COMPLETELY INTEGRABLE BI-HAMILTONIAN SYSTEMS

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Abstract

We study the geometry of completely integrable bi-Hamiltonian systems, and in particular, the existence of a bi-Hamiltonian structure for a completely integrable Hamiltonian system. We show that under some natural hypothesis, such a structure exists in a neighborhood of an invariant torus if, and only if, the graph of the Hamiltonian function is a hypersurface of translation, relative to the affine structure determined by the action variables. This generalizes a result of Brouzet for dimension four.

KEY WORDS: *Bi-Hamiltonian system; completely integrable system.*

0. Introduction

The study of completely integrable Hamiltonian systems, i.e., systems admitting a complete sequence of first integrals, started with the pionneering work of Liouville (1855) on finding local solutions by quadratures. We have now a complete picture of the geometry of such systems, which in its modern presentation is due to Arnol'd (1988). A major flaw in the Arnol'd-Liouville theory is that it provides no indication on how to obtain first integrals, and this is one of the main reasons for the growing interest on bi-Hamiltonian systems (systems admitting two compatible Hamiltonian formulations). For a given bi-Hamiltonian system, a result due to Magri (1978) shows how to construct a whole hierarchy of first integrals. Under an additional assumption, one can show that Magri's theorem yields a complete sequence of first integrals. Moreover, this assumption may be formulated in a way that still makes sense in the setting of infinite dimensional systems. Therefore, if one wants to extend the notion of complete integrability, the following natural question arises: Given a completely integrable Hamiltonian system, does the complete sequence of first integrals arise from a second Hamiltonian structure via Magri's theorem?

This problem was first studied by Magri and Morosi (1984) in a set of unpublished notes, which seems to contain an incorrect answer. More recently, Brouzet in his "thése de doctorat" studied the same question when the dimension equals four, and showed that the answer in general is negative (Brouzet, 1990). Our work might be considered an improvement on their results. Theorem 3.1 below shows that a second Hamiltonian structure exists if and only if the Hamiltonian function satisfies a certain geometric condition. This

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condition is expressed invariantly in terms of the action-angle variables as a restriction on the Hamiltonian, and suggests some important remarks for the study of finite dimensional systems.

A related problem was considered by De Filippo et al. (1984). They show that a completely integrable system always has a bi-Hamiltonian formulation, but we remark that, in general, neither of the Poisson structures coincides with the given one. The situation we are considering here is more restrictive. We fix the original Hamiltonian structure, and ask under what conditions there exists a second Hamiltonian structure such that the sequence of first integrals is obtained via Magri's result.

This paper is organized as follows. In section 1, we review the Arnol'd-Liouville theory of completely integrable systems, and we recall the basic facts concerning bi-Hamiltonian systems. In section 2, starting with a bi-Hamiltonian system for which Magri's theorem gives a complete sequence of first integrals, we construct a set of coordinates which have the property of splitting both the Hamiltonian and the action variables. In section 3, we interpret this property as a geometric condition on the graph of the Hamiltonian, and show how it can be used to rule out the existence of a second Hamiltonian structure. Conversely, we show that any completely integrable system whose Hamiltonian satisfies this condition has a bi-Hamiltonian formulation. In section 4, we give several applications of these results. In particular, we give an example of a completely integrable Hamiltonian system, which is not bi-Hamiltonian in the sense just described, but admits a degenerate bi-Hamiltonian structure. This explains why a bi-Hamiltonian formulation is not known for many of the classical examples of completely integrable systems: in general, degenerate Poisson pairs will be required.

1. Magri's Theorem

In this section we recall, without proof, some basic facts from the theory of bi-Hamiltonian systems and completely integrable systems.

A **Poisson structure** on a manifold M is a 2-vector Λ, i.e., a section of the second exterior bundle $\bigwedge^2 T(M)$, satisfying the closure condition

$$\left[\Lambda, \Lambda\right] = 0, \tag{1.1}$$

where [,] denotes the Schouten bracket. This is equivalent to defining a Lie algebra structure on $C^\infty(M)$, whose bracket $\{,\}_\Lambda$, called the **Poisson bracket**, determines for each fixed $H \in C^\infty(M)$ a derivation $G \to \{H, G\}_\Lambda$ of $C^\infty(M)$. The relation between the two definitions is given by the formula

$$\{H, G\}_\Lambda = \Lambda(dH, dG),$$

and condition (1.1) is just a restatement of the Jacobi identity. The pair (M, Λ) is called a **Poisson Manifold**. This definition is due to Lichnerowitz (1977).

