A GEOMETRIC APPROACH TO CONN'S LINEARIZATION THEOREM

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Dedicated to Alan Weinstein

ABSTRACT. We give a soft geometric proof of the classical result due to Conn stating that a Poisson structure is linearizable around a singular point (zero) at which the isotropy Lie algebra is compact and semisimple.

INTRODUCTION

Recall that a **Poisson bracket** on a manifold M is a Lie bracket $\{\cdot, \cdot\}$ on the space $C^\infty(M)$ of smooth functions on M, satisfying the derivation property

$$\{fg, h\} = f\{g, h\} + g\{f, h\}, \quad f, g, h \in C^\infty(M).$$

Let us fix a **zero** of the Poisson bracket, i.e., a point $x_0 \in M$ where $\{f, g\}(x_0) = 0$, for all functions $f, g \in C^\infty(M)$. Then $T^*_{x_0} M$ becomes a Lie algebra with the Lie bracket:

$$[\mathrm{d}_{x_0} f, \mathrm{d}_{x_0} g] := \mathrm{d}_{x_0} \{f, g\}.$$

This Lie algebra is called the **isotropy Lie algebra** at x_0 and will be denoted by $\mathfrak{g}_{x_0}$. Equivalently, the tangent space $T_{x_0} M = \mathfrak{g}^*_{x_0}$ carries a canonical linear Poisson bracket called the **linear approximation** at x_0. The *linearization problem* for $(M, \{\cdot, \cdot\})$ around x_0 is the following:

- Is there a Poisson diffeomorphism $\phi : U \to V$ from a neighborhood $U \subset M$ of x_0 to a neighborhood $V \subset T_{x_0} M$ of 0?

When ϕ exists, one says that the Poisson structure is **linearizable** around x_0. The deepest known linearization result is the following theorem due to Conn [4]:

Theorem 1. *Let* $(M, \{\cdot, \cdot\})$ *be a Poisson manifold with a zero* $x_0 \in M$. *If the isotropy Lie algebra* $\mathfrak{g}_{x_0}$ *is semisimple of compact type, then* $\{\cdot, \cdot\}$ *is linearizable around* x_0.

Note that there exists a simple well-known criterion to decide if $\mathfrak{g}_{x_0}$ is semisimple of compact type: its *Killing form* K must be negative definite.

The proof given by Conn in [4] is analytic. He uses a combination of Newton's method with smoothing operators, as devised by Nash and Moser, to construct a converging change of coordinates. This proof is full of difficult estimates and, in spite of several attempts to find a more geometric argument, it is the only one available up to now. See, also, the historical comments at the end of this paper.

In this paper we will give a soft geometric proof of this result using Moser's path method. At the heart of our proof is an integration argument and an averaging argument. The averaging enters into the proof in a similar fashion to the proofs of other linearization theorems, such as Bochner's Linearization Theorem for actions of compact Lie group around fixed points. Our proof gives a new geometric insight to

the theorem, clarifies the compactness assumption, and should also work in various other situations. More precisely, the proof consists of the following four steps:

Step 1: *Moser's path method.* Using a Moser's path method, we prove a *Poisson version* of Moser's theorem (see Theorem 2), which is inspired by the work of Ginzburg and Weinstein [14]. It reduces the proof of Conn's Theorem to showing that the 2nd Poisson cohomology around x_0 vanishes.

Step 2: *Reduction to integrability around a fixed point.* Using the vanishing of cohomology for proper Lie groupoids and the general Van Est theorem relating groupoid and algebroid cohomology [5], we show that it is enough to prove integrability of the Poisson structure around a fixed point x_0.

Step 3: *Reduction to the existence of symplectic realizations.* Using the equivalence of integrability in the Poisson case and the existence of complete symplectic realizations [7], we show that it is enough to construct a symplectic realization of a neighborhood of x_0 with the property that the fiber over x_0 is 1-connected and compact.

Step 4: *Existence of symplectic realizations.* The same path space used in [6] to determine the precise obstructions to integrate a Lie algebroid and to explicitly construct an integrating Lie groupoid, yields that a neighborhood of x_0 admits the desired symplectic realization.

The fact that the tools that we use only became available recently probably explains why it took more than 20 years to find a geometric proof of Conn's Theorem.

The four sections that follow describe each of the steps in the proof. We conclude the paper with two appendices: the first one contains an auxiliary proposition on foliations (which is used in the last step), while in the second one make some historical remarks.

Finally, we would like to mention that our method works in other situations as well. A similar linearization result around symplectic leaves instead of fixed points is being worked out in [17]. The analogue of Conn's Theorem for Lie algebroids (conjectured in [25] and proved in [20]) can also be proved by our method, the only missing step being the proof of the vanishing conjecture of [8] (one must replace the Poisson cohomology of Step 1 by the deformation cohomology of [8]). Details will be given elsewhere. It would also be interesting to find a similar geometric proof of the smooth Levi decomposition theorem of Monnier and Zung [20].

Step 1: Moser's path method

Let us start by recalling that a Poisson bracket $\{\cdot, \cdot\}$ on M can also be viewed as a bivector field $\pi \in \Gamma(\wedge^2 TM)$ with zero Schouten bracket $[\pi, \pi] = 0$. One determines the other through the relation

$$\pi(\mathrm{d}f \wedge \mathrm{d}g) = \{f, g\}, \quad (f, g \in C^\infty(M)).$$

Recall also, that the **Poisson cohomology** of M (with trivial coefficients) is the cohomology of the complex $(\mathfrak{X}^k(M), \mathrm{d}_\pi)$, where $\mathfrak{X}^k(M)$ is the space of k-vector fields, and the differential is defined by

$$\mathrm{d}_\pi \theta := [\pi, \theta].$$

When x_0 is a zero of π, we can consider the **local Poisson cohomology groups** $H_\pi^k(M; x_0)$. By this we mean the Poisson cohomology group of the germ of (M, π) at x_0, i.e., the group $\varinjlim H_\pi^k(U)$ obtained by taking the direct limit of the ordinary Poisson cohomology groups of U, when U runs over the filter of open neighborhoods of x_0.

