

The Classifying Lie Algebroid of a Geometric Structure

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Chapter 1

Introduction

In this chapter we will give an outline of the results presented in this thesis. We then describe the content of each chapter. We end this introduction by stating the conventions that will be in force throughout the thesis.

1.1 Outline of the Main Results

This thesis deals with symmetries, invariants and moduli spaces of geometric structures of *finite type*. By a geometric structure of finite type, we mean any object on a smooth manifold which determines, and is determined by a coframe on a (possibly different) manifold. Recall that a **coframe** on an n -dimensional manifold M is a set $\{\theta^1, \dots, \theta^n\}$ of everywhere linearly independent 1-forms on M . Examples of these geometric structures include:

1. Finite type G -structures (Definition 3.3.5), which are determined by the tautological form of its last non-trivial prolongation;
2. Cartan geometries (Definition 6.2.4), which are determined by a Cartan connection, viewed as a coframe, on a principal bundle $P \rightarrow M$;
3. Linear connections ∇ on the tangent bundle TM of a manifold, which might be assumed to preserve some tensor (or a G -structure) on M . These connections are characterized by the coframe on the frame bundle $\mathcal{B}(M)$ of M (or a G -structure $\mathcal{B}_G(M)$ over M) given by its tautological form and the connection form associated to ∇ .

The first problem that we will be interested in is that of determining when two such geometric structures are (locally) isomorphic. This problem is known as the *equivalence problem*. When two geometric structures of the same kind have been properly characterized by coframes $\{\theta^i\}$ and $\{\bar{\theta}^i\}$ on n -dimensional manifolds M and \bar{M} , the (local) equivalence problem takes the form:

Problem 1.1.1 (Equivalence Problem) *Does there exist a (locally defined) diffeomorphism $\phi : M \rightarrow \bar{M}$ satisfying*

$$\phi^* \bar{\theta}^i = \theta^i$$

for all $1 \leq i \leq n$?

The solution to this problem is based on the fact that exterior differentiation and pullbacks commute. Taking exterior derivatives and using the fact that $\{\theta^i\}$ is a coframe, we may write the **structure equations**

$$d\theta^k = \sum_{i < j} C_{ij}^k(x) \theta^i \wedge \theta^j \quad (1.1.1)$$

for uniquely defined functions $C_{ij}^k \in C^\infty(M)$ known as **structure functions**. Analogously, we may write

$$d\bar{\theta}^k = \sum_{i < j} \bar{C}_{ij}^k(\bar{x}) \bar{\theta}^i \wedge \bar{\theta}^j.$$

It is then clear from

$$\phi^* d\bar{\theta}^k = d\phi^* \bar{\theta}^k,$$

that a necessary condition for equivalence is that

$$\bar{C}_{ij}^k(\phi(x)) = C_{ij}^k(x).$$

Thus, the structure functions can be seen as invariants of the coframe.

In the opposite direction, we may consider the problem of determining when a given a set of functions can be realized as the structure functions of some coframe. This problem was proposed and solved by Cartan in [9] (see also the Appendix of [5]). Its precise formulation can be stated as:

Problem 1.1.2 (Cartan's Realization Problem) *One is given:*

- an integer $n \in \mathbb{N}$,
- an open set $X \subset \mathbb{R}^d$,
- a set of functions $C_{ij}^k \in C^\infty(X)$ with indexes $1 \leq i, j, k \leq n$,
- and a set of functions $F_i^a \in C^\infty(X)$ with $1 \leq a \leq d$

and asks for the existence of

- an n -dimensional manifold M ;
- a coframe $\{\theta^i\}$ on M ;
- a map $h : M \rightarrow X$

satisfying

$$d\theta^k = \sum_{i < j} C_{ij}^k(h(m)) \theta^i \wedge \theta^j \quad (1.1.2)$$

$$dh^a = \sum_i F_i^a(h(m)) \theta^i. \quad (1.1.3)$$

We will also be interested in answering the following questions:

Local Classification Problem What are all germs of coframes which solve Cartan's realization problem?

Local Equivalence Problem When are two such germs of coframes equivalent?

We will present complete solutions to both of these problems.

Before we proceed with the outline of the problems treated in this thesis, let us present a simple, but motivating example:

Example 1.1.3 *Suppose that we are interested in the equivalence and classification problem for coframes whose structure functions are constant. In terms of Cartan's realization problem, this setting corresponds to the case where $d = 0$, so that X is a point (in particular, the functions F_i^a vanish identically).*

Necessary conditions for the existence of a solution to the realization problem are obtained by

$$d^2\theta^k = 0 \quad (k = 1, \dots, n).$$

They imply that $-C_{ij}^k$ are the structure constants of a Lie algebra \mathfrak{g} , i.e., there exists a Lie algebra \mathfrak{g} with a basis $\{e_1, \dots, e_n\}$ such that

$$[e_i, e_j] = -\sum_k C_{ij}^k e_k.$$

This condition is also sufficient. In fact, if G is any Lie group which has \mathfrak{g} as its Lie algebra, then the components of its \mathfrak{g} -valued right invariant Maurer-Cartan form ω_{MC} , when written in terms of the basis $\{e_1, \dots, e_n\}$, have C_{ij}^k as its structure functions.

Now let (M, θ^i, h) be any other solution of the realization problem. If we define a \mathfrak{g} -valued 1-form on M by

$$\theta = \sum_i \theta^i e_i,$$

then the structure equation (1.1.1) for the coframe $\{\theta^i\}$ implies that θ satisfies the Maurer-Cartan equation

$$d\theta + \frac{1}{2}[\theta, \theta] = 0.$$

It is then well known that, by the universal property of the Maurer-Cartan forms on Lie groups, there exists a locally defined diffeomorphism onto a neighborhood of the identity

$$\psi : M \rightarrow G$$

such that

$$\theta = \psi^* \omega_{MC}.$$

Thus, at least locally, there is only one solution to the realization problem, up to equivalence.