A Poisson structure determines in a natural way a vector bundle morphism $J : T^*(M) \to T(M)$, so that to each smooth function $H \in C^\infty(M)$ there is associated a vector field $X_H(x) = J(x)dH(x)$. One calls H a **Hamiltonian function**, X_H the associated **Hamiltonian vector field** and the triple (M, Λ, H) a **Hamiltonian system**. The integral curves of X_H are the solutions of Hamilton's equations of motion

$$\dot{x} = JdH.$$

At each $x \in M$ the rank of the linear transformation $J(x)$ defines the **rank** of the Poisson structure Λ. If the rank is everywhere constant and equal to the dimension of M, then *dim* M is even and J is a vector bundle isomorphism . In this case we can associate to Λ, in a natural way, a non-degenerate, closed, 2-form ω, and we have the usual approach to classical mechanics through symplectic geometry (Abraham and Marsden, 1978), (Arnol'd, 1988). More generally, one can show that a Poisson manifold decomposes into a foliation by symplectic manifolds, and therefore is a natural setting for the study of families of mechanical systems. For details on these constructions see Weinstein (1983).

Given a Hamiltonian system (M^{2n}, ω, H) on a symplectic manifold, we say that the system is **completely integrable** if there exists n independent functions $I_1 = H, I_2, \ldots, I_n$, with pairwise vanishing Poisson brackets. The geometry of completely integrable systems is described by the so-called Arnol'd-Liouville theorem.

Theorem 1.1. *(Arnol'd, 1988) Let $\pi : M \to \mathbb{R}^n$ be the fibration $x \to (I_1(x), \ldots, I_n(x))$. Then:*

(i) π is a Lagrangian fibration and each connected component of $\pi^{-1}(c)$ is a cylinder. There exists affine coordinates $(\theta^1, \ldots, \theta^n)$ on $\pi^{-1}(c)$ which straightens out the Hamiltonian flow, i.e., $\dot{\theta}^i = const.$

(ii) If $\pi^{-1}(c)$ is connected and compact then it is a topological torus $\mathbb{T}^n$. There exists a neighborhood U of $\pi^{-1}(c)$ and a trivialization $(s^1, \ldots, s^n, \theta^1, \ldots, \theta^n) : U \to \mathbb{R}^n \times \mathbb{T}^n$ such that $\omega = \sum_i ds^i \wedge d\theta^i$.

The variables (s^i, θ^i) are the **action-angle variables**. Explicit formulas for the action variables can be obtained as follows. Since each torus $\pi^{-1}(c)$ is a Lagrangian submanifold, on some neighborhood U of this level set the symplectic form is exact: $\omega = d\alpha$. The action variables are obtain by integration along a basis $(\gamma_1, \ldots, \gamma_n)$ of 1-cycles of the torus:

$$s_i = \frac{1}{2\pi} \oint_{\gamma_i} \alpha.$$

The action variables define a canonical integral affine structure, i.e., they are unique up to translations and integral linear transformations of $\mathbb{R}^n$ (Arnol'd and Givental, 1988). Note that the Arnol'd-Liouville theory tells nothing about how to find a full set of first integrals, so we turn now to the theory of bi-Hamiltonian systems.

By a **Poisson bi-structure** on a manifold M we mean a compatible pair (Λ_1,Λ_2) of Poisson structures on M, i.e., Λ_1 and Λ_2 are 2-vectors on M such that

$$[\Lambda_1, \Lambda_1] = [\Lambda_2, \Lambda_2] = 0 \tag{1.2}$$

$$[\Lambda_1, \Lambda_2] = 0. \tag{1.3}$$

A **bi-Hamiltonian system** is prescribed by specifying two Hamiltonian functions H_1, $H_2 \in C^\infty(M)$ satisfying

$$X = J_1 dH_1 = J_2 dH_2, \tag{1.4}$$

where J_i, i=1,2, are the bundle maps determined by the Λ_i. The vector field X is said to be a **bi-Hamiltonian vector field**. We shall only consider **non-degenerate** pairs (Λ_1,Λ_2) where at least one of the Poisson structures is symplectic. The treatment of degenerate cases is a much more difficult task of which little is known (see Oevel and Ragnisco, 1989). Let us assume that Λ_1 is symplectic, then we define the **recursion operator** associated with the pair (Λ_1,Λ_2) to be the (1,1)-tensor N given by[2]:

$$\Lambda_2(\alpha, \cdot) = N\Lambda_1(\alpha, \cdot), \qquad \forall \alpha \in \Omega^1(M).$$

Recall that for any (1,1)-tensor N its Nijenhuis torsion T_N, is the (1,2)-tensor[2]

$$T_N(X,Y) \equiv L_{NX}(NY) - NL_{NX}(Y) - NL_X(NY) + N^2 L_X(Y), \quad \forall X, Y \in \mathfrak{X}(M) \tag{1.5}$$

Using the compatibility condition (1.3) one obtains:

Theorem 1.2. *(Magri and Morosi, 1984)*

(i) The Nijenhuis torsion of the recursion operator N associated with a Poisson bi-structure (Λ_1,Λ_2) vanishes.

(ii) For any bi-Hamiltonian vector field $X \in \mathfrak{X}(M)$, the eigenvalues of the recursion operator N form a commuting family of first integrals with respect to both brackets.

