is to the integration of Poisson homogeneous spaces.

3.1. Integrability of Poisson quotients. We now turn to the integrability of Poisson quotients. We start with the following remark:

Proposition 3.1. *Let $G \times M \rightarrow M$ be a Poisson action of a complete Poisson-Lie group on an integrable Poisson manifold. The lifted action $G \times \Sigma(M) \rightarrow \Sigma(M)$ is proper (respectively, free) if and only if the original action $G \times M \rightarrow M$ is proper (respectively, free).*

Proof. For the proof, we use the following simple fact: Let a Lie group G act smoothly on manifolds P and Q and let $\phi : P \rightarrow Q$ be a G -equivariant map. If the action on Q is free (respectively, proper), then the action on P is free (respectively, proper).

Now, we just need to observe that the identity section $\varepsilon : M \rightarrow \Sigma(M)$ and the target map $\mathbf{t} : \Sigma(M) \rightarrow M$ are G -equivariant maps. \square

From now on we will assume that the action $G \times M \rightarrow M$ is proper and free, so that the lifted action is also proper and free. For a Poisson action $G \times M \rightarrow M$ the space of G -invariant functions is a Poisson subalgebra $C^\infty(M)^G \subset C^\infty(M)$. It follows that if the action is proper and free, this space can be identified with $C^\infty(M/G)$, so that M/G has a reduced Poisson structure π_{red} such that the quotient map $M \rightarrow M/G$ is a Poisson map.

Using our lifted action one can construct a symplectic groupoid integrating M/G :

Theorem 3.2. *Let (G, π_G) be a complete Poisson-Lie group, (M, π) an integrable Poisson manifold and $G \times M \rightarrow M$ a Poisson action which is proper and free. Then the symplectic reduced space $(J^{-1}(e)/G, \Omega_{\text{red}})$ gives a symplectic groupoid over M/G which integrates the reduced Poisson manifold $(M/G, \pi_{\text{red}})$.*

Proof. By Proposition 3.1, the lifted action of G on $\Sigma(M)$ is proper and free. It follows that its momentum map $J : \Sigma(M) \rightarrow G^*$ is a groupoid morphism which is a submersion. Therefore, its kernel $J^{-1}(e) \subset \Sigma(M)$ is a Lie subgroupoid. The equivariance of the momentum map, implies that $J^{-1}(e)$ is G -invariant. Moreover, the twisted multiplicativity property (1) guarantees that the restricted action of G on $J^{-1}(e)$ is by groupoid automorphisms. We conclude that the quotient $J^{-1}(e)/G$ is a Lie groupoid over M/G .

Now, by the Lu-Weinstein reduction theorem, there exists a reduced symplectic form Ω_{red} on $J^{-1}(e)/G$. It follows from the multiplicativity of Ω , that Ω_{red} is also a multiplicative 2-form, so that $(J^{-1}(e)/G, \Omega_{\text{red}})$ is a symplectic groupoid over M/G and the source map $s : J^{-1}(e)/G \rightarrow M/G$ is a Poisson morphism. Hence, $(J^{-1}(e)/G, \Omega_{\text{red}})$ integrates $(M/G, \pi_{\text{red}})$. \square

Theorem 3.2 raises a natural question: does $\Sigma(M/G)$ coincide with the symplectic reduction:

$$\Sigma(M)//G := J^{-1}(e)/G?$$

Since we already know this is a symplectic groupoid integrating the quotient M/G , the question amounts to decide if the source fibers are 1-connected or not. This problem can be handled by the same method used in [8] for the case $\pi_G = 0$.

First, we observe that the source fibers of $J^{-1}(e)$ need not be connected. Let $J^{-1}(e)^0$ be the unique source connected Lie subgroupoid of $\Sigma(M)$ integrating the Lie algebroid $j^{-1}(0) \subset T^*M$:

$$J^{-1}(e)^0 = \{[a] \in \Sigma(M) : j(a(t)) = 0, \forall t \in [0, 1]\}.$$

If G is connected, $J^{-1}(e)^0$ is G -invariant. The action of G on $J^{-1}(e)^0$ is by automorphisms, so we have a groupoid morphism $\Phi : J^{-1}(e)^0 \rightarrow J^{-1}(e)^0/G$ which induces the Lie algebroid morphism $\phi : j^{-1}(0) \rightarrow j^{-1}(0)/G \cong T^*(M/G)$. On the other hand, the Lie algebroid morphism ϕ integrates to a morphism of source 1-connected groupoids $\widehat{\Phi} : \mathcal{G}(j^{-1}(0)) \rightarrow \Sigma(M/G)$ which covers the homomorphism Φ (here, $\mathcal{G}(j^{-1}(0))$ denotes the source 1-connected groupoid integrating $j^{-1}(0)$). This leads to a commutative diagram:

$$(13) \quad \begin{array}{ccccc} K_M & \hookrightarrow & \mathcal{G}(j^{-1}(0)) & \xrightarrow{\widehat{p}} & J^{-1}(e)^0 \\ \downarrow \widehat{\Phi} & & \downarrow \widehat{\Phi} & & \downarrow \Phi \\ K_{M/G} & \hookrightarrow & \Sigma(M/G) & \xrightarrow{p} & J^{-1}(e)^0/G \end{array}$$

where K_M and $K_{M/G}$ are group bundles over M and M/G , respectively, with discrete fibers. Now the same argument as in [8], for the case where $\pi_G = 0$, shows that the group bundle K_M measures how symplectization and reduction fail to commute. More precisely, we recover [16, Proposition 5.3]:

Theorem 3.3. *Let (G, π_G) be a connected complete Poisson-Lie group, (M, π) an integrable Poisson manifold and $G \times M \rightarrow M$ a Poisson action which is proper and free. Then symplectization and reduction commute if and only if the discrete groups*

$$K_m := \frac{\{a : I \rightarrow j^{-1}(0) \mid a \text{ is an cotangent loop such that } a \sim 0_m\}}{\{\text{cotangent homotopies with values in } j^{-1}(0)\}}$$

are trivial, for all $m \in M$.

We refer to [8] for more details on the proof.

3.2. Integration of hamiltonian actions. Let us turn now to the study of hamiltonian actions $G \times M \rightarrow M$. The following remark is due to Xu [17]:

Proposition 3.4. *If the action $G \times M \rightarrow M$ is a hamiltonian action with momentum map $\mu : M \rightarrow G^*$ then $J : \Sigma(M) \rightarrow G^*$ is exact:*

$$J(x) = \mu(\mathbf{t}(x))\mu(\mathbf{s}(x))^{-1}.$$

The momentum map $\mu : M \rightarrow G^*$ defines a Lie algebroid morphism $\psi : \mathfrak{g} \ltimes M \rightarrow T^*M$ by setting:

$$(14) \quad (\xi, m) \mapsto \mathbf{d}_m \langle \xi, \mu \rangle.$$

Now we have:

Proposition 3.5. *Assume that the Lie algebroid morphism $\psi : \mathfrak{g} \times M \rightarrow T^*M$ given by (14) integrates to a groupoid morphism $\Psi : G \times M \rightarrow \Sigma(M)$. Then the lifted G -action on $\Sigma(M)$ is a twisted inner action, i.e., it is given by:*

$$(15) \quad gx = \Psi(g, \mathbf{t}(x)) \cdot x \cdot \Psi(g^{J(x)}, \mathbf{s}(x))^{-1}.$$

Proof. One checks that formula (15) defines a G -action on $\Sigma(M)$ which is twisted multiplicative (relative to $J : \Sigma(M) \rightarrow G^*$). The corresponding G -action induced on the Lie algebroid $A = T^*M$ is just the cotangent lifted action of $G \times M \rightarrow M$. By Proposition 2.2, it follows that (15) must coincide with the lifted G -action on $\Sigma(M)$. \square

Let us assume now that the hamiltonian $G \times M \rightarrow M$ is proper and free. Then the quotient $M//G := \mu^{-1}(e)/G$ is a Poisson submanifold of M/G . Is this Poisson submanifold integrable? What is a symplectic groupoid integrating it? The following result gives an answer to these questions:

Theorem 3.6. *Let $G \times M \rightarrow M$ be a proper and free hamiltonian action with momentum map $\mu : M \rightarrow G^*$ and assume that the Lie algebroid morphism $\psi : \mathfrak{g} \times M \rightarrow T^*M$ given by (14) integrates to a groupoid morphism $\Psi : G \times M \rightarrow \Sigma(M)$. Then, there exists a hamiltonian action of $G \times G$ on $\Sigma(M)$ which is proper and free, and its symplectic quotient given by:*

$$\Sigma(M)//G \times G := J^{-1}(e)|_{\mu^{-1}(e)}/G \times G \subset \Sigma(M)//G,$$

is a symplectic subgroupoid integrating the Poisson submanifold $M//G \subset M/G$.