In general, when the structure functions are not constant, they cannot determine a Lie algebra. Instead, it will be shown that solutions to Cartan's problem exist if and only if the initial data to the problem determines a *Lie algebroid*, which will be called the **classifying Lie algebroid** of the realization problem.

A Lie algebroid is vector bundle $A \rightarrow X$ which generalizes both Lie algebras and tangent bundles of manifolds. Its global counterpart is a Lie groupoid,

which is a manifold \mathcal{G} equipped with two surjective submersions \mathbf{s} and \mathbf{t} to X , as well as a partially defined smooth multiplication for which there are identity elements and such that each element of \mathcal{G} has an inverse (see Chapter 2 for precise definitions). Two important features of a Lie algebroid are that (i) it determines an integrable (singular) distribution on X , whose integral manifolds are called **leaves** or **orbits** of A , and (ii) that over each point x in X there is a Lie algebra called the **isotropy Lie algebra** at x .

Inspired by the example presented above, we propose the following solution to Cartan's realization problem. A necessary condition for the existence of a realization is the existence of a classifying Lie algebroid. Assume, for simplicity, that this Lie algebroid comes from a Lie groupoid \mathcal{G} (see Section 2.3). In order to prove that this necessary condition is also sufficient, we introduce Maurer-Cartan forms on Lie groupoids. It then follows that the Maurer-Cartan form ω_{MC} induces a coframe on each \mathbf{s} -fiber which gives a solution to the realization problem. Moreover, we deduce a universal property of the Maurer-Cartan form on Lie groupoids which implies that every solution to Cartan's problem is locally equivalent to one of these. This settles the questions proposed so far in this introduction.

One of the main advantages of using our Lie algebroid approach, instead of Cartan's original methods, is that, while Cartan only obtains an existence result, the classifying Lie algebroid (or its local Lie groupoid) gives rise to a recipe for constructing explicit solutions to the problem. These solutions are the \mathbf{s} -fibers of the groupoid, equipped with the restriction of the Maurer-Cartan forms.

It turns out that the classifying Lie algebroid of a realization problem satisfies some very interesting properties. To state a few:

1. To every point on X there corresponds a germ of a coframe which is a solution to the realization problem;
2. Two such germs of coframes are equivalent if and only if they correspond to the same point in X ;
3. The isotropy Lie algebra at a point x of the classifying Lie algebra is isomorphic to the *symmetry Lie algebra* of the corresponding germ of coframe, i.e., to the Lie algebra of germs of vector fields which preserve the germ of coframe.

So far we have only discussed local aspects of the theory. We will also be interested in global issues. The first of these problems that we will solve is the *globalization problem*:

Problem 1.1.4 (Globalization Problem) *Given a Cartan's problem with initial data (n, X, C_{ij}^k, F_i^a) and two germs of coframes θ_0 and θ_1 which solve the problem, does there exist a global connected solution (M, θ, h) to the realization problem for which θ_0 is the germ of θ at a point $p_0 \in M$ and θ_1 is the germ of θ at a point $p_1 \in M$?*

Of course, the solution to this problem relies, again, on understanding the classifying Lie algebroid. What we will show is that two germs of coframes which belong to the same global connected realization correspond to points in the same orbit of A . Moreover, if the Lie algebroid comes from a Lie groupoid,

then the converse is also true, namely, if two germs of coframes correspond to the same orbit of A , then they belong to the same global connected realization. It follows that the classifying Lie algebroid also gives us information about the moduli space of global solutions of a realization problem.

Another global problem that we will deal with is the global equivalence problem. In general, the classifying Lie algebroid does not distinguish between a realization and its universal covering space. If (M, θ, h) is a realization and $\pi : \tilde{M} \rightarrow M$ is a covering map, then $(\tilde{M}, \pi^*\theta, \pi^*h)$ is also a realization called the **induced realization**. It will also be called a **realization cover** of (M, θ, h) .

It is then natural to consider the following equivalence relation on the set of realizations of a Cartan's problem. Two realization (M_1, θ_1, h_1) and (M_2, θ_2, h_2) are said to be **globally equivalent, up to covering** if they have a common realization cover (M, θ, h) , i.e.,

$$\begin{array}{ccc} & (M, \theta, h) & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ (M_1, \theta_1, h_1) & & (M_2, \theta_2, h_2) \end{array}$$

This leads to:

Problem 1.1.5 (Global Classification Problem) *What are all the solutions of a Cartan's realization problem up to global equivalence, up to covering?*

Again, in order to solve this problem we will need an *integrability* assumption, namely that the classifying Lie algebroid for the realization problem (or, more precisely, its restriction to each orbit) comes from a Lie groupoid. In this case, we will show that any solution to Cartan's problem is globally equivalent, up to covering, to an open set of an \mathfrak{s} -fiber of the Lie groupoid. This result generalizes both of the main theorems concerning global equivalences presented in [27].

The classifying Lie algebroid can also be used to produce invariants of a geometric structure of finite type, as well as to recover classical results about their symmetry groups. In fact, with a suitable regularity assumption, any coframe θ on a manifold M gives rise to a Cartan's realization problem (n, X, C_{ij}^k, F_i) and a map $h : M \rightarrow X$ which makes (M, θ, h) into a realization problem. In this case, however, we must allow X to be a manifold instead of simply an open set of \mathbb{R}^d . This does not impose any extra difficulties to the problem, and we still obtain a classifying Lie algebroid A over X , which turns out to be *transitive*. We can then see the coframe θ as a Lie algebroid morphism $\theta : TM \rightarrow A$ which covers h .