The proof of this result (cf. Magri and Morosi, 1984), is based on the recursion relations

$$J_1 dI_{k+1} = J_2 dI_k, \quad k = 1, 2, \ldots,$$

where $I_k \equiv \frac{1}{k} tr\ N^k$. The I_k's are also first integrals, so we conclude that there is a family of first integrals which are solutions of the systems of first order p.d.e.'s $dF = N^* dG$. In the next section, we will use the restrictions imposed by this p.d.e. to study the converse problem, concerning the existence of a bi-Hamiltonian formulation for a completely integrable system. This p.d.e. was first studied by Olver (1990), who used it to give a local classification of bi-Hamiltonian systems.

A first, rather trivial remark, is the following: any completely integrable Hamiltonian system X_H is bi-Hamiltonian in a neighborhood of an invariant torus (De Filippo et al.,1984). The point is that in such a neighborhood we can find coordinates, say $(x^1, \ldots, x^{2n})$, where $X_H = \frac{\partial}{\partial x^1}$, so it is easy to construct some bi-Hamiltonian formulation for X_H. However, in general this coordinates will not be canonical, and this artificially constructed Poisson pair has no direct relationship with the original Poisson structure. In practice, some natural

[2] The symbols $\mathfrak{X}(M)$ and $\Omega^1(M)$ denote, resp., the spaces of vector fields and differential 1-forms on M. Also L will denote the Lie derivative.

Poisson structure is known and one seeks a second one that might give the integrability of the system. This is the problem we consider now.

2. Splitting Variables for Completely Integrable bi-Hamiltonian Systems

In this section we look at a completely integrable Hamiltonian system (M^{2n}, ω, H) and ask if the complete sequence of integrals arises from a second Poisson structure via Magri's theorem (recall that a symplectic form determines a canonical Poisson structure).

The local problem is more or less trivial and is not so interesting from the point of view of the theory of integrability. We shall look rather at neighborhood of a fixed invariant torus. By the Arnold-Liouville theorem a tubular neighborhood of the torus can be described by choosing action-angle variables (s^i, θ^i), so without loss of generality we can assume that M^{2n} is a product $\mathbb{R}^n \times \mathbb{T}^n$, where the original torus is identified with $\{0\} \times \mathbb{T}^n$ and $H = H(s^1, \dots, s^n)$, $\omega = \sum_i ds^i \wedge d\theta^i$. The canonical projection $\pi : M \to \mathbb{R}^n$ is a Lagrangian fibration, so each torus $\pi^{-1}(x)$ is a Lagrangian submanifold and $Ker\ d_m\pi = (Ker\ d_m\pi)^\perp$. We make the following assumption.

(ND). *det* $(\partial^2 H/\partial s^i \partial s^j) \neq 0$ *in a dense set.*

This non degeneracy condition will be used most often in the following form: any first integral depends only on the action variables. In fact, (ND) implies that the non-resonant tori are dense in a neighborhood of $\{0\} \times \mathbb{T}^n$, but any first integral is constant on any such torus.

We want to investigate the existence of a second Poisson structure, possibly degenerate, giving the complete integrability of the system. Thus we consider an additional assumption:

(BH). *The system is bi-Hamiltonian with diagonalizable recursion operator N, having functionally independent real eigenvalues $\lambda_1, \dots, \lambda_n$*[3]

By Magri's theorem the sequence $\lambda_1, \dots, \lambda_n$ is a complete sequence of first integrals of the system.

Given a point $m \in M$ we denote by $E_\lambda(m)$ the (real) eigenspace of the recursion operator N belonging to a eigenvalue λ in the spectrum $\sigma(N) = \{\lambda_1, \dots, \lambda_n\}$. For diagonalizable (1,1)-tensors with vanishing Nijenhuis torsion we have the following classical result (Nijenhuis, 1951):

Proposition 2.1. *For any subset $S \subset \sigma(N)$ the distribution $m \to \oplus_{\lambda \in S} E_\lambda(m)$ is integrable. If $\mu \in \sigma(N) \backslash S$, then μ is an integral of the distribution.*

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[3] *It is possible to relax this condition by assuming only distinct eigenvalues. All the results that follow still hold in this more general setting. Here we consider only the functionally independent case in view of Magri's theorem and in order to keep technical details to a minimum.*

Proof:

Let X_λ and Y_μ be eigenvectors of N corresponding to eigenvalues λ and μ. From expression (1.5) for the Nijenhuis torsion of N we find:

$$0 = T_N(X_\lambda, Y_\mu) = (N - \lambda I)(N - \mu I)[X_\lambda, Y_\mu] + (\lambda - \mu)\big\{(X_\lambda \cdot \mu)Y_\mu - (Y_\mu \cdot \lambda)X_\lambda\big\} \quad (2.1)$$

If one applies $(N - \lambda I)(N - \mu I)$ to both sides of this equation one gets:

$$[X_\lambda, Y_\mu] \in Ker\ (N - \lambda I)^2(N - \mu I)^2.$$

But N is diagonalizable, so we have

$$\begin{aligned} Ker\ (N - \lambda I)^2(N - \mu I)^2 &= Ker\ (N - \lambda I)^2 \oplus Ker\ (N - \mu I)^2 \\ &= Ker\ (N - \lambda I) \oplus Ker\ (N - \mu I), \end{aligned}$$

and the first part of the proposition follows. If $\lambda \neq \mu$, (2.1) now gives:

$$X_\lambda \cdot \mu = Y_\mu \cdot \lambda = 0,$$

so the second part also follows.