Theorem 2. *Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold with a zero x_0. Assume that the Lie algebra cohomology groups $H^1(\mathfrak{g}_{x_0})$ and $H^1(\mathfrak{g}_{x_0}, \mathfrak{g}_{x_0})$ vanish. If $H^2_\pi(M; x_0) = 0$, then $\{\cdot, \cdot\}$ is linearizable at x_0.*

For the proof, we will apply a Poisson version of Moser's path method. Since this is a local result, we can assume that $M = \mathbb{R}^n$ and $x_0 = 0$. Also, to simplify the notation we denote by $\mathfrak{g}$ the isotropy Lie algebra at 0. We consider the path of Poisson structures π_t on $\mathbb{R}^n$ defined by the formula

$$\pi_t(x) = \frac{1}{t}\pi(tx), \quad (t \in [0, 1]).$$

Then $\pi_1 = \pi$, while $\pi_0 = \pi_{\text{lin}}$ is the linearization of π at the origin. Moser's method will give us an isotopy $\{\phi_t\}$, $0 \le t \le 1$, defined in a neighborhood of the origin, and such that

$$(\phi_t)_*\pi_t = \pi_{\text{lin}}, \quad (t \in [0, 1]).$$

Therefore ϕ_1 will be the desired linearization map. To construct ϕ_t let us consider the bivector field $\dot{\pi}_t := \frac{d\pi_t}{dt}$.

Lemma 1. *There exists a vector field X around the origin $0 \in \mathbb{R}^n$, whose first jet at 0 vanishes, such that*

(1) $$\mathcal{L}_X\pi = -\dot{\pi}_1,$$

Proof. Differentiating the equation $[\pi_t, \pi_t] = 0$ with respect to t, we obtain

$$d_\pi\dot{\pi}_1 = [\pi, \dot{\pi}_1] = 0,$$

so $\dot{\pi}_1$ is a Poisson 2-cocycle. Hence its restriction to a ball around the origin will be exact, i.e., we find a vector field Y on the ball such that

$$\dot{\pi}_1 = d_\pi Y.$$

This relation has two consequences:

(a) Since $\dot{\pi}_1$ vanishes at 0, if we evaluate both sides on a pair of 1-forms and set $x = 0$, we see that $Y_0([\alpha, \beta]) = 0$, for $\alpha, \beta \in \mathfrak{g}$. Since $H^1(\mathfrak{g}) = 0$ (i.e. $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$), we conclude that $Y_0 = 0$. Let Y_{lin} be the linearization of Y at the origin.

(b) Since $\dot{\pi}_1$ has zero linearization at the origin, the linearization at 0 of our equation becomes

$$d_{\pi_{\text{lin}}}Y_{\text{lin}} = 0.$$

Note that the complex $\mathfrak{X}^\bullet_{\text{lin}}(V)$ of linear multi-vector fields on $V = \mathbb{R}^n$, endowed with $d_{\pi_{\text{lin}}}$, identifies canonically with the Eilenberg-Chevalley complex $C^\bullet(\mathfrak{g}, \mathfrak{g})$ with coefficients in the adjoint representation. Hence, Y_{lin} becomes a 1-cocycle in this complex; since $H^1(\mathfrak{g}, \mathfrak{g}) = 0$, Y_{lin} must be exact, so $Y_{\text{lin}} = d_{\pi_{\text{lin}}}v$ for some $v \in \mathfrak{g} = C^0(\mathfrak{g}, \mathfrak{g}) \cong \mathfrak{X}^0_{\text{lin}}(V)$. The vector field $X = Y - d_\pi v$ has the desired properties. $\square$

Proof of Theorem 2. If X is a vector field as in the Lemma, consider the time-dependent vector field $X_t(x) := \frac{1}{t^2}X(tx)$. From (1) we obtain immediately that

$$\mathcal{L}_{X_t}\pi_t = -\dot{\pi}_t.$$

Let ϕ_t be the flow of X_t. Since $X_t(0) = 0$, we see that ϕ_t is defined in some neighborhood V of the origin for $0 \le t \le 1$. Also, we compute:

$$\frac{d}{dt}(\phi_t)_*\pi_t = (\phi_t)_*\left(\mathcal{L}_{X_t}\pi_t + \frac{d\pi_t}{dt}\right) = 0.$$

We conclude that ϕ_t is a diffeomorphism of V with the desired property. $\square$

STEP 2: REDUCTION TO INTEGRABILITY AROUND A FIXED POINT

In this section we explain the statement and we prove the following proposition which, when combined with Theorem 2, reduces the proof of Conn's Theorem to integrability around a fixed point:

Proposition 1. *Let (M, π) be a Poisson manifold, $x_0 \in M$ a fixed point. If some neighborhood of x_0 is integrable by a Hausdorff Lie groupoid with 1-connected s-fibers, then*

$$H_\pi^2(M, x_0) = 0.$$

More precisely,

(i) *There exist arbitrarily small neighborhoods V of x_0 which are integrable by Hausdorff proper groupoids $\mathcal{G} \rightrightarrows V$ with homological 2-connected fibers.*
(ii) *For any such V, $H_\pi^1(V) = H_\pi^2(V) = 0$.*

The geometric object behind the Poisson brackets which provides the bridge between Poisson geometry and Lie-group type techniques is the **cotangent Lie algebroid** $A = T^*M$ and the associated groupoid $\mathcal{G}(A)$ (see [6, 7]). For a Poisson manifold M we will denote by $\Sigma(M, \pi) = \mathcal{G}(T^*M)$ its associated groupoid. We recall that $\Sigma(M)$ is defined as the set of cotangent paths in M modulo cotangent homotopies, and that it is a *topological groupoid* with 1-simply connected s-fibers. A Poisson manifold M is said to be **integrable** if the associated Lie algebroid T^*M is integrable. This happens iff $\Sigma(M, \pi)$ is a *Lie groupoid*. In this case, $\Sigma(M, \pi)$ carries a natural symplectic structure, that makes it into a **symplectic groupoid**.