On the one hand, we can use the classifying Lie algebroid of a fixed coframe to furnish simple proofs of classical results about the dimension of the symmetry Lie group of the geometric structure being studied. These are all immediate corollaries of the fact that the Lie algebra of the symmetry group is isomorphic to the isotropy Lie algebra at a point of the classifying Lie algebroid. As an example, we prove three theorems which are present in [21].

On the other hand, we will advocate the general philosophy that the coframe, viewed as a morphism, should pullback invariants of the classifying Lie algebroid to invariants of the coframe. We will illustrate this point of view with two

examples of cohomological invariants: the *basic cohomology of a coframe* and the *modular class of a coframe*.

We can summarize this outline of the thesis by saying that there are essentially two ways in which we use the existence of a classifying Lie algebroid for a Cartan's realization problem.

- For a *class* of geometric structures of finite type whose moduli space (of germs) is finite dimensional, we can set up a Cartan's realization problem whose classifying Lie algebroid gives us information about its local equivalence and classification problems, as well as its globalization and global classification problems.
- For a *single* geometric structure of finite type, we can set up a Cartan's realization problem whose classifying Lie algebroid describes the symmetries of the geometry, and also provides a recipe for producing invariants of the structure.

To conclude this thesis we will present a series of examples which illustrate the results obtained before.

1.2 Contents

We now give a brief discussion of the content of each chapter.

Chapter 2 In Chapter 2, we will introduce the concepts of Lie algebroids and Lie groupoids. Since these objects will play a fundamental role throughout the thesis, we have decided to state all the results that will be used. We begin by defining Lie algebroids and then proceed to present several examples. Instead of presenting an extensive list, we focus on the examples which will be relevant for this thesis. We then give a thorough description of Lie algebroid morphisms in two equivalent ways, both of which will be useful. The second section contains the definition of a Lie groupoid and its basic properties. Section 2.3 deals with the Lie theory for Lie algebroids and Lie groupoids. We begin by showing how to obtain a Lie algebroid out of a Lie groupoid. It turns out that not every Lie algebroid arises in this way, and the obstructions obtained in [12] for this to happen are described at the end of the section. Along the way, we state the Lie algebroid versions of Lie's first and second theorems. The chapter ends with a list of all the examples of Lie groupoids which will be needed in the rest of the thesis.

Chapter 3 The third chapter of this thesis is about G -structures. These will be the main source of examples of geometric structures to be considered. The chapter begins with the definition and some examples of G -structures. We then describe their equivalence problem and find a necessary condition to solve it. To be able to obtain more refined necessary conditions for equivalence, we introduce in Section 3.3 the method of prolongation. This will lead us to the concept of G -structures of *finite type*, for which our results are valid. We finish the chapter with a description of the structure functions and structure equations for G -structures of finite type. We first deal with the case of G -structures of type 1, and then we go on to the

general case of G -structures of type k . The results obtained here will be the key ingredients for stating and solving classification problems for these geometric structures.

Chapter 4 While the previous chapters included mostly background material, this chapter is the core of the thesis. We begin by describing the equivalence problem for coframes, as stated in [27] and [33]. The analyses of necessary conditions for solving this problem will lead us to Cartan's realization problem. We will then show that if a solution to the realization problem exists, then its initial data determines a Lie algebroid which we call the classifying Lie algebroid. In order to solve Cartan's problem, we introduce Maurer-Cartan forms on Lie groupoids and prove their local universal property. By adding an extra topological hypothesis, we are also able to obtain a global universal property of these forms. When this is established, we proceed to solve the local classification problem and to describe the symmetries of a realization. The chapter finishes with a discussion of some global issues. In fact, we solve the globalization problem and the global equivalence problem.

Chapter 5 In Chapter 5 we specialize to the case where the coframe comes from a finite type G -structure. In this case, the structure equations of the coframe have the particular format described at the end of Chapter 3. It follows that the classifying Lie algebroid also has the particular feature that it comes equipped with an infinitesimal action of \mathfrak{g} , the Lie algebra of G , by inner algebroid automorphisms. We begin by describing the case of G -structures of type 1. This case will be used, at the end of the chapter, to solve the realization problem for general finite type G -structures. Before we do so, however, we give a geometric description of what happens in the best possible scenario, i.e., when the classifying Lie algebroid A is integrable by a Lie groupoid \mathcal{G} , and when the infinitesimal action of \mathfrak{g} on A can be integrated to a free and proper action of G on \mathcal{G} . This is the content of Section 5.3.

Chapter 6 This chapter presents several applications of the existence of a classifying Lie algebroid associated to a realization problem. It begins with a slight generalization of the realization problem presented before, to the case where X is a manifold, instead of an open set of some \mathbb{R}^d . This generalization will be shown to be useful in concrete examples, and in proving classical results about the symmetry group of geometric structures. We focus on three theorems which are present in [21]. The chapter ends with a discussion of how to obtain invariants of geometric structures out of invariants of the classifying Lie algebroid.