$\square$

For $i = 1, \ldots, n$, we denote the foliations associated with the integrable distributions $m \to E_{\lambda_i}(m)$ and $m \to E_{\lambda_1}(m) \oplus \ldots \oplus \widehat{E_{\lambda_i}(m)} \oplus \ldots \oplus E_{\lambda_n}(m)$ by $\mathcal{F}_i$ and $\mathcal{D}_i$, respectively (here $\hat{E}$ means omit the factor E). It is obvious from the definitions that $\mathcal{F}_i$ and $\mathcal{D}_i$ are transversal. In fact, we have the following stronger result:

Proposition 2.2. *If $i \neq j$ the foliations $\mathcal{F}_i$ and $\mathcal{F}_j$ are ω-orthogonal. In particular, one has $\mathcal{F}_i^\perp = \mathcal{D}_i$.*

Proof:

If X_λ and X_μ are eigenvectors of N corresponding to distinct eigenvalues λ and μ we find:

$$\begin{aligned} \lambda\omega(X_\lambda, X_\mu) &= \omega(\lambda X_\lambda, X_\mu) \\ &= \omega(N X_\lambda, X_\mu) \\ &= \omega(X_\lambda, N X_\mu) \\ &= \omega(X_\lambda, \mu X_\mu) = \mu\omega(X_\lambda, X_\mu). \end{aligned}$$

Thus $\omega(X_\lambda, X_\mu) = 0$, so E_λ and E_μ are ω-orthogonal.

$\square$

The foliations $\mathcal{F}_i$ are invariant under the Hamiltonian flow:

Lemma 2.3. *Suppose $\lambda_i \in \sigma(N)$ and X is a vector field tangent to $\mathcal{F}_i$. Then $[X_H, X]$ is also tangent to $\mathcal{F}_i$.*

Proof:

Because of (BH) the Lie derivative $L_{X_H} N$ vanishes. Therefore, if $X \in E_{\lambda_i}$ one finds:

$$\begin{aligned} N[X_H, X] &= N L_{X_H} X \\ &= L_{X_H}(NX) \\ &= L_{X_H}(\lambda_i X) = (X_H \cdot \lambda_i) X + \lambda_i [X_H, X]. \end{aligned}$$

But λ_i is a constant of the motion, so $[X_H, X] \in E_{\lambda_i}$.

□

¿From proposition 2.1 we know that each λ_i is an integral of $\mathcal{D}_i$. On the other hand, (BH) implies that each λ_i is a constant of the motion, and so by (ND) depends only on the action variables. Since the λ_i's are functionally independent we can use them as new "action" variables $(y_1, \ldots, y_n)$ on $\mathbb{R}^n$ to obtain:

Proposition 2.4. *The foliations $\mathcal{D}_i$ on M project to (n-1)-dimensional foliations Δ_i on $\mathbb{R}^n$ which are pairwise transversal. In particular, there are coordinate functions (y^i) on $\mathbb{R}^n$, such that each $y^i \circ \pi$ is constant on the leaves of $\mathcal{D}_i$.*

In the sequel we will not distinguish between y^i and $y^i \circ \pi$. Our interest in the new coordinates (y^i) lies in the following splitting result which is the basis of this work.

Theorem 2.5. *In the new coordinates (y^i) one has the following splittings:*

$$H(y^1, \ldots, y^n) = H_1(y^1) + \ldots + H_n(y^n) \qquad s^i(y^1, \ldots, y^n) = S_1^{(i)}(y^1) + \ldots + S_n^{(i)}(y^n).$$

Proof:

Denote by φ^i, $i = 1 \ldots, n$, conjugate coordinates in M to the coordinates y^i, so we have the Poisson bracket relations

$$\{y^i, y^j\} = 0 \quad , \quad \{y^i, \varphi^j\} = \delta_{ij} \quad , \quad \{\varphi^i, \varphi^j\} = 0 \quad i, j = 1, \ldots, n.$$

Explicitly, one finds

$$\theta^i = \sum_{j=1}^n \frac{\partial y^j}{\partial s^i} \varphi^j. \tag{2.2}$$

We claim that in the new coordinates one has

$$T\mathcal{F}_i = span \left\{ X_{y^i}, X_{\varphi^i} + \sum_{j=1}^n a_{ij} X_{y^j} \right\} \qquad (a_{ii} = 0). \tag{2.3}$$