Proof. We will only sketch a proof of this result. Further details will be available in [7]. First one defines an action of $G \times G$ on $\Sigma(M)$ by setting:

$$(g_1, g_2)x = \Psi(g_1, \mathbf{t}(x)) \cdot x \cdot \Psi(g_2^{J(x)}, \mathbf{s}(x))^{-1}.$$

This action is hamiltonian, with momentum map:

$$\bar{\mu} : \Sigma(M) \rightarrow G^* \times G^*, \quad x \mapsto (\mu(\mathbf{t}(x)), \mu(\mathbf{s}(x))^{-1}).$$

Notice that restriction of this action to the diagonal in $G \times G$ yields the lifted G -action.

Next one checks that the $G \times G$ -action on $\Sigma(M)$ is proper, free, and $(e, e) \in G^* \times G^*$ is a regular value of the momentum map. Hence, we have the symplectic quotient:

$$\Sigma(M)//G \times G = \bar{\mu}^{-1}((e, e))/G \times G = J^{-1}(e)|_{\mu^{-1}(e)}/G \times G.$$

Note that $J^{-1}(e)|_{\mu^{-1}(e)} \subset \Sigma(M)$ is a Lie subgroupoid. One can show that its product structure descends to the quotient $\Sigma(M)//G \times G$, so that this is a symplectic groupoid. Finally, to finish the proof, one verifies that its source (respectively, target) map is a Poisson (respectively, anti-Poisson) map. \square

Remark 3.7. The quotient $M//G$ is still defined when $e \in G^*$ is a regular value of $\mu : M \rightarrow G^*$ and the action on the level set $\mu^{-1}(e)/G$ is proper and free. In this case, one can check that the groupoid given in the proposition stills gives an integration of $M//G$. However, now M/G need not be a smooth manifold and it may not make sense to speak of the groupoid $\Sigma(M)//G$.

For a compact Poisson-Lie group G a result of Ginzburg and Weinstein [9] states that there is a Poisson diffeomorphism $e : \mathfrak{g}^* \rightarrow G^*$. Moreover, for hamiltonian actions of compact Poisson-Lie groups we have the following result of Alekseev [1]:

Theorem 3.8. *Let G be a compact, 1-simple, Poisson-Lie group and let $G \times M \rightarrow M$ be a Poisson action with momentum map $\mu : M \rightarrow \mathfrak{g}^*$. There is a Poisson structure on M , gauge equivalent to the original one, for which the action of G is by Poisson diffeomorphisms with momentum map $e^{-1} \circ \mu : M \rightarrow \mathfrak{g}^*$.*

Our description of the lifted G -action allows us to explain this result at the level of the symplectic groupoid. Let us denote by π the original Poisson structure on M and by $\tilde{\pi}$ the gauge equivalent Poisson structure. This means that π and $\tilde{\pi}$ have the same symplectic leaves and that there is a closed 2-form $B \in \Omega^2(M)$ such that the symplectic structures on a leaf S differ by the pullback of B to S (see [3]). It follows ([3, Theorem 4.1]) that the symplectic groupoids of (M, π) and $(M, \tilde{\pi})$ have the same groupoid structure, while the symplectic forms are related by:

$$\tilde{\Omega} = \Omega + \mathfrak{t}^*B - \mathfrak{s}^*B.$$

Our next result describes the lifted G -action on the symplectic groupoid $(\Sigma(M), \tilde{\Omega})$:

Theorem 3.9. *For a compact, 1-simple, Poisson-Lie group G acting on an integrable Poisson manifold (M, π) , the G -action on $(M, \tilde{\pi})$ lifts to a hamiltonian G -action on $(\Sigma(M), \tilde{\Omega})$ with momentum map $e^{-1} \circ \mu \circ \mathfrak{t} - e^{-1} \circ \mu \circ \mathfrak{s} : \Sigma(M) \rightarrow \mathfrak{g}^*$. Moreover, this action is inner and is explicitly given by:*

$$gx = \Psi(g, \mathfrak{t}(x)) \cdot x \cdot \Psi(g, \mathfrak{s}(x))^{-1}.$$

The proof is more or less straightforward. We refer to [7] for more details.