Proof of Proposition 1. Let's assume that U is a neighborhood of x_0 which is integrable by an Hausdorff Lie groupoid $\mathcal{G} \rightrightarrows U$. The fiber of the source map $s : \mathcal{G} \to U$ above x_0 is a Lie group integrating $\mathfrak{g}_{x_0}$, so it is compact and 1-connected. Hence, by Reeb stability, there exists a neighborhood V_0 of x_0 such that $s^{-1}(V_0)$ is diffeomorphic to the product $V_0 \times G$. If we let $V = t(s^{-1}(V_0)) \subset U$ be the saturation of V_0, the restriction $\mathcal{G}_V$ of $\mathcal{G}$ to V will be a groupoid whose source map has compact, 1-connected, fibers: using right translations, each fiber will be diffeomorphic to $s^{-1}(x_0) \simeq G$. Moreover, a compact Lie group has the same rational homology type as a product of odd dimensional spheres, so G is automatically homological 2-connected, so the s-fibers are also homological 2-connected.

The proof of the second part is a combination of two classical results on Lie groups which have been extended to Lie groupoids. The first result states that the differentiable cohomology (defined using groups cocycles which are smooth) vanishes for compact groups, and this follows immediately by averaging. This result immediately extends to groupoids, i.e. $H^*_{\mathrm{diff}}(\mathcal{G}) = 0$ for any proper groupoid $\mathcal{G}$ ([5], Proposition 1).

The second result is the Van Est isomorphism. As explained in [5], differentiable group(oid) cocycles can be differentiated and they give rise to Lie algebra(oid) cocycles. The resulting map $\Phi : H^k_{\mathrm{diff}}(\mathcal{G}) \to H^k(A)$, called also the Van Est map, is an isomorphism for degree $k \leq n$ provided the s-fibers of $\mathcal{G}$ are homological n-connected ([5], Theorem 3). Again, the proof is just an extension of the classical proof of Van Est.

If we apply these two results to our groupoid $\mathcal{G} \rightrightarrows V$, the second part of the proposition follows since the Poisson cohomology of V coincides with the Lie algebroid cohomology of $A = T^*V$. $\qquad\qquad\square$

STEP 3: REDUCTION TO THE EXISTENCE OF SYMPLECTIC REALIZATIONS

In the previous step, we have reduced the proof of Conn's Theorem to integrability around a fixed point. The integrability of a Poisson manifold (M, π) is strongly related to the existence of symplectic realizations.

Recall that a symplectic realization of (M, π) consists of a symplectic manifold S together with a Poisson map $\nu : S \to M$ which is a surjective submersion. One calls it complete if for any complete Hamiltonian vector field X_f on M, the vector field $X_{\nu^*(f)}$ is complete. It is known that the existence of complete symplectic realizations is equivalent to integrability (Theorem 8 in [7]), but that depends on subtleties regarding the (required) Hausdorffness conditions on S which are not relevant for us since we are interested on Hausdorff Lie groupoids. Instead, in this paper we do require S to be Hausdorff and we extract from [7] the following result. In the statement we use the following conventions: for a symplectic realization $\nu : S \to M$ we denote by $\mathcal{F}(\nu)$ the foliation of S by the (connected components of the) fibers of ν, and $\mathcal{F}(\nu)^{\perp}$ is its symplectic orthogonal. Also, we recall that a foliation is simple if it is induced by a submersion.

Theorem 3. *A Poisson manifold (M, π) is integrable by a Hausdorff Lie groupoid with 1-connected s-fibers if and only if it admits a complete symplectic realization $\nu : S \to M$ with the property that the foliation $\mathcal{F}(\nu)^{\perp}$ is simple and has simply-connected leaves.*

Proof. One direction is clear: the source map of a Lie groupoid as in the statement provides the desired symplectic integration (the symplectic orthogonals of the s-fibers are the t-fibers). Assume now that $\nu : S \to M$ is a symplectic integration as in the statement. Theorem 8 in [7] insures that $\Sigma = \Sigma(M, \pi)$ is smooth (but possibly non-Hausdorff). A simple remark on the proof of the cited theorem implies that, under our hypothesis, Σ is actually Hausdorff. Recall the main steps of the proof : the assignment $X_f \mapsto X_{\nu^*(f)}$ induces an action of the Lie algebroid T^*M on S which integrates to an action of the Lie groupoid Σ on S and that the associated action groupoid is homeomorphic to the monodromy groupoid of $\mathcal{F}(\nu)^{\perp}$, which we denote by $\mathcal{G}(\mathcal{F}^{\perp})$. In other words, we have

$$\Sigma \times_M S \cong \mathcal{G}(\mathcal{F}^{\perp}).$$

where the fibered product is over s and ν. Since the right hand side is smooth, it follows easily [7] that Σ is smooth as well and the previous homeomorphism is a diffeomorphism. Finally, note that $\mathcal{F}^{\perp}$ is induced by a submersion $\pi : S \to B$, for some manifold B, and that its leaves are simply connected. Therefore, we see that $\mathcal{G}(\mathcal{F}^{\perp}) = S \times_B S$ is Hausdorff. We conclude that Σ is Hausdorff as well. $\square$

Remark 1. The proof actually shows that the conditions on $\mathcal{F}^{\perp}$ can be replaced by the fact that it has no vanishing cycles.

The following corollary reduces the proof of Conn's Theorem to the existence of symplectic realizations around a fixed point:

Corollary 1. *Let (M, π) be a Poisson manifold, $x_0 \in M$ a fixed point and assume that a neighborhood U of x_0 admits a symplectic realization $\nu : S \to U$ with the property that $\nu^{-1}(x_0)$ is simply connected and compact. Then there exists a neighborhood of x_0 which is integrable by a Hausdorff Lie groupoid with 1-connected s-fibers.*

Proof. Note that $\nu^{-1}(x_0)$ is a Lagrangian submanifold of S. Therefore, $\nu^{-1}(x_0)$ is a compact, 1-connected, leaf of $\mathcal{F}^{\perp}(\nu)$. By Reeb stability, nearby leaves are compact, 1-connected and $\mathcal{F}^{\perp}(\nu)$ is simple. Hence we can apply Theorem 3. $\square$

STEP 4: EXISTENCE OF SYMPLECTIC REALIZATIONS

The proof of Conn's Theorem can now be concluded by proving:

Theorem 4. *Let (M, π) be a Poisson manifold, $x_0 \in M$ a fixed point, and assume that the isotropy Lie algebra $\mathfrak{g}$ at x_0 is semi-simple of compact type, with associated simply connected Lie group G. Then there exists a symplectic realization $\nu : S \to U$ of some open neighborhood U of x_0 such that $\nu^{-1}(x_0) = G$.*