To prove this observe that by proposition 2.4 each y^i is constant on the leaves of $\mathcal{F}_j$ $(i \neq j)$, while by proposition 2.2 $(T\mathcal{F}_i)^\perp = T\mathcal{D}_i = T\mathcal{F}_1 + \ldots + \widehat{T\mathcal{F}_i} + \ldots + T\mathcal{F}_n$, so we have $X_{y^i} \in T\mathcal{F}_i$. This shows that

$$T\mathcal{F}_i = span \left\{ X_{y^i}, bX_{\varphi^i} + \sum_{j=1}^{n} a_{ij} X_{y^j} + \sum_{j=1}^{n} c_{ij} X_{\varphi^j} \right\}.$$

where we can assume $a_{ii} = c_{ii} = 0$. Since $X_{y^i} \in T\mathcal{F}_i$, each X_{y^j} is orthogonal to $T\mathcal{F}_i$ for $i \neq j$, so it follows that:

$$0 = \omega(X_{y^j}, bX_{\varphi^i} + \sum_{k=1}^{n} a_{ik} X_{y^k} + \sum_{k=1}^{n} c_{ik} X_{\varphi^k}) = c_{ij} \qquad (i \neq j).$$

We conclude that

$$T\mathcal{F}_i = span \left\{ X_{y^i}, bX_{\varphi^i} + \sum_{j=1}^{n} a_{ij} X_{y^j} \right\}.$$

Finally, if the coefficient b vanishes at some $m \in M$, then $T_m\mathcal{F}_i \subset Ker\ d_m\pi$, and from proposition (2.2) we get that

$$T_m\mathcal{F}_i \subset Ker\ d_m\pi = (Ker\ d_m\pi)^\perp \subset (T_m\mathcal{F}_i)^\perp = T_m\mathcal{D}_i,$$

which contradicts the transversality of $\mathcal{F}_i$ and $\mathcal{D}_i$. Thus we can assume $b = 1$ and (2.3) follows.

Using (2.3), we can find the expression for N in the coordinates (y^i, φ^i). The final result is:

$$N = \begin{pmatrix} \Lambda & 0 \\ B & \Lambda \end{pmatrix} \quad \text{where} \quad \Lambda = diag\ (\lambda_1, \ldots, \lambda_n)\ ,\ B_{ij} = (\lambda_j - \lambda_i) a_{ji} \tag{2.3a}$$

Recall now that (BH) assures the existence of a second Hamiltonian $\tilde{H}$ such that

$$N^* d\tilde{H} = dH,$$

and by (ND) we must have $\tilde{H} = \tilde{H}(y^1, \ldots, y^n)$. Thus the first n equations of this system reduce to

$$y^i \frac{\partial \tilde{H}}{\partial y^i} = \frac{\partial H}{\partial y^i}, \quad i = 1, \ldots, n. \tag{2.4}$$

By crossing differentiating (2.4) we see that $\partial^2 H / \partial y^j \partial y^i = 0$ $(i \neq j)$, which proves the splitting for H.

The analogous splitting for s^i is proved by showing that X_{s^i} is a bi-Hamiltonian vector field in the set of points where all λ_j's are nonzero. Then we can repeat the argument of the last paragraph to show that the splitting holds on this set. But, by (ND), it must hold everywhere. Now, if all the λ_i's are nonzero the second Poisson structure is symplectic, and X_{s^i} is bi-Hamiltonian provided $L_{X_{s^i}} N = 0$. In the original variables (s^i, θ^i) one has

$X_{s^i} = \partial/\partial\theta^i$ so the Lie derivative of N vanishes if one can show that its entries do not depend on the θ^i's. This is proved in two steps:

(i) In the variables (y^i, φ^i) the entries of N do not depend on the φ^i's;
Since $\lambda_i = y^i$, we see from (2.3a) that the assertion will follow provided that $a_{ij} = a_{ij}(y^1, \dots, y^n)$. By lemma 2.3 and (2.3) we have

$$\Big[X_H, X_{\varphi^i} + \sum_{j=1}^{n} a_{ij} X_{y^j}\Big] = -X_{\{H,\varphi^i\}} + \sum_{j=1}^{n} \big(X_H(a_{ij}) X_{y^j} - a_{ij} X_{\{H,y^j\}}\big) \in T\mathcal{F}_i$$

Since $\{H, \varphi^i\} = \frac{\partial H}{\partial y^i}$, $\{H, y^j\} = 0$, we conclude that

$$\sum_{j=1}^{n} \Big(-\frac{\partial^2 H}{\partial y^i \partial y^j} + X_H(a_{ij})\Big) X_{y^j} \in T\mathcal{F}_i.$$

But we have shown already that $\partial^2 H/\partial y^i \partial y^j = 0$ $(i \neq j)$, so by (2.3) we also have $X_H(a_{ij}) = 0$ $(i \neq j)$. Finally from (ND) we conclude that $a_{ij} = a_{ij}(y^1, \dots, y^n)$.

(ii) In the variables (s^i, θ^i) the entries of N do not depend on the θ^i's.
Because of the form of the transformation $(s^i, \theta^i) \to (y^i, \varphi^i)$ (cf. (2.2)), we see from (i) that the entries of N when written in the variables (s^i, θ^i) are at most linear in the θ^i's. But those entries are well defined functions on the tori, so in fact they do not depend on the θ^i's.