3.3. Poisson homogeneous spaces. Let (G, π_G) be a Poisson-Lie group. We say that a Poisson manifold (M, π) is a *Poisson homogeneous space* if there exists a Poisson action $G \times M \rightarrow M$ which is transitive. In this section, we will give a simple description of the symplectic groupoid $\Sigma(M)$, in the case where the Poisson structure vanishes at some point, by applying Theorem 2.7.

Let $m_0 \in M$ be the point where the Poisson structure vanishes. We identify M with G/H , where $H = G_{m_0}$ is the isotropy group at m_0 . Under this identification, the Poisson action of G on G/H is by left translations and the projection $q : G \rightarrow G/H \cong P$ becomes a Poisson map. Then H is a closed subgroup of G with Lie algebra \mathfrak{h} such that its annihilator \mathfrak{h}^\perp is a Lie subalgebra of \mathfrak{g}^* , that is, H is a coisotropic subgroup of G . In the sequel we will assume that H is connected. Our final result describes the symplectic groupoid integrating the Poisson manifold G/H . For this we need the following definition (compare with [2]):

Definition 3.10. Let G be a Poisson-Lie group and let $H \subset G$ be a closed, connected, coisotropic subgroup with Lie algebra $\mathfrak{h} \subset \mathfrak{g}$. We say that the pair (G, H) is **relatively complete** if the annihilator $\mathfrak{h}^\perp \subset \mathfrak{g}^*$ integrates to a closed subgroup $H^\perp \subset G^*$ and the right dressing action $\mathfrak{g}^* \rightarrow \mathfrak{X}(G)$ restricted to \mathfrak{h}^\perp integrates to an action of H^\perp on H .

For a relatively complete pair (G, H) we have:

Theorem 3.11. *Let G be a Poisson-Lie group and let $H \subset G$ be a closed, connected, coisotropic subgroup such that the pair (G, H) is relatively complete. Then $(J^{-1}(H^\perp)/H, \Omega_{\text{red}})$ is a symplectic groupoid integrating the Poisson homogeneous space G/H .*

Proof. As usual, we let $\Sigma(G)$ be the symplectic groupoid of the Poisson manifold G and we let $J : \Sigma(G) \rightarrow G^*$ the momentum map for the local action of G on $\Sigma(G)$ obtained by lifting the action of G on itself by left translations (recall that we are not assuming that G is complete, so this is only a local action).

Because the action by translations is free, the lifted action is also free and J is a submersion. Since $H^\perp \subset G^*$ is assumed to be a closed subgroup, it follows that $J^{-1}(H^\perp) \subset \Sigma(G)$ is a Lie subgroupoid. Moreover, using that J is a Poisson submersion and that H^\perp is coisotropic, so is $J^{-1}(H^\perp)$.

By our relative completeness assumption, the action of H on G by left translations can be integrated to a *global action* of H on $J^{-1}(H^\perp) \subset \Sigma(G)$. This follows from the explicit formula (11), which only uses the dressing action of H^\perp on H .

Now the twisted multiplicativity property (6) shows that the quotient $J^{-1}(H^\perp)/H$ inherits a groupoid structure over G/H . This quotient is also a symplectic manifold, and the source (respectively target) map is a Poisson (respectively, anti-Poisson), so we conclude that it is a symplectic groupoid integrating G/H . \square

Example 3.12. Let us consider the case where G is a complete Poisson-Lie group. Then the symplectic groupoid $\Sigma(G)$ is isomorphic to the transformation groupoid $G^* \times G$, associated with the dressing action of G^* on G . The lift of the action of G on itself by left translations to the symplectic groupoid $\Sigma(G)$ is:

$$g \cdot (u, h) = ({}^g u, g^u h).$$

The momentum map $J : \Sigma(G) \rightarrow G^*$ is projection on the factor G^* and Theorem 3.11 says that:

$$J^{-1}(H^\perp)/H = (H^\perp \times G)/H,$$

is a symplectic groupoid integrating G/H .

Note that general Poisson homogeneous spaces of the form G/H are always integrable. However, in general, to construct their symplectic groupoids one requires more complicated procedures than the one given in Theorem 3.11.

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