We first recall some of the general properties of $\Sigma(M)$ (see [6]). To construct it as a topological space and possibly as a smooth manifold (in the integrable case), we consider the Banach manifold $P(T^*M)$ consisting of paths $a : I \to T^*M$ of class C^2, with the topology of uniform convergence of a map together with its derivatives. Inside this Banach manifold we have the space of cotangent paths:

$$X := \{a \in P(T^*M) : \pi^\sharp(a(t)) = \frac{\mathrm{d}}{\mathrm{d}t}p(a(t))\},$$

where $p : T^*M \to M$ is the bundle projection. Then X is a submanifold of $P(T^*M)$ which carries a canonical foliation $\mathcal{F}$: two cotangent paths a_0 and a_1 belong to the same leaf if they are cotangent homotopic. This foliation has finite codimension and leaf space precisely $\Sigma(M)$. Concatenation of paths, makes $\Sigma(M)$ into a topological groupoid which is smooth precisely when M is integrable.

The symplectic structure on $\Sigma(M)$ is a consequence of the following general property: the restriction of the canonical symplectic form of $P(T^*M) \simeq T^*P(M)$ to X has kernel $\mathcal{F}$ and is invariant under the holonomy of $\mathcal{F}$. We conclude, also, that any transversal to $\mathcal{F}$ carries a symplectic structure invariant under the (induced) holonomy action. Therefore, the quotient of such a transversal by the holonomy action gives a symplectic manifold, provided the quotient is smooth. Unfortunately, achieving smoothness is difficult (and it would imply integrability directly). Instead, we will perform a quotient modulo only *some* holonomy transformations, so that the result is smooth, and we will see that this is enough for our purposes.

Proof. First of all, we consider the source map $s : X \to M$ which sends a cotangent path $a(t)$ to its initial base point $p(a(0))$. This is a smooth submersion, and we look at the fiber $Y = s^{-1}(x_0)$. Since x_0 is a zero of π, Y is saturated by leaves of $\mathcal{F}$ and we set $\mathcal{F}_Y = \mathcal{F}|_Y$. The quotient $G = Y/\mathcal{F}_Y$ is the 1-connected Lie group integrating the isotropy Lie algebra $\mathfrak{g}_{x_0}$, so it is compact. Moreover, note that we can canonically identify Y with paths in the Lie group G which start at the origin, so that the quotient map $Y \to Y/\mathcal{F}_Y = G$ sends a path to its end point. Also, two points in Y belong to the same leaf of $\mathcal{F}_Y$ if the corresponding paths are homotopic relative to the end points. Since the first and second homotopy groups of G vanish, the leaves of $\mathcal{F}_Y$ are 1-connected fibers of a locally trivial fibration $Y \to G$ with compact base. For the local triviality: using right translations by contracting homotopies, one finds even that the restriction of our bundle to any contractible open is trivializable. By the proposition one can find:

(i) a transversal $T_X \subset X$ to the foliation $\mathcal{F}$ such that $T_Y := Y \cap T_X$ is a complete transversal to $\mathcal{F}_Y$ (i.e., intersects each leaf of $\mathcal{F}_Y$ at least once).

(ii) a retraction $r : T_X \to T_Y$.

(iii) an action of the holonomy of $\mathcal{F}_Y$ on $r : T_X \to T_Y$ along $\mathcal{F}$.

Moreover, the orbit space $S := T_X / \mathrm{Hol}_{T_Y}(\mathcal{F}_Y)$ is a smooth (Hausdorff) manifold. Notice that the source map induces a map $\nu : S \to U$, where U is an open neighborhood of x_0. Also, $\nu^{-1}(x_0) = Y/\mathcal{F}_Y = G$ is compact. It follows that S carries a symplectic form and that $\nu : S \to U$ is a Poisson map, so it satisfies all the properties in the statement of the theorem. $\qquad \square$

The aim of this section is to prove the proposition below which was used in the proof of Theorem 4. This proposition concerns the "transversal geometry" of a foliation of finite codimension, so in spite of being a result about foliations of Banach manifolds it is essentially of a finite dimensional nature. We follow the approach first proposed by Haefliger which consists of of giving a meaning to "transversal geometry" by using the language of étale groupoids (the holonomy groupoid of the foliation restricted to a complete transversal).

Proposition 2. *Let $\mathcal{F}$ be a foliation of finite codimension on a Banach manifold X and let $Y \subset X$ be a submanifold which is saturated with respect to $\mathcal{F}$ (i.e., each leaf of $\mathcal{F}$ which hits Y is contained in Y). Assume that:*

(H0) *The holonomy of $\mathcal{F}$ at all points in Y is trivial.*
(H1) *The foliation $\mathcal{F}_Y := \mathcal{F}|_Y$ is simple, i.e. its leaves are the fibers of a submersion $p : Y \to B$ into a compact manifold B.*
(H2) *The fibration $p : Y \to B$ is locally trivial.*

Then one can find:

(i) *a transversal $T_X \subset X$ to the foliation $\mathcal{F}$ such that $T_Y := Y \cap T_X$ is a complete transversal to $\mathcal{F}_Y$ (i.e., intersects each leaf of $\mathcal{F}_Y$ at least once).*
(ii) *a retraction $r : T_X \to T_Y$.*
(iii) *an action of the holonomy of $\mathcal{F}_Y$ on $r : T_X \to T_Y$ along $\mathcal{F}$.*

Moreover, the orbit space $T_X / \mathrm{Hol}_{T_Y}(\mathcal{F}_Y)$ is a smooth (Hausdorff) manifold.

Remark 2. The proposition states that under some conditions one can lift germ-wise actions to actual actions. A similar, more familiar, situation occurs with group actions: given a discrete group Γ and a group homomorphism from Γ to the group of germs of diffeomorphisms of some Euclidean space $\mathbb{R}^k$, preserving the origin, then one can promote this germ-wise action to an actual action of Γ on some open neighborhood of the origin. The proposition can be seen as an instance of a more general phenomena of this type, where one considers actions of (proper) étale groupoids instead of (finite) discrete ones. Even in the case of finite groups, although the proof is straightforward, the details are somewhat tedious.