□

Remark 2.6. In Morandi et al. (1990), a result closely related to theorem 2.5 was obtained (cf. prop. 3.22). They use the Lagrangian approach, and prove separability of the Hamiltonian with respect to both Hamiltonian structures. However, they fail to recognize the importance of the action-angle variables, which as we will see in the next sections play a central role.

3. A Geometric Interpretation

Let $(x^1, \dots, x^{n+1})$ be affine coordinates in a (n+1)-dimensional affine space $\mathbb{A}^{n+1}$. A hypersurface in $\mathbb{A}^{n+1}$ is called a **hypersurface of translation** if it admits a parameterization of the form:

$$(y^1, \dots, y^n) \to x^l(y^1, \dots, y^n) = a_1^l(y^1) + \dots + a_n^l(y^n) \quad (l = 1, \dots, n+1) \tag{3.1}$$

This generalizes Darboux's definition for n=2: a surface of translation is a surface obtained by parallel translating a curve along another curve (Darboux, 1887). The results of the previous section lead to the following geometric picture:

Theorem 3.1. *A completely integrable Hamiltonian system is bi-Hamiltonian (satisfying (BH)) if and only if the graph of the Hamiltonian function is a hypersurface of translation, relative to the affine structure determined by the action variables.*

Proof:

The 'only if' part was the subject of the previous section. Now assume that (M^{2n}, ω, H) is a completely integrable system and that *graph* H is a hypersurface of translation relative to the action variables (s^i), so it has a parameterization of the form (3.1), with $x^i = s^i, i = 1, \ldots, n$ and $x^{n+1} = H$. We can choose the parameters (y^i) so that the Hamiltonian takes the simple form

$$H(y^1, \ldots, y^n) = y^1 + \ldots + y^n.$$

If $(\varphi^1, \ldots, \varphi^n)$ are coordinates conjugate to the $(y^1, \ldots, y^n)$, we define a second Poisson structure by the formula

$$\Lambda_2 = \sum_{i=1}^{n} y^i \frac{\partial}{\partial y^i} \wedge \frac{\partial}{\partial \varphi^i}.$$

One checks easily that the two Poisson structures are compatible, and that the recursion operator is given by

$$N = \sum_{i=1}^{n} y^i \Big(\frac{\partial}{\partial y^i} \otimes dy^i + \frac{\partial}{\partial \varphi^i} \otimes d\varphi^i \Big)$$

It is now clear from the expression of the Hamiltonian function in the y-coordinates that $L_{X_H} N = 0$, so the vector field X_H is bi-Hamiltonian.

□

Remark 3.2. Note that the notion of the graph of the Hamiltonian being a hypersurface of translation, is invariantly associated with the system, since it is defined relative to the affine structure determined by the action-angle variables. It should not be confused with separability of the Hamiltonian system, or of the Hamilton-Jacobi equation, which are coordinate dependent. For example, consider a system with Hamiltonian $H(s^1, s^2) = s^1(1 + (s^2)^2)$, expressed in action-angle variables. It will be shown below that *graph* H is not an hypersurface of translation. But if we introduce new canonical coordinates $(y^1, y^2, \varphi^1, \varphi^2)$ with $y^1 = H(s^1, s^2)$, the Hamiltonian system splits in the new coordinates into two independent two-dimensional Hamiltonian systems.

It arises the problem of recognizing when is *graph* H a hypersurface of translation. For this purpose, we introduce the Hessian metric g on $\mathbb{R}^n$, which is defined with respect to the affine coordinates (x^i) by the formula:

$$g = \sum_{i,j} \frac{\partial^2 H}{\partial x^i \partial x^j} \; dx^i dx^j \tag{3.2}$$

In the (non-affine) coordinates (y^α) the metric g is given by:

$$g=\sum_{\alpha,\beta}\Big(\frac{\partial^2 H}{\partial y^\alpha \partial y^\beta}-\frac{\partial H}{\partial x^k}\frac{\partial^2 x^k}{\partial y^\alpha \partial y^\beta}\Big)dy^\alpha dy^\beta, \tag{3.3}$$

and so, by (3.1), diagonalizes:

$$\frac{\partial^2 H}{\partial y^\alpha \partial y^\beta}=0 \quad , \quad \frac{\partial^2 x^k}{\partial y^\alpha \partial y^\beta}=0 \quad (\alpha\neq\beta). \tag{3.3}$$

Defining the coordinate vector fields $Y_\alpha=\partial/\partial y^\alpha$, these conditions on the metric can be written in the form

$$g(Y_\alpha,Y_\beta)=0 \qquad (\alpha\neq\beta), \tag{3.4}$$

$$Y_\alpha(Y_\beta(x^k))=0 \qquad (\alpha\neq\beta). \tag{3.5}$$

We conclude that the existence of vector fields $Y_\alpha, \alpha=1,\ldots,n$, satisfying (3.4) and (3.5) is a necessary and sufficient condition for *graph* H to be an hypersurface of translation. Note that we can find the n-tuples of vector fields satisfying (3.4) by solving an eigenvalue problem. This will define the (Y_α)'s up to multiplicative factors C_α, and equations (3.5) then form a system of first order linear p.d.e's for each of these factors. Our recognition problem is then reduced to investigate the local solvability of these equations. Suppose for example that $Y_\alpha=C_\alpha\sum_i A_{\alpha i}\partial/\partial x^i$. Then (3.5) gives:

$$Y_\alpha(C_\beta)A_{\beta k}+C_\beta Y_\alpha(A_{\beta k})=0 \quad (\alpha\neq\beta). \tag{3.6}$$

