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A Note on Poisson Symmetric Spaces†

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Abstract

We introduce the notion of a Poisson symmetric space and the associated infinitesimal object, a symmetric Lie bialgebra. They generalize corresponding notions for Lie groups due to V. G. Drinfel'd. We use them to give some geometric insight to certain Poisson brackets that have appeared before in the literature.

1 Motivation

Let us recall briefly the best-known examples of Poisson manifolds. The most basic example is that of a symplectic manifold (M,ω), where the Poisson bracket is defined by

$$\{f_1,f_2\} =< \omega, X_{f_1}\wedge X_{f_2} >, \qquad f_1,f_2\in C^\infty(M) \tag{1}$$

and where for each $f \in C^\infty(M)$ we have introduced the vector field X_f satisfying $X_f \lrcorner\, \omega = df$ [1]. Every Poisson manifold foliates into symplectic manifolds [16].

The second well known class of Poisson brackets are the linear Poisson brackets on a vector space, which arise as follows. Let $\mathfrak{g}$ be a Lie algebra and for each $\xi\in\mathfrak{g}^*$ view the differential $d_\xi f$ of $f\in C^\infty(\mathfrak{g}^*)$ as an element of $\mathfrak{g}$. Then we have the linear Poisson bracket

$$\{f_1,f_2\}(\xi) =< \xi, [d_\xi f_1, d_\xi f_2] >, \quad \xi\in\mathfrak{g}^*, f_1,f_2\in C^\infty(\mathfrak{g}^*). \tag{2}$$

Moreover, every linear Poisson bracket has this form [16].

The next class one might consider is the class of quadratic Poisson brackets on a Lie group G. They can be described as follows. Let $\mathfrak{g} = Lie(G)$ be the Lie algebra of G and assume that $\mathfrak{g}$ has an ad-invariant inner product $(\ ,\ )$. Then to each skew-symmetric solution $A\colon \mathfrak{g}\to\mathfrak{g}$ of the classical Yang-Baxter equation[1]

$$\bigodot_{x,y,z} [[Ax,Ay]-A[x,y]_A, z] = 0 \tag{3}$$

corresponds the quadratic Poisson bracket

$$\{f_1,f_2\} = (A(\tilde\nabla f_1),\tilde\nabla f_2) - (A(\nabla f_1),\nabla f_2) \qquad f_1,f_2\in C^\infty(G) \tag{4}$$

where we have denoted by ∇ and $\tilde\nabla$ the right and left differentials on G and we have set $[x,y]_A \equiv [Ax,y]+[x,Ay]$.

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[1]The symbol $\bigodot$ denotes a cyclic sum over its indexes.

Bracket (4) is important in the theory of integrable systems. Among the applications of this bracket one has:

-Poisson brackets for scattering and transition matrices [6];

-hidden symmetries of integrable systems [5], [15];

-second Poisson bracket for integrable systems [12], [13];

-quantum groups [4], [8];

There is a geometric interpretation of bracket (4), due to Drinfel'd [3], which asserts that these brackets are naturally associated with Poisson Lie groups.

In this note we consider the following extension of bracket (4), which has been study by several authors [11], [12], [13], [14]. Let $R: \mathfrak{g} \to \mathfrak{g}$ be a solution of (3) such that its skew-symmetric part $A = \frac{1}{2}(R - R^*)$ also solves (3). Then if $S = \frac{1}{2}(R + R^*)$ is the symmetric part of R, we have the Poisson bracket on G

$$\{f_1, f_2\} = (A(\tilde{\nabla} f_1), \tilde{\nabla} f_2) - (A(\nabla f_1), \nabla f_2) + (S(\nabla f_1), \tilde{\nabla} f_2) - (S(\tilde{\nabla} f_1), \nabla f_2). \tag{5}$$

This bracket appears in connection with the applications mentioned above when some symmetry is present in the system. We will now show that bracket (5) is closely related with a geometric object which we call a Poisson symmetric space. This parallels Drinfel'd interpretation of bracket (4).