Remark 3. In this proposition, by an action of the holonomy of $\mathcal{F}_Y$ on $r : T_X \to T_Y$ we mean an action of the holonomy groupoid of $\mathcal{F}_Y$ restricted to T_Y, denoted $\mathrm{Hol}_{T_Y}(\mathcal{F}_Y)$, on the map $r : T_X \to T_Y$ (recall that groupoids act on smooth maps over the space of units). Also, when we say "along the leaves of $\mathcal{F}$" we mean that the orbits of the action lie inside the leaves of $\mathcal{F}$.

In the situation described by the proposition, $\mathcal{F}_Y$ is simple and the action can be made more explicit in the following way. The action is given by a smooth family of diffeomorphisms $h_{x,y} : r^{-1}(x) \to r^{-1}(y)$ defined for any $x, y \in T_Y$ with $p(x) = p(y)$, satisfying $h_{y,z} \circ h_{x,y} = h_{x,z}$ and $h_{x,x} = I$. Also, the action being along the leaves of $\mathcal{F}$ means that $h_{x,y}(u)$ and u are in the same leaf of $\mathcal{F}$, for any $u \in r^{-1}(x)$.

Remark 4. In a preliminary version of this paper, we stated the proposition without (H0) and requiring instead in condition (H1) the leaves of $(Y, \mathcal{F}_Y)$ to be simply connected. This would be enough for the application we have in mind since the fibers of our fibration p are 1-connected, a consequence of the fact that $\pi_2(G) = 0$ for any Lie group G. This improved version resulted from a question of Ezra Getzler, who asked if one could use the much simpler fact that $H^2(G) = 0$ (for a proof of this, see e.g. [12]). Indeed, the condition $H^2(G) = 0$ does imply (H0) for our fibration.

We now turn to the (straightforward but tedious) detailed proof of Proposition 2. We will consider cross-sections of the fibration $p : Y \to B$ whose fibers are the

leaves of $\mathcal{F}_Y$. A cross-section $\sigma : U \to Y$, defined over an open set $U \subset B$, can be identified with its image $\sigma(U) \subset Y$, which is a transversal to $\mathcal{F}_Y$.

Given a cross-section $\sigma : U \to Y$, by a **transversal tubular neighborhood** of σ we mean:

$$
\begin{array}{ccc}
E & \xrightarrow{\quad\tilde{\sigma}\quad} & X \\
& \searrow{\scriptstyle r} \quad \nearrow{\scriptstyle \sigma} & \\
& U &
\end{array}
$$

where $r : E \to U$ is a vector bundle and $\tilde{\sigma} : E \to X$ is an embedding defining a tubular neighborhood of $\sigma(U)$ in some transversal T to $\mathcal{F}$ containing $\sigma(U)$. Hence $\tilde{\sigma}|_U = \sigma$ and $\tilde{\sigma}(E)$ is an open subset of T. We will assume that the vector bundle comes equipped with a norm $||\ ||$. The proof of existence of transversal tubular neighborhoods can be found in [16]. Similarly, one can talk about a **transversal partial tubular neighborhood** of σ (*loc. cit.* pp. 109): in this case $\tilde{\sigma}$ is only defined on an open neighborhood of the zero-section in E. Any such transversal partial neighborhood contains a transversal tubular neighborhood (*loc. cit.*). By abuse of notation we write $\tilde{\sigma} : E \to T$ for a transversal partial tubular neighborhood, even if it is only defined in a open neighborhood of the zero section in E. Also, we have the following extension property which follows from general properties of tubular neighborhoods (see, e.g., Exercise 3, pp. 118 in [16]).

Lemma 2. *Let $\sigma : U \to Y$ be a cross-section, V and W opens in U such that $\overline{V} \subset W \subset \overline{W} \subset U$ (where the closures are in B). Let also T be a transversal to $\mathcal{F}$ containing $\sigma(U)$. Assume that $\tilde{\sigma}_W : E_W \to T$ is a transversal tubular neighborhood of $\sigma|_W$. Then there exists a transversal tubular neighborhood $\tilde{\sigma} : E \to T$ of σ, defined on some vector bundle E over U, such that $E_W|_V = E|_V$ (as vector bundles) and $\tilde{\sigma} = \tilde{\sigma}_W$ on $E|_V$.*

A **homotopy of two cross-sections** $\sigma_0, \sigma_1 : U \to Y$ defined over the same open set $U \subset B$ is a smooth family $\{\sigma_t : t \in [0,1]\}$ of cross sections over U connecting σ_0 and σ_1. Since the fibration $p : Y \to B$ is locally trivial with connected fibers it follows that any two cross-sections over a small enough contractible open set are homotopic.

Let $\sigma = \{\sigma_t\}$ be a homotopy between two cross-sections $\sigma_0, \sigma_1 : U \to Y$. Two transversal partial tubular neighborhoods $\tilde{\sigma}_i : E \to X$ of σ_i ($i \in \{0,1\}$) are said to be σ-**compatible** if the map

$$ \tilde{\sigma}_0(e) \overset{h}{\longmapsto} \tilde{\sigma}_1(e) $$

(defined for $e \in E$ in the intersection of the domains of $\tilde{\sigma}_i$) has the following properties:

(a) x and $h(x)$ are in the same leaf of $\mathcal{F}$ for all x;
(b) the germ of h at each point $\sigma_0(u)$, where $u \in U$, coincides with the holonomy germ of the foliation $\mathcal{F}$ along the path $t \mapsto \sigma_t(u)$.

Lemma 3. *Let $\sigma_0, \sigma_1 : U \to Y$ be two cross-sections over an open $U \subset B$ connected by a homotopy $\sigma = \{\sigma_t\}$. Let $\tilde{\sigma}_0 : E \to X$ be a transversal partial tubular neighborhood above σ_0 and let T be a transversal to $\mathcal{F}$ containing $\sigma_1(U)$. Then, for any $V \subset B$ open with $\overline{V} \subset U$, there exists*

  (i) *an open subset $F \subset E|_V$ containing V (so that $\tilde{\sigma}_0|_F$ is a transversal partial tubular neighborhood of $\sigma_0|_V$);*
  (ii) *a transversal partial tubular neighborhood $\tilde{\sigma}_1 : F \to T$ of $\sigma_1|_V$;*

such that $\tilde{\sigma}_0|_F$ and $\tilde{\sigma}_1$ are $\sigma|_V = \{\sigma_t|_V\}$-compatible.