Since we are interested in non-zero, local, solutions of this equation, we obtain the following integrability conditions

$$Y_\alpha(A_{\beta k})A_{\beta l}-A_{\beta k}Y_\alpha(A_{\beta l})=0 \quad (\alpha\neq\beta, k\neq l). \tag{3.7}$$

A more invariant way of describing the above conditions can be given as follows. Denote by ∇ the law of covariant differentiation associated with the connection defined by the affine structure on $\mathbb{R}^n$. Then the Hessian metric defined by H is given invariantly by the expression $g(X,Y)=\nabla_X\nabla_Y H$. To solve our recognition problem we seek an orthonormal basis $(Y_1,\ldots,Y_n)$ for the tangent bundle $T(M)$, satisfying

$$\nabla_{Y_\beta}Y_\alpha=0 \quad (\alpha\neq\beta),$$

i.e., such that each vector field Y_α can be obtained by parallel transport along the integral curves of any other $Y_\beta, (\alpha\neq\beta)$.

4. Applications

4.1 Counter-examples. A simple way of constructing completely integrable Hamiltonian systems possessing no bi-Hamiltonian formulation is to consider the standard model $M\simeq$

$\mathbb{R}^n \times \mathbb{T}^n$ with $\omega = \sum_{i=1}^n ds^i \wedge d\theta^i$, and choose any Hamiltonian function $H = H(s^1, \ldots, s^n)$ whose graph is not a hypersurface of translation. For example, take

$$H(s^1, \ldots, s^n) = s^1 + s^1(s^2)^2 + (s^3)^2 + \ldots + (s^n)^2.$$

The Hessian matrix has eigenvectors

$$\begin{aligned} Y_{1,2} &= C_{1,2}\Big(2s^2 \frac{\partial}{\partial s^1} + \big(s^1 \pm \sqrt{(s^1)^2 + 4(s^2)^2}\,\big)\frac{\partial}{\partial s^2}\Big), \\ Y_j &= C_j \frac{\partial}{\partial s^j}, \qquad\qquad j = 3, \ldots, n. \end{aligned}$$

It is easy to check that conditions (3.7) are not satisfied. For example, if we let $\alpha = 1, \beta = 2, k = 1, l = 2$ we obtain

$$Y_1(A_{21})A_{22} - A_{21}Y_1(A_{22}) = -4(s^2)^2 + 12s^1(s^2)^2\sqrt{(s^1)^2 + 4(s^2)^2} \neq 0.$$

We conclude that *graph* H is not a hypersurface of translation. Brouzet's original counter-example corresponds to the case n=2 (see Brouzet, 1990).

A counter-example with some physical meaning is obtained by considering the perturbed Kepler problem. In spherical coordinates (r, θ, ϕ), where θ denotes the colatitude and ϕ the azimuth, the Hamiltonian takes the form

$$H = \frac{1}{2}\Big({p_r}^2 + \frac{{p_\theta}^2}{r^2} + \frac{{p_\phi}^2}{r^2 \sin^2\theta}\Big) - \frac{1}{r} + \frac{\varepsilon}{2r^2}.$$

Two additional integrals, Poisson commuting with H, are the total angular momentum

$$L^2 \equiv {p_\theta}^2 + \frac{{p_\phi}^2}{\sin^2\theta},$$

and the component of the angular momentum along the polar axis

$$M \equiv p_\phi.$$

For $h < 0$, each common level set $\{H = h, L = l, M = m\}$ is an embedded 3-torus on the phase space. The action variables are obtained by integration along a basis $(\gamma_1, \gamma_2, \gamma_3)$ of 1-cycles for this torus:

$$s_i = \frac{1}{2\pi}\oint_{\gamma_i} p_r dr + p_\theta d\theta + p_\phi d\phi$$

We obtain

$$\begin{aligned} s_\phi &= \frac{1}{2\pi}\oint p_\phi d\phi = m \\ s_\theta &= \frac{1}{2\pi}\oint p_\theta d\theta = \frac{1}{2\pi}\oint \sqrt{l^2 - \frac{m^2}{\sin^2\theta}}\,d\theta = l - m \\ s_r &= \frac{1}{2\pi}\oint p_r dr = \frac{1}{2\pi}\oint \sqrt{2h + \frac{2}{r} - \frac{l^2 + \varepsilon}{r^2}}\,dr = \frac{1}{\sqrt{-2h}} - \sqrt{l^2 + \varepsilon} \end{aligned}$$

We conclude that the Hamiltonian, when written in action variables, takes the form

$$H = -\frac{1}{2\big(s_r + \sqrt{(s_\phi + s_\theta)^2 + \varepsilon}\ \big)^2}.$$

A more or less tedious computation, similar to the one in the previous example, shows that conditions (3.7) once again are not satisfied. Also we note that for the unperturbed Kepler problem ($\varepsilon = 0$) the graph of the Hamiltonian is a surface of translation, and so it has a bi-Hamiltonian formulation (on the other hand, one can show that the relativistic Kepler problem also does not have a bi-Hamiltonian formulation).