2 Poisson symmetric spaces

A **Poisson Lie group** is a Lie group G with a Poisson bracket such that group multiplication $G \times G \to G$ is a Poisson map. The associated infinitesimal objects are the Lie bialgebras $(\mathfrak{g}, \mathfrak{g}^*)$. For the elementary facts on the theory of Poisson Lie groups we refer the reader to [10], [15].

Recall that every Lie group G is a symmetric space: $G \simeq (G \times G)/H$, where $H \subset G \times G$ (the diagonal) is the fixed point set of the involutive automorphism Θ of $G \times G$ $(g_1, g_2) \mapsto (g_2, g_1)$ and the isomorphism is given explicitly by $(g_1, g_2)H \mapsto g_1 g_2^{-1}$.

We would like this construction to work in the Poisson category. For this we let G be a Poisson Lie group with Poisson bivector Λ^G and we consider on $G \times G$ the Poisson bivector $\Lambda^G \oplus (-\Lambda^G)$. Then we check that $G \times G$ is a Poisson-Lie group, $(G \times G)/H$ is a Poisson manifold and $G \simeq (G \times G)/H$ as Poisson manifolds. Moreover, the involution $\Theta: G \times G \to G \times G$ satisfies

$$\{f_1 \circ \Theta, f_2 \circ \Theta\}_{G \times G} = -\{f_1, f_2\}_{G \times G} \circ \Theta, \quad f_1, f_2 \in C^\infty(G \times G) \tag{6}$$

(note the $-$ sign!).

DEFINITION 2.1.

(i) A **Poisson symmetric Lie group** *is a pair* (G, Θ) *where* G *is a Poisson Lie group and* $\Theta: G \to G$ *is an involutive Poisson Lie group anti-morphism.*

(ii) A **symmetric Lie bialgebra** *is a triple* $(\mathfrak{g}, \mathfrak{g}^*, \theta)$ *where* $(\mathfrak{g}, \mathfrak{g}^*)$ *is a Lie bialgebra and* $\theta: \mathfrak{g} \to \mathfrak{g}$ *is an involutive Lie bialgebra anti-morphism.*

There is a 1-1 correspondence between simply-connected Poisson symmetric Lie groups and symmetric Lie bialgebras. The proof is the same as in the Poisson Lie group case (see [10]). In the following result we denote by $H \subset G$ the connected component of the identity of the fixed point set of Θ, and by $\mathfrak{h}$ its Lie algebra. Also G^* denotes the dual Poisson Lie group to G [15].

THEOREM 2.1.

(i) Let $M = G/H$ be the symmetric space associated with a Poisson symmetric Lie group (G, Θ). Then there is a unique Poisson structure on M such that $\pi: G \to M$ is a Poisson map.

(ii) Let $H^\perp \subset G^*$ be the Lie subgroup with Lie algebra the annihilator $\mathfrak{h}^\perp \subset \mathfrak{g}^*$. The leaves of the symplectic foliation of M are the orbits of the action of $H^\perp$ on M induced from the dressing action of G^* on G.

These concepts fit well into the classical theory of symmetric spaces [9]. For example, the usual duality for symmetric pairs works in the Poisson category, so that to each symmetric Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*, \theta)$ one can associate a **dual symmetric Lie bialgebra** $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}^*, \tilde{\theta})$. For the details of this construction and the proof of the results above see [7].

Example. Let $G = SL(n, \mathbb{R})$ with bracket

$$\{s_{ij}, s_{kl}\}_G = (\mathrm{sgn}(i-k) - \mathrm{sgn}(l-j))s_{il}s_{kj} \tag{7}$$

and let $\Theta: G \to G \ : L \mapsto (L^t)^{-1}$. Then (G, Θ) is a Poisson symmetric pair. The associated symmetric Lie bialgebra is $(\mathfrak{g}, \mathfrak{g}^*, \theta)$ where $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$, $\theta(x) = -x^t$ and $\mathfrak{g}^*$ can be identified with $\mathfrak{g}$ with bracket $[x, y]_A$, where

$$Ax = \begin{cases} x & \text{if } x \text{ is upper triangular} \\ 0 & \text{if } x \text{ is diagonal} \\ -x & \text{if } x \text{ is lower triangular} \end{cases} \tag{8}$$

For the dual symmetric Lie bialgebra one has $\tilde{\mathfrak{g}} = \mathfrak{su}(n)$ and $\tilde{\mathfrak{g}}^* = \mathfrak{b}$ the sub-algebra of $\mathfrak{sl}(n, \mathbb{C})$ of upper triangular matrices with real diagonal elements.