Chapter 7 The final chapter is dedicated to several examples which illustrate the results obtained throughout the thesis. We begin by deducing the structure equations and structure functions of an arbitrary torsion-free connection on a G -structure. It turns out, however, that the moduli space of all torsion-free connections on a fixed G -structure may be infinite dimensional, and thus, not treatable by our methods. On the other hand, there are many interesting classes of these connections which can be treated. The first example we present is that of constant curvature torsion-free

connections. In particular, we exhibit the classifying Lie algebroid for flat torsion-free connections on arbitrary G -structures and for constant curvature Riemannian metrics on \mathbb{R}^2 . We then look at the space of locally symmetric torsion-free connections. The existence of a classifying Lie algebroid for these connections can be used to show that every symmetric Berger Lie algebra is the holonomy Lie algebra of some torsion-free connection. This will be discussed in Section 7.4. Our main class of examples, however, are the special symplectic manifolds. We will give a detailed description of the classifying Lie algebroid for these manifolds and use it to prove results about their symmetries and moduli space. We also show how to find examples of such manifolds. We remark that even though most of the results presented about special symplectic connections are not new, our approach differs from the original one. Our starting point is the construction of the classifying Lie algebroid from where all other properties are deduced, thus making the results more natural and less mysterious.

1.3 Conventions

This thesis takes place in the C^∞ category. Thus, unless explicitly stated otherwise, all our manifolds are smooth, second countable, and Hausdorff, and all our maps are also smooth. The only exception is the total space \mathcal{G} of a Lie groupoid, for which important examples force us to allow it to be a non-Hausdorff manifold. On the other hand, the \mathfrak{s} -fibers and \mathfrak{t} -fibers of a Lie groupoid, as well as the base manifold X are always assumed to be Hausdorff.

Chapter 2

Lie Algebroids and Lie Groupoids

This chapter is devoted to the study of Lie algebroids and Lie groupoids. Since these will play an important role, we have decided to state all the results that will be needed. For more details and proofs, we refer the reader to [8], [26], [23], [24] or [15].

2.1 Lie Algebroids

This section is about Lie algebroids. A Lie algebroid should be thought of as a substitute to the tangent bundle of manifold, whenever a (possibly singular) geometric structure is present.

2.1.1 Definition and First Examples

Definition 2.1.1 A *Lie algebroid* over a manifold X is a (real) vector bundle $A \rightarrow X$ equipped with a Lie bracket $[\cdot, \cdot]$ on its space of sections $\Gamma(A)$ and a bundle map $\# : A \rightarrow TX$ called the **anchor** of A satisfying

$$[\alpha, f\beta] = f[\alpha, \beta] + (\#(\alpha)f)\beta \text{ for all } \alpha, \beta \in \Gamma(A) \text{ and } f \in C^\infty(X).$$

The second condition above is known as the **Leibniz identity**. It implies that the anchor, seen as a map from $\Gamma(A)$ to $\mathfrak{X}(X)$, is a Lie algebra homomorphism, i.e.,

$$\#([\alpha, \beta]_A) = [\#(\alpha), \#(\beta)]_{\mathfrak{X}(X)} \text{ for all } \alpha, \beta \in \Gamma(A).$$

From this identity, it follows also that the kernel of $\#$ over a point $x \in X$ is a Lie algebra with the induced bracket. It is called the **isotropy Lie algebra** at x . Another important property of Lie algebroids is that the image of $\#$ is always an *integrable distribution* in the sense of Sussman [34]. In general, this distribution may be singular, i.e., its rank is *not* constant on X . Its maximal integral leaves are known as **orbits** or **leafs** of A .

For any Lie algebroid $A \rightarrow X$, we can define a differential $d_A : \Gamma(\wedge^\bullet A^*) \rightarrow \Gamma(\wedge^{\bullet+1} A^*)$ by

$$\begin{aligned} d_A \eta(\alpha_0, \dots, \alpha_k) &= \sum_{i=1}^k (-1)^i \#(\alpha_i) \cdot \eta(\alpha_0, \dots, \hat{\alpha}_i, \dots, \alpha_k) + \\ &\quad + \sum_{0 \leq i < j \leq k} (-1)^{i+j} \eta([\alpha_i, \alpha_j], \alpha_0, \dots, \hat{\alpha}_i, \dots, \hat{\alpha}_j, \dots, \alpha_k) \end{aligned}$$

which makes $\Gamma(\wedge^\bullet A^*)$ into a complex. In fact, there is a one-to-one correspondence between Lie algebroid structures on a vector bundle $E \rightarrow X$ and degree one derivations of the algebra $\Gamma(\wedge^2 E^*)$. The cohomology $H^\bullet(A)$ of the complex $(\Gamma(\wedge^\bullet A^*), d_A)$ will be called the **Lie algebroid cohomology of A** .

It will be useful to have a description of Lie algebroids in terms of local coordinates and local basis of sections. So let (x_1, \dots, x_d) be a local coordinate system on $U \subset X$ and $\alpha_1, \dots, \alpha_r$ a local basis of sections of A over U . We define the structure functions $C_{ij}^k, F_i^a \in C^\infty(U)$ of A relative to these local coordinates and basis of sections by

$$[\alpha_i, \alpha_j] = \sum_k C_{ij}^k(x) \alpha_k \quad (2.1.1)$$

$$\# \alpha_i = \sum_a F_i^a(x) \frac{\partial}{\partial x_a}. \quad (2.1.2)$$

Then, the conditions defining a Lie algebroid are translated into the following partial differential equations

$$\sum_{b=1}^d \left(F_i^b \frac{\partial F_j^a}{\partial x_b} - F_j^b \frac{\partial F_i^a}{\partial x_b} \right) = \sum_{l=1}^r C_{ij}^l F_l^a \quad (2.1.3)$$

for all $1 \leq i, j \leq r, 1 \leq a \leq d$ and

$$\begin{aligned} \sum_{b=1}^d \left(F_j^b \frac{\partial C_{kl}^i}{\partial x_b} + F_k^b \frac{\partial C_{lj}^i}{\partial x_b} + F_l^b \frac{\partial C_{jk}^i}{\partial x_b} \right) = \\ \sum_{m=1}^r (C_{mj}^i C_{kl}^m + C_{mk}^i C_{lj}^m + C_{ml}^i C_{jk}^m) \end{aligned} \quad (2.1.4)$$

for all $1 \leq i, j, k, l \leq r$.