These counter-examples help one understand why a bi-Hamiltonian formulation is not known for many of the classical integrable systems. In general, one will require some degenerate Poisson pair in a higher dimension manifold. This is the situation, for example, in the R-matrix approach (see Oevel and Ragnisco, 1989). In the case of the perturbed Kepler problem, the Hamilton-Jacobi equation can be solved by separation of variables, and it follows from some recent work of Rauch-Wojciechowski (1991) that this system admits a degenerate bi-Hamiltonian formulation in a higher dimensional manifold.

4.2 Example. Consider a symmetric top rotating freely about a fixed point. As in the general theory of tops (see for example Bobenko *et al.* (1989) and references therein), it can be realized as an Hamiltonian system for the Lie-Poisson structure on $\mathfrak{e}(3)$, the Lie algebra of the group of motions of three dimensional Euclidean space. For the usual coordinates $(M_1, M_2, M_3, p_1, p_2, p_3)$ on $\mathfrak{e}(3)$, the Poisson bracket is defined by the relations

$$\{M_i, M_j\} = \varepsilon_{ijk} M_k, \qquad \{M_i, p_j\} = \varepsilon_{ijk} p_k, \qquad \{p_i, p_j\} = 0,$$

and the Hamiltonian is given by

$$H = \frac{1}{2I_1}({M_1}^2 + {M_2}^2) + \frac{1}{2I_3}{M_3}^2.$$

Note that this bracket is degenerate, the algebra of Casimirs being generated by $C_1 = \sum_i {p_i}^2$ and $C_2 = \sum_i p_i M_i$. Since M_3 provides an additional first integral commuting with H, it follows that the system is completely integrable when restricted to any symplectic leaf of the Kirilov foliation. Let us consider the symplectic leaf $C_1 = 1, C_2 = 0$. It can be identified with the tangent bundle $T\mathbb{S}^2$ of the unit sphere on $\mathbb{R}^3$, and this suggests introducing new variables $(\theta, \varphi, p_\theta, p_\varphi)$ given by

$$\begin{array}{lll} p_1 = \cos\theta\cos\varphi & p_2 = \cos\theta\sin\varphi & p_3 = \sin\theta \\ M_1 = p_\varphi\tan\theta\cos\varphi - p_\theta\sin\varphi & M_1 = p_\varphi\tan\theta\sin\varphi + p_\theta\cos\varphi & M_3 = p_\varphi \end{array}$$

This coordinates are canonical and the Hamiltonian function takes the form

$$H = {p_\varphi}^2\Big(\frac{1}{2I_1}\tan^2\theta + \frac{1}{2I_3}\Big) + \frac{1}{2I_1}{p_\theta}^2.$$

For $h/m^2 > 1/2I_3$, the level surfaces $\{H = h, M_3 = m\}$ are embedded 2-tori in the fixed symplectic leaf. The action variables are computed in the usual way:

$$s_1 = \frac{1}{2\pi}\oint_{\gamma_1} p_\phi d\phi = \frac{1}{2\pi}\oint m\ d\phi = m$$

$$s_2 = \frac{1}{2\pi}\oint_{\gamma_2} p_\theta d\theta = \frac{1}{2\pi}\oint \sqrt{2hI_1 - m^2\frac{I_1}{I_3} - m^2\tan^2\theta}\ d\theta = \sqrt{2hI_1 - m^2\frac{I_1 - I_3}{I_3}} - m$$

The expression for the Hamiltonian in the action variables is

$$H = \frac{I_1 - I_3}{2I_1 I_3}{s_1}^2 + \frac{1}{2I_1}(s_1 + s_2)^2,$$

so according to theorem 3.1 the system possesses a bi-Hamiltonian formulation. It is easy to see that the second Poisson structure (cf. proof of theorem 3.1) is given by:

$$\Lambda_2 = \frac{I_1 - I_3}{2I_1 I_3}{s_1}^2\frac{\partial}{\partial s_1}\wedge\frac{\partial}{\partial \theta_1} + \Big(\frac{I_1 - I_3}{2I_1 I_3}{s_1}^2 - \frac{1}{2I_1}(s_1+s_2)^2\Big)\frac{\partial}{\partial \theta_1}\wedge\frac{\partial}{\partial s_2} + \frac{1}{2I_1}(s_1+s_2)^2\frac{\partial}{\partial s_2}\wedge\frac{\partial}{\partial \theta_2}.$$

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