Proof. Fix σ_0, σ_1, $\tilde{\sigma}_0 : E \to X$ and T as in the statement. As a temporary terminology, we say that an open subset $V \subset U$ is *good* if $\overline{V} \subset U$ and the conclusion of the lemma holds for V. An open subset of a good open set is also good.

We first show that any $u \in U$ admits a good open neighborhood. Consider the holonomy transformation along the path $\sigma^u(t) := \sigma(t, u)$ from the transversal $\tilde{\sigma}_0(E)$ to the transversal T. This is the germ of a diffeomorphism h_u, defined in some neighborhood of $\sigma_0(u)$, which can be taken of the form $\tilde{\sigma}_0(F)$ for some open set $F \subset E$ containing u. Choosing F a small enough open ball (relative to $|| \ ||$) and setting $\tilde{\sigma}_1 := h_u \circ \tilde{\sigma}_0|_F$, we conclude that V is a good open set.

Let V be an arbitrary open set with $\overline{V} \subset U$. We can find a cover of $\overline{V}$ by good open sets, so we can extract a finite subcover $\{U_i : 1 \le i \le p\}$ of $\overline{V}$ in which each U_i is good. We prove by induction on p that V must be a good open set. Obviously, the result holds if $p = 1$. For the induction step, assume the assertion is true for $p - 1$ and assume that $\overline{V}$ is covered by p good open sets $U_i \subset U$. Choose another cover $\{V_i\}$ of V with $\overline{V}_i \subset U_i$. Then, by the induction hypothesis, $U_1 = V_1$ and $U_2 := V_2 \cup \ldots \cup V_p$ will be good open sets. Moreover, $\overline{V} \subset U_1 \cup U_2$, so all that remains to show is the case $p = 2$.

Let $U_1, U_2 \subset U$ be good opens sets and $\overline{V} \subset U_1 \cup U_2 \subset U$. We need to show that V is a good open set. Let $F_i \subset E$, $\tilde{\sigma}_i : F_i \to T$ be the associated transversal partial tubular neighborhoods. Consider also the induced maps $h_i : \tilde{\sigma}_0(F_i) \to \tilde{\sigma}_i(F_i)$. We consider two new open sets V_i such that $\overline{V} \subset V_1 \cup V_2$ and $\overline{V}_i \subset U_i$. Compactness of $\overline{V}_i$ shows that we can find $R > 0$ such that:

$$x \in E|_{V_i}, ||x|| < R \Longrightarrow x \in F_i.$$

Due to the properties of h_i (properties (a) and (b) above), we see that h_1 and h_2 coincide in a neighborhood of $\sigma_0(u)$ in $\sigma_0(U)$. Hence, choosing eventually a smaller R, we may assume that

$$x \in E|_{V_1 \cap V_2}, ||x|| < R \Longrightarrow \tilde{\sigma}_1(x) = \tilde{\sigma}_2(x).$$

It follows that $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ will glue on

$$F = \{x \in E_{V_1 \cup V_2} : ||x|| < R\}$$

and the resulting transversal partial tubular neighborhood will have the desired properties so that V is a good open set. $\qquad \square$

For the next lemma, we introduce the following notation. A **$\mathcal{F}$-data** is a tuple

$$(2) \qquad\qquad \{U_i, \sigma_i, \tilde{\sigma}_i, \sigma_{(i,j)}, E : 1 \le i, j \le k\}$$

consisting of the following:

(a) $\{U_i : i = 1, \ldots, k\}$ is a family of open sets in B and E is a vector bundle over $U = U_1 \cup \cdots \cup U_k$.
(b) $\sigma_i : U_i \to Y$ are cross-sections and $\sigma_{(i,j)}$ are homotopies between $\sigma_i|_{U_i \cap U_j}$ and $\sigma_j|_{U_i \cap U_j}$.
(c) $\tilde{\sigma}_i : E|_{U_i} \to X$ are transversal tubular neighborhoods over σ_i such that the restrictions of $\tilde{\sigma}_i$ and $\tilde{\sigma}_j$ to $E|_{U_i \cap U_j}$ are $\sigma_{i,j}$-compatible for all i and j.

Assume now that:

(i) $U_{k+1} \subset B$ is another open set, $\sigma_{k+1} : U_{k+1} \to Y$ is a cross-section above U_{k+1} and T is a transversal to $\mathcal{F}$ containing $\sigma_{k+1}(U_{k+1})$.
(ii) for each $1 \le i \le k$ we have a homotopy $\sigma_{(i,k+1)}$ between $\sigma_i|_{U_i \cap U_{k+1}}$ and $\sigma_{k+1}|_{U_i \cap U_{k+1}}$.

Let $V_i \subset B$ be open sets with

$$\overline{V}_i \subset U_i \quad (1 \le i \le k+1)$$

and set $V = V_1 \cup \ldots V_k$, $V' = V \cup V_{k+1}$. Then:

Lemma 4. *Under the above assumptions, one can find a vector bundle E' over V' together with an embedding $\phi : E'|_V \hookrightarrow E|_V$ of bundles, as well as a map $\tilde{\sigma}'_{k+1} : E'|_{V_{k+1}} \to X$ which is a transversal tubular neighborhood of $\sigma_{k+1}|_{V_{k+1}}$ inside T such that*

$$\{V_i, \sigma_i, \tilde{\sigma}_i \circ \phi, \sigma_{(i,j)}|_{V_i \cap V_j}, E' : 1 \leq i, j \leq k+1\}$$

is a $\mathcal{F}$-data.

Proof. For $1 \leq i \leq k+1$, choose open sets $\overline{V}_i \subset V'_i \subset \overline{V}'_i \subset U_i$. We can apply Lemma 3 to:

- the restrictions of σ_i and σ_{k+1} to $U_i \cap U_{k+1}$ and the homotopy $\sigma_{(i,k+1)}$.
- the transversal tubular neighborhood to $\sigma_i|_{U_i \cap U_{k+1}}$ which is the restriction of $\tilde{\sigma}_i$ to $E|_{U_i \cap U_{k+1}}$.
- the open set $V'_i \cap V'_{k+1}$ whose closure is inside $U_i \cap U_{k+1}$.