We now present a few examples of Lie algebroids. Our purpose here is not to present an extensive list of examples, but to describe those that will be used elsewhere in this thesis.

Example 2.1.2 (Tangent Bundles) *The simplest example of a Lie algebroid is the tangent bundle TX of a manifold X , equipped with its Lie bracket of vector fields, and the identity map as the anchor.*

Example 2.1.3 (Lie Algebras) *Another extreme example of a Lie algebroid occurs when X is a point. In this case, A is simply a vector space equipped with a Lie bracket, i.e., a Lie algebra.*

Example 2.1.4 (Foliations) Let \mathcal{F} be a (regular) foliation on X , so that $T\mathcal{F} \subset TX$ is an involutive distribution of constant rank. Then $T\mathcal{F}$ is a Lie algebroid over X whose bracket is the Lie bracket of vector fields on X , and whose anchor is the inclusion $i : T\mathcal{F} \rightarrow TX$. Note that the orbits of this Lie algebroid are precisely the leaves of \mathcal{F} .

Example 2.1.5 (Atiyah Algebroid) Let

$$\begin{array}{c} P \xrightarrow{G} \\ \pi \downarrow \\ X \end{array}$$

be a principal G -bundle. We obtain a Lie algebroid, known as the **Atiyah algebroid** of the principal bundle as follows: Take A to be the quotient TP/G , which is a vector bundle over X . We can identify the sections of A with right invariant vector fields on P . Since the Lie bracket of invariant vector fields is again invariant, we obtain a bracket on $\Gamma(A)$. The anchor of A is simply the map $[\pi_*] : A \rightarrow TX$ induced by π_* .

Note that this Lie algebroid is transitive (i.e., the orbit of a point x is the whole base X of A), and its isotropy Lie algebras are isomorphic to \mathfrak{g} , the Lie algebra of G .

Example 2.1.6 (Infinitesimal Actions) Let

$$\psi : \mathfrak{g} \rightarrow \mathfrak{X}(X)$$

be an infinitesimal action of a Lie algebra \mathfrak{g} on X . We obtain a Lie algebroid, called the **transformation Lie algebroid** associated to the action by setting A to be the trivial vector bundle $X \times \mathfrak{g}$, with anchor $\#(x, \alpha) = \psi(\alpha)|_x$ and bracket

$$[\alpha, \beta](x) = [\alpha(x), \beta(x)]_{\mathfrak{g}} + (\psi(\alpha) \cdot \beta)(x) - (\psi(\beta) \cdot \alpha)(x),$$

where we are regarding a section of A as a smooth map $X \rightarrow \mathfrak{g}$.

In this case, the leaves of A coincide with the orbits of the \mathfrak{g} -action and the isotropy Lie algebras coincide with the isotropy of the action.

Example 2.1.7 (Poisson Manifolds) A **Poisson structure** on a manifold X is a Lie bracket $\{ , \}$ on the space of smooth functions on X , $C^\infty(X)$, which satisfies the Leibniz identity

$$\{f, gh\} = g\{f, h\} + \{f, g\}h \text{ for all } f, g, h \in C^\infty(X).$$

Equivalently, we may define a Poisson structure as a bivector field $\Pi \in \Gamma(\wedge^2 TX)$ such that $[\Pi, \Pi] = 0$, where $[,]$ denotes the Schouten bracket on multi-vector fields. The correspondence between both definitions is given by $\Pi(df, dg) = \{f, g\}$. A **Poisson manifold** is a manifold X equipped with a Poisson structure Π .

Given a Poisson manifold (X, Π) , we can construct a Lie algebroid, called the **cotangent Lie algebroid** as follows: We take A to be the cotangent bundle of X , T^*X . The Poisson structure then determines a map

$$\Pi^\sharp : T^*X \rightarrow TX, \quad \Pi^\sharp(\alpha)(\beta) = \Pi(\alpha, \beta) \text{ for all } \alpha, \beta \in T^*X,$$

which we take to be the anchor of A . The Lie bracket on $\Gamma(A) = \Omega^1(X)$ is then defined by

$$[\alpha, \beta] = \mathcal{L}_{\Pi^\sharp(\alpha)}\beta - \mathcal{L}_{\Pi^\sharp(\beta)}\alpha - d(\Pi(\alpha, \beta)).$$

It is the unique bracket on $\Omega^1(X)$ such that $[df, dg] = d\{f, g\}$.

It is important to note that the restriction of the Poisson bivector Π to each leaf of A is non-degenerate, and thus induces a canonical symplectic form on each leaf. Explicitly, if L is a leaf of A , we have

$$\omega_L(\xi_1, \xi_2) = ((\Pi|_L)^\sharp)^{-1}(\xi_1)(\xi_2) \text{ for all } \xi_1, \xi_2 \in \mathfrak{X}(L).$$

For this reason, the foliation on X induced by a Poisson structure is called a **symplectic foliation**.

There are two examples of Poisson manifolds that will appear in this thesis, which we now present.

Example 2.1.8 (Lie-Poisson Manifolds) Let \mathfrak{g} be a Lie algebra. Then \mathfrak{g}^* carries a natural Poisson structure, known as its **Kostant-Kirillov-Souriau Poisson structure**, by setting

$$\{f, g\}(\rho) = \rho([d_\rho f, d_\rho g]) \text{ for all } f, g \in C^\infty(\mathfrak{g}^*),$$

where we have identified $T_\rho^*\mathfrak{g}^*$ with \mathfrak{g} . The Poisson manifold $(\mathfrak{g}^*, \{, \})$ is called a **Lie-Poisson manifold**. Its cotangent Lie algebroid $A = T^*\mathfrak{g}^*$ can be identified with the transformation Lie algebroid induced by the co-adjoint action of \mathfrak{g} on \mathfrak{g}^* .

Example 2.1.9 (Cosymplectic Submanifolds) Let (X, Π) be a Poisson manifold. A submanifold Y is called a **cosymplectic submanifold** if it intersects each leaf L of $A = T^*X$ transversely, and the restriction of the symplectic form ω_L to $T_y Y \cap T_y L_y$ is non-degenerate for all $y \in Y$. Equivalently, we may express this condition as

$$TX|_Y = TY \oplus (TY)^\perp,$$

where $TY^\perp \subset TX$ denotes the image by Π^\sharp of the annihilator of TY .

If Y is a cosymplectic submanifold of (X, Π) then it carries a natural Poisson bivector, called the **coinduced Poisson structure**, which may be defined as

$$\Pi_Y(\xi_1, \xi_2) = \Pi(\iota^{-1}(\xi_1), \iota^{-1}(\xi_2)),$$

where ι denotes the restriction to W of the natural projection $T^*X \rightarrow T^*Y$.

2.1.2 Lie Algebroid Morphisms

Let $A \rightarrow X$ and $B \rightarrow Y$ be Lie algebroids. We will now give two equivalent definitions of Lie algebroid morphism. The first one appeared in [20] to which we refer for a detailed exposition on Lie algebroid morphisms.

Definition 2.1.10 A **morphism of Lie algebroids** is a bundle map

$$\begin{array}{ccc} A & \xrightarrow{F} & B \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

that is compatible with the anchors and the brackets.

Let us explain what we mean by compatibility. A bundle map (F, f) from A to B is **compatible with the anchors** if

$$\begin{array}{ccc} A & \xrightarrow{F} & B \\ \# \downarrow & & \downarrow \# \\ TX & \xrightarrow{f_*} & TY. \end{array}$$

The problem of defining compatibility with the brackets is that in general the bundle map (F, f) does not induce a map at the level of sections. The way to overcome this difficulty is to work on the pullback bundle f^*B . Note that each $\alpha \in \Gamma(A)$ can be pushed forward to a section $F(\alpha) \in \Gamma(f^*B)$. Also, each section $\beta \in \Gamma(B)$ can be pulled back to $f^*\beta \in \Gamma(f^*B)$. Now, if α and $\tilde{\alpha}$ belong to $\Gamma(A)$, their image in $\Gamma(f^*B)$ can be written (non-uniquely) as

$$\begin{aligned} F(\alpha) &= \sum_i u_i(f^*\beta_i) \\ F(\tilde{\alpha}) &= \sum_j \tilde{u}_j(f^*\tilde{\beta}_j) \end{aligned}$$

where $u_i, \tilde{u}_j \in C^\infty(X)$ and $\beta_i, \tilde{\beta}_j \in \Gamma(B)$.

We will say that (F, f) is **compatible with the brackets** if

$$\begin{aligned} F([\alpha, \tilde{\alpha}]_A) &= \sum_{i,j} u_i \tilde{u}_j (f^*[\beta_i, \tilde{\beta}_j]_B) + \\ &+ \sum_j (\#\alpha)(\tilde{u}_j) f^*\tilde{\beta}_j - \sum_i (\#\tilde{\alpha})(u_i) f^*\beta_i \end{aligned} \quad (2.1.5)$$

for all $\alpha, \tilde{\alpha} \in \Gamma(A)$. We observe that the definition above doesn't depend on the choice of decomposition of $F(\alpha)$ and $F(\tilde{\alpha})$.

There is a more invariant way of expressing the bracket compatibility (2.1.5) which will serve as inspiration for the main results of this thesis. Let ∇ be an arbitrary connection on the vector bundle $B \rightarrow Y$, and denote by $\bar{\nabla}$ the induced connection on f^*B defined by

$$\bar{\nabla}_\xi \left(\sum_i u_i(f^*\beta_i) \right) = \sum_i u_i f^*(\nabla_{f_*\xi} \beta_i) + \sum \xi(u_i) f^*\beta_i. \quad (2.1.6)$$

Define the torsion of the pullback connection $\bar{\nabla}$ by

$$T_{\bar{\nabla}} \left(\sum u_i f^*\beta_i, \sum \tilde{u}_j f^*\tilde{\beta}_j \right) = \sum_{i,j} u_i \tilde{u}_j f^* T_{\nabla}(\beta_i, \tilde{\beta}_j) \quad (2.1.7)$$

where the torsion of a connection on a Lie algebroid is defined by

$$T_{\nabla}(\beta_i, \tilde{\beta}_j) = \nabla_{\#\beta_i} \tilde{\beta}_j - \nabla_{\#\tilde{\beta}_j} \beta_i - [\beta_i, \tilde{\beta}_j]. \quad (2.1.8)$$

Finally, for a bundle map $(F, f) : A \rightarrow B$ we define a tensor $R_F \in \Gamma(\wedge^2 A^* \otimes f^*B)$ by

$$R_F(\alpha, \tilde{\alpha}) = \bar{\nabla}_{\#\alpha} F(\tilde{\alpha}) - \bar{\nabla}_{\#\tilde{\alpha}} F(\alpha) - F([\alpha, \tilde{\alpha}]) - T_{\bar{\nabla}}(F(\alpha), F(\tilde{\alpha})). \quad (2.1.9)$$

A straight forward calculation shows that whenever (F, f) is compatible with anchors, then R_F does not depend on ∇ . Moreover, (F, f) is a morphism of Lie algebroids if and only if $R_F \equiv 0$.

The second equivalent definition of a morphism of Lie algebroids that we will now present is due to Vaintrob [35] and is based on viewing a Lie algebroid as a certain kind of supermanifold.

A bundle map

$$\begin{array}{ccc} A & \xrightarrow{F} & B \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

induces a pullback map $F^* : \Gamma(B^*) \rightarrow \Gamma(A^*)$ defined by

$$\langle (F^*\phi)_x, \alpha_x \rangle = \langle (\phi \circ f)(x), F(\alpha_x) \rangle$$

for $\phi \in \Gamma(B^*)$, $x \in X$ and $\alpha \in \Gamma(A)$. We can extend this map to

$$F^* : \Gamma(\wedge^\bullet B^*) \rightarrow \Gamma(\wedge^\bullet A^*)$$

where we set in degree zero $F^*(h) = h \circ f$ for $h \in C^\infty(X)$.

Proposition 2.1.11 (Vaintrob [35]) *A bundle map (F, f) from A to B is a Lie algebroid morphism if and only if the map*

$$F^* : (\Gamma(\wedge^\bullet B^*), d_B) \rightarrow (\Gamma(\wedge^\bullet A^*), d_A)$$

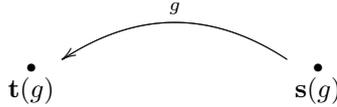
is a chain map.

2.2 Lie Groupoids

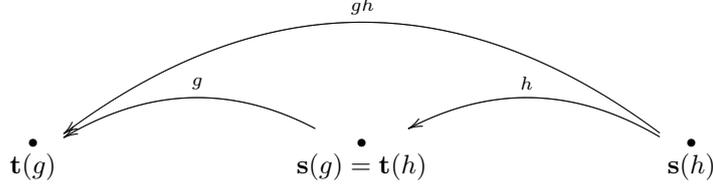
The global counterpart to Lie algebroids are Lie groupoids. A concise definition of a groupoid is a small category in which all morphisms are invertible. We can expand this definition into:

Definition 2.2.1 *A Lie groupoid (denoted by $\mathcal{G} \rightrightarrows X$) over a manifold X consists of two manifolds: X , called the base of the groupoid, and \mathcal{G} (in general non-Hausdorff), called the total space of the groupoid and the following smooth structure functions:*

1. *two surjective submersions $\mathbf{s}, \mathbf{t} : \mathcal{G} \rightarrow X$ called source and target respectively. Each point g in \mathcal{G} should be seen as an arrow joining $\mathbf{s}(g)$ to $\mathbf{t}(g)$*



2. a smooth multiplication m defined on $\mathcal{G}^{(2)} = \{(g, h) \in \mathcal{G} \times \mathcal{G} : \mathbf{s}(g) = \mathbf{t}(h)\}$ and denoted by $m(g, h) = gh$ satisfying $\mathbf{s}(gh) = \mathbf{s}(h)$ and $\mathbf{t}(gh) = \mathbf{t}(g)$



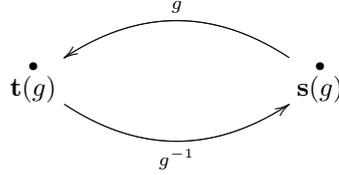
This multiplication is required to be associative, i.e., the product $(gh)k$ is defined if and only if $g(hk)$ is defined and in this case $(gh)k = g(hk)$.

3. an embedding $\mathbf{1} : X \rightarrow \mathcal{G}$ called identity section which satisfies $\mathbf{1}_x g = g$ for all g in $\mathbf{t}^{-1}(x)$ and $g \mathbf{1}_x = g$ for all g in $\mathbf{s}^{-1}(x)$ (in particular $\mathbf{s} \circ \mathbf{1} = Id_M = \mathbf{t} \circ \mathbf{1}$)



and

4. a diffeomorphism $i : \mathcal{G} \rightarrow \mathcal{G}$, $g \mapsto g^{-1}$ called inversion which satisfies $\mathbf{s} \circ i = \mathbf{t}$, $\mathbf{t} \circ i = \mathbf{s}$ and $g g^{-1} = \mathbf{1}_{\mathbf{s}(g)}$, $g g^{-1} = \mathbf{1}_{\mathbf{t}(g)}$



We define the isotropy Lie group at $x \in X$ as the set of arrows in \mathcal{G} that start and end at x , i.e.,

$$\mathcal{G}_x = \{g \in \mathcal{G} : \mathbf{s}(g) = x = \mathbf{t}(g)\}.$$

We also define the orbit of x as the set of points that can be reached from x by an arrow in \mathcal{G} , i.e.,

$$\mathcal{G} \cdot x = \{\mathbf{t}(g) : g \in \mathbf{s}^{-1}(x)\}.$$

The orbits of \mathcal{G} are immersed, not necessarily connected submanifolds.

We shall call a Lie groupoid **s-connected** if its \mathbf{s} -fibers are connected, and **s-simply connected** if its \mathbf{s} -fibers are connected and simply connected.

We shall describe several examples of Lie groupoids in Section 2.4.

2.3 Lie Theory

To every Lie groupoid there is an associated Lie algebroid. As a vector bundle, take A to be the restriction to the identity section (which we identify with X) of the vector bundle formed by vectors tangent to the \mathbf{s} -fibers of \mathcal{G} , i.e., denoting

$\ker \mathbf{s}_* = T^s\mathcal{G}$ then $A = T^s\mathcal{G} \big|_X$. The sections of A are then identified with right invariant \mathbf{s} -vertical vector fields on \mathcal{G}

$$\Gamma(A) \cong \mathfrak{X}_{\text{inv}}^s(\mathcal{G}). \quad (2.3.1)$$

A simple computation shows that

$$[\mathfrak{X}_{\text{inv}}^s(\mathcal{G}), \mathfrak{X}_{\text{inv}}^s(\mathcal{G})]_{\mathfrak{X}(\mathcal{G})} \subset \mathfrak{X}_{\text{inv}}^s(\mathcal{G}). \quad (2.3.2)$$

Thus, the Lie bracket of vector fields on \mathcal{G} induces a Lie bracket on $\Gamma(A)$. Finally, defining the anchor $\#$ of A as the restriction to A of \mathbf{t}_* one obtains on A the structure of a Lie algebroid. The isotropy Lie algebra of A at a point x is the Lie algebra of the isotropy Lie group \mathcal{G}_x of \mathcal{G} . Moreover, if \mathcal{G} is source connected, then it's orbit's coincide with the orbits of A .

Unlike Lie algebras, not every Lie algebroid arises as the Lie algebroid of a Lie groupoid. When this occurs, we say that A is integrable. Obstructions to integrability are well known. We briefly summarize these results at the end of this section, and we refer the reader to [12] for a detailed discussion of the integrability problem for Lie algebroids, and [13] for the particular case of Poisson manifolds. Other aspects of Lie theory do however apply to Lie groupoids and Lie algebroids. We now state two of these results that will be used in this thesis. For proofs, see [26] or [12].

Proposition 2.3.1 (Lie I) *Let \mathcal{G} be a Lie groupoid with Lie algebroid A . Then there exists a unique \mathbf{s} -simply connected Lie groupoid $\tilde{\mathcal{G}}$ whose Lie algebroid is also A .*

Proposition 2.3.2 (Lie II) *Let $A \rightarrow X$ and $B \rightarrow Y$ be integrable Lie algebroids. Denote by $\mathcal{G}(A)$ the source simply connected Lie groupoid integrating A and by \mathcal{H} any Lie groupoid integrating B . If $\Phi : A \rightarrow B$ is a morphism of Lie algebroids covering $f : X \rightarrow Y$, then there exists a unique morphism of Lie groupoids $F : \mathcal{G}(A) \rightarrow \mathcal{H}$ also covering f such that*

$$d_{1_x} F(v) = \Phi(v) \quad (2.3.3)$$

for all $x \in X$ and $v \in T_{1_x}^s \mathcal{G}(A)$. In this case we say that F integrates Φ .

Lie's third theorem for Lie algebras, which states that every finite dimensional Lie algebra is integrable, does not hold for Lie algebroids. We now give a brief description of the obstructions to integrate a Lie algebroid, and state the criteria for integrability that will be used in this thesis.

We begin by describing a canonical source connected and source simply connected *topological* groupoid $\mathcal{G}(A)$ associated to any Lie algebroid A , called the **Weinstein groupoid** of A .

Definition 2.3.3 • *An **A-path** on a Lie algebroid A is a path $a : I \rightarrow A$ such that*

$$\frac{d}{dt} \pi(a(t)) = \#a(t).$$

We will denote the space of A -paths by $P(A)$.

- *An **A-homotopy** between A -paths a_0 and a_1 is a Lie algebroid morphism $T(I \times I) \rightarrow A$ which covers a (standard) homotopy with fixed end-points between the base paths $\pi(a_i(t))$.*

- The **Weinstein groupoid of A** is the topological groupoid

$$\mathcal{G}(A) = P(A)/\sim,$$

where \sim denotes the equivalence relation of A -homotopy. The multiplication on $\mathcal{G}(A)$ is given by concatenation of paths and the source and target maps are the projections of the end-points of the A -path.

Remark 2.3.4 The set of A -paths can be identified with the Lie algebroid morphisms $TI \rightarrow A$. It has the structure of an infinite dimensional Banach manifold.

In order to describe the obstruction for $\mathcal{G}(A)$ to be a Lie groupoid, and hence of A being integrable, denote by \mathfrak{g}_x the isotropy Lie algebra of A at x , and by $Z(\mathfrak{g}_x)$ its center.

Definition 2.3.5 The **monodromy group of A at x** is the set

$$N_x(A) = \{v \in Z(\mathfrak{g}_x) : v \text{ is } A\text{-homotopic to the constant zero path}\} \subset A_x.$$

It is a remarkable fact that there is a **monodromy map**

$$\partial : \pi_2(L, x) \rightarrow \mathcal{G}(\mathfrak{g}_x)$$

whose image is precisely the monodromy subgroup $N_x(A)$, where L is the leaf of A through x , $\mathcal{G}(\mathfrak{g}_x)$ is the simply connected Lie group integrating the Lie algebra \mathfrak{g}_x and where we have identified $N_x(A) \subset Z(\mathfrak{g}_x)$ with an abelian subgroup of $\mathcal{G}(\mathfrak{g}_x)$. We are now able to state Lie's third theorem for Lie algebroids [12].

Theorem 2.3.6 (Lie III) Let $A \rightarrow X$ be a Lie algebroid. The following statements are equivalent:

1. A is integrable.
2. The Weinstein groupoid $\mathcal{G}(A)$ is a Lie groupoid.
3. The monodromy groups are uniformly discrete.