3 Formulas for Poisson brackets

We consider a Lie algebra $\mathfrak{g}$ with an invariant inner product $(\ ,\ )$, such that the cohomology $H^1(\mathfrak{g}, V)$, where V is the $\mathfrak{g}$-module $\mathfrak{g} \wedge \mathfrak{g}$, vanishes. For example, a classical Lie algebra satisfies these hypothesis. Our objective is to show that, under these circumstances, (5) corresponds to the bracket on a Poisson symmetric space $M = G/H$.

THEOREM 3.1.

(i) Let $\mathfrak{g}$ be a Lie algebra as above. The Lie bialgebras $(\mathfrak{g}, \mathfrak{g}^*)$ are in one-to-one correspondence with skew-symmetric solutions of the Yang-Baxter equation (3).

(ii) If G is a Poisson Lie group with Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ the bracket on G is given by formula (4).

For a proof see [14], [15]. The correspondence in theorem 3.1 is obtained by identifying $\mathfrak{g}$ with $\mathfrak{g}^*$ via $(\ ,\ )$ and letting $[x, y]_* = [x, y]_A$.

Now let $M = G/H$ be a Poisson symmetric space with associated symmetric Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*, \theta)$ and assume $\mathfrak{g}$ satisfies the hypothesis above. Then $[x, y]_* = [x, y]_A$ and one can check that

$$\theta A = -A\theta. \tag{9}$$

Finally recall that the Cartan immersion is the map $\iota: M \to G$ defined by

$$\iota(gH) = g\Theta(g^{-1}). \tag{10}$$

This is a totally geodesic immersion whose image is $P = \exp(\mathfrak{p})$, where $\mathfrak{p}$ is the -1-eigenspace of the involution $\theta: \mathfrak{g} \to \mathfrak{g}$.

THEOREM 3.2. *The formula*

$$\{f_1, f_2\}_P = (A(\tilde{\nabla} f_1), \tilde{\nabla} f_2) - (A(\nabla f_1), \nabla f_2) + (\theta A(\nabla f_1), \tilde{\nabla} f_2) - (\theta A(\tilde{\nabla} f_1), \nabla f_2) \tag{11}$$

defines a Poisson bracket on P, such that the following diagram is a commuting diagram of Poisson maps

$$\begin{tikzcd} G \arrow[r, "\hat{\iota}"] \arrow[d, "\pi"'] & P \\ G/K \arrow[ur, "\iota"'] & \end{tikzcd}$$

REMARK 3.1.

(i) Formula (11) is a special case of formula (5) when we let $R = A+S$ with $S = \theta(A+I)$ (use (9) to check that S is symmetric) and hence gives a partial geometric interpretation of this formula;

(ii) In [2] formula (11) is given and for some examples of R-matrices the authors note that (9) is satisfied. However, the geometric interpretation given here is missing.

4 Further developments

There are several directions one might proceed. We mention the following:

- Study Hamiltonian systems on P defined by Hamiltonians that have a central extension to G. These furnish examples of integrable systems (the Toda lattice is one such example). This study was essentially started in [2];

- Find a more intrinsic definition of a Poisson symmetric space. In other words, suppose given a bracket on a symmetric space. When does the space became Poisson symmetric ?

- Classify Poisson symmetric spaces. This is equivalent, in the Riemannian case, to find the skew-symmetric solutions of Yang-Baxter that satisfy (9).

A more detailed version of the results announced here and some other results along these directions will appear elsewhere.

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