It gives a transversal tubular neighborhood of $\sigma_{k+1}|_{V'_i \cap V'_{k+1}}$, denoted

$$\tilde{\sigma}^{(i)}_{k+1} : E'_{i,k+1} \to T$$

defined on some $E'_{i,k+1} \subset E|_{V'_i \cap V'_{k+1}}$, an open set containing $V'_i \cap V'_{k+1}$.

Now choose open sets $\overline{V}_i \subset V''_i \subset \overline{V}''_i \subset V'_i$. Since the closure of $V''_i \cap V''_{k+1}$ is compact, we find $R_i > 0$ such that

$$x \in E|_{V''_i \cap V''_{k+1}}, \|x\| < R_i \Longrightarrow x \in E'_{i,k+1}.$$

Next, for each $1 \leq i, j \leq k$, the restrictions of $\tilde{\sigma}^{(i)}_{k+1}$ and $\tilde{\sigma}^{(j)}_{k+1}$ to $E'_{i,k+1} \cap E'_{j,k+1}$ are transversal partial tubular neighborhoods above the same cross-section $\sigma_{k+1}|_{V'_i \cap V'_j}$. Moreover, they are σ-compatible, where σ is the concatenation of the homotopies $\sigma_{(i,k+1)}$, $\sigma_{(j,i)}$ and $\sigma_{(k+1,j)}$. Since all paths $\sigma^u(-) = \sigma(u,-)$ induced by the homotopy σ are inside leaves of $\mathcal{F}_Y$ and the holonomy of $\mathcal{F}$ along loops inside Y is trivial (the first assumption in the statement), the holonomy germs induced by the closed loops $\sigma^u_{(k+1,j)} \circ \sigma^u_{(j,i)} \circ \sigma^u_{(i,k+1)}$ are trivial. We conclude that

$$\{x \in E'_{i,k+1} \cap E'_{j,k+1} : \tilde{\sigma}^{(i)}_{k+1}(x) = \tilde{\sigma}^{(j)}_{k+1}(x)\}$$

contains an open subset in $E'_{i,k+1} \cap E'_{j,k+1}$ containing $V'_i \cap V'_j \cap V'_{k+1}$. Again, we can find constants $R_{i,j}$ such that:

$$x \in E'_{i,k+1}|_{V''_i \cap V''_j \cap V''_{k+1}} \cap E'_{j,k+1}|_{V''_i \cap V''_j \cap V''_{k+1}}, \|x\| < R_{i,j} \Longrightarrow \tilde{\sigma}^{(i)}_{k+1}(x) = \tilde{\sigma}^{(j)}_{k+1}(x).$$

Let us set:

$$R = \min\{R_i, R_{i,j} : 1 \leq i, j \leq k\}, \quad E'' = \{x \in E : \|x\| < R\}.$$

The maps $\tilde{\sigma}^i_{k+1}$ glue together to give a smooth map defined on $E''|_{V'' \cap V''_{k+1}}$ where

$$V'' = V''_1 \cup \ldots \cup V''_k.$$

Denote this map by $\tilde{\sigma}''_{k+1}$. Consider now $\lambda : [0,\infty) \to [0,1)$ be a diffeomorphism equal to the identity near 0 and define the embedding

$$\phi : E \to E'', h(x) = R \frac{\lambda(\|x\|)}{\|x\|} x.$$

Composing with this embedding, we obtain

$$\tilde{\sigma}''_{k+1} : E|_{V''_{k+1} \cap V''} \to T$$

which is a transversal tubular neighborhood of $\sigma_{k+1}|_{V''_{k+1} \cap V''}$. We can now apply Lemma 2 to find:

(i) a vector bundle E_{k+1} over V_{k+1} such that $E_{k+1}|_{V_{k+1} \cap V} = E|_{V_{k+1} \cap V}$.

(ii) a transversal tubular neighborhood $\tilde{\sigma}'_{k+1}$ of $\sigma_{k+1}|_{V_{k+1}}$ defined on the entire E_{k+1} and which coincides with $\tilde{\sigma}''_{k+1}$ on $E|_{V_{k+1} \cap V}$.

Finally, if we let E' be the vector bundle over $V' = V \cup V_{k+1}$ obtained by gluing $E|_V$ (over V) and E_{k+1} (over V_{k+1}), we have obtained the desired $\mathcal{F}$-data. $\qquad \square$

Proof of Proposition 2. Let $\mathcal{U}^{(1)} = \{U_1^{(1)}, \dots, U_n^{(1)}\}$ be a finite good cover of B in the sense of [2]. Since B is compact, good covers exist- actually, any cover can be refined by a finite good cover (see *loc. cit.*). Hence, refining covers consisting of opens over which $p : Y \longrightarrow B$ is trivial, we may assume that there are cross-sections $\sigma_i : U_i^{(1)} \to Y$, with the image of each σ_i is inside some transversal T_i of $\mathcal{F}$. Since all intersections $U_i^{(1)} \cap U_j^{(1)}$ are contractible and the fibers of $p : Y \to B$ are connected, there are homotopies $\sigma_{(i,j)}$ between $\sigma_i|_{U_i \cap U_j}$ and $\sigma_j|_{U_i \cap U_j}$. Finally, we choose new good covers $\mathcal{U}^{(k)} = \{U_1^{(k)}, \dots, U_n^{(k)}\}$, $k = 1, \dots, n$, with the property

$$\overline{U}_i^{(k+1)} \subset U_i^{(k)} \quad (i, k = 1, \dots, n).$$

We then apply inductively Lemma 4: at each step one gets a vector bundle over $U_1^{(k)} \cup \dots \cup U_k^{(k)}$ and an $\mathcal{F}$-data. For $k = n$, we obtain a vector bundle over B, a complete transversal to $\mathcal{F}_Y$ (the images of the $U_i^{(n)}$'s by the cross sections) and the transversal to $\mathcal{F}$ (the transversal tubular neighborhoods of the final $\mathcal{F}$-data).

It only remains to show that $T_X / \mathrm{Hol}_{T_Y}(\mathcal{F}_Y)$ is Hausdorff manifold. This can be checked directly (for instance, if the starting transversals T_i are chosen so that their closures are disjoint, then the quotient is just the resulting vector bundle E over B). Here we indicate a more conceptual argument which is based on general properties of groupoids and their representations (in the sense of spaces on which the groupoid act). For Morita equivalences, we refer to [18]. First of all, representations can be transported along Morita equivalences and, provided the groupoids and the Morita equivalences used are Hausdorff, the Hausdorff property of representations is preserved by this transport. Secondly, since $\mathcal{F}_Y$ is induced by the submersion $p : Y \to B$, the groupoid $\mathrm{Hol}_{T_Y}(\mathcal{F}_Y) \rightrightarrows T_Y$ is Morita equivalent to the trivial groupoid $B \rightrightarrows B$, via the bimodule $T_Y \xleftarrow{\mathrm{id}} T_Y \xrightarrow{p} B$. Finally, one just remarks that under this equivalence, $T_X / \mathrm{Hol}_{T_Y}(\mathcal{F}_Y)$ is the representation of $B \rightrightarrows B$ which corresponds to the representation T_X of $\mathrm{Hol}_{T_Y}(\mathcal{F}_Y)$. $\qquad \square$

APPENDIX 2: HISTORICAL REMARKS

The study of the linearization problem for Poisson brackets was initiated by Alan Weinstein in the foundational paper [23]. There, he states the problem and he shows that the formal linearization problem can be reduced to a cohomology obstruction. If the isotropy Lie algebra is semisimple this obstruction vanishes. For analytic linearization he conjectured that, provided the isotropy Lie algebra is semisimple, this can always be achieved, a result later proved by Conn [3].

In [23], Alan Weinstein also considers the smooth linearization problem. He gives an example of a smooth, non-linearizable, Poisson bracket with isotropy Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. The situation is remarkable similar to the case of Lie algebra actions, and this counter-example is analogous to the example of a non-linearizable smooth action of $\mathfrak{sl}(2, \mathbb{R})$, given by Guillemin and Sternberg in [13]. By contrast, he suggests that linearization when the isotropy is $\mathfrak{so}(3)$ could be proved as follows (see [23], page 539):

> *The first step would be to use the theorems of Reeb and Moussu to "linearize" the foliation by symplectic areas. Next, a "volume preserving Morse lemma" would be used to put in standard form the*

> *function which measures the symplectic area of the leaves. Finally,*
> *the deformation method of Moser and Weinstein would have to be*
> *applied to each symplectic leaf, with care taken to assure regularity*
> *at the origin.*

The proof sketched was actually implemented by Dazord in [9]; it is less known that this result and a proof of it can be traced back to the thesis of Reeb [21] (but in the dual language of completely integrable Pfaffian forms- which, in dimension 3, correspond to bivectors which are Poisson). Weinstein goes on to conjecture that smooth linearization can be achieved for compact semisimple isotropy. This again was proved to be so by Conn in [4].

Conn's proof of smooth linearization is a highly non-trivial analytic argument. He views the effect of changes of coordinates upon the Poisson tensor as a non-linear partial differential operator. A combination of Newton's method with smoothing operators, as devised by J. Nash and J. Moser, is used to construct successive approximations to the desired linearizing coordinates. The linearized equations that need to be solved at each step are non-degenerate and overdetermined (the operator differentiates only along the symplectic foliation). However, by working at the level of Lie algebra cohomology of $\mathfrak{g}$ with coefficients in the space of smooth functions on $\mathfrak{g}^*$, Conn is able to find accurate solutions to the linearized equations. This involves many estimates on the Sobolev norms, which are defined from the the Killing form, and so take advantage of its invariance, non-degeneracy and definiteness.

After Conn's work was completed attention turned to other Lie algebras. In [24], Weinstein showed that semisimple Lie algebras of real rank greater than one are non-linearizable, in general. The case of real rank 1, with the exception of $\mathfrak{sl}(2,\mathbb{R})$, remains open. In [10], Dufour studied linearization when the isotropy belongs to a certain class of Lie algebras, called non-resonant, which allowed him to classify all the 3-dimensional Lie algebras that entail linearizability. Dufour and Zung proved formal and analytic linearization for the Lie algebra of affine transformations $\mathfrak{aff}(n)$ [11]. There are also examples of Poisson structures for which linearization can be decided only from knowledge of its higher order jets (see [1]). More recently, a Levi decomposition for Poisson brackets, generalizing linearization, has been introduced by Wade ([22], formal category), Zung ([27], analytic category) and Zung and Monnier ([20], smooth category). The methods are entirely similar to the ones of Weinstein and Conn. A survey of these results can be found in [19].

In spite of Conn's master work, the question remained if a simple, more geometric, proof of smooth linearization would be possible. In the Introduction of [25], Alan Weinstein writes:

> *Why is it so hard to prove the linearizability of Poisson structures*
> *with semisimple linear part? Conn published proofs about 15 years*
> *ago in a pair of papers full of elaborate estimates (...) no simpli-*
> *fication of Conn's proofs has appeared.*
> *This is a mystery to me, because analogous theorems about lin-*
> *earizability of actions of semisimple groups near their fixed points*
> *were proven (...) using a simple averaging.*

In this paper he goes on to propose to use Lie algebroid/groupoid theory to tackle this and other linearization problems. After this work, it become clear that this would indeed be the proper setup for a geometric proof of linearization. However, his attempt would not be successful because some of the techniques needed were not available yet. Some basic results on proper groupoids, as well as a full understanding of the integrability problem for Poisson manifolds and Lie algebroids was missing, and this was done later by us in [5, 6, 7].

In the end, the geometric proof we have given here, is really a combination of classical results on Lie groups extended to the groupoid context. Once the groupoid is brought into the picture, one has the usual differential geometric machinery at hand, and hence also all the standard techniques to deal with linearization problems one finds in different contexts (Moser trick, Van Est argument, Reeb stability, averaging). It is curious that the methods used are so close to the proof suggested by Alan Weinstein for the case of $\mathfrak{so}(3)$, that we have quoted above. A general setup to discuss linearization problems and its relation to deformation problems will be given elsewhere (work in progress).

Finally, note that it would be possible to combine our Proposition 1 with the linearization theorem for proper groupoids around fixed points (see [26, 28]), to obtain another proof of Conn's theorem (this would be a geometric-analytic proof, since the linearization of proper groupoids also involves some estimates.)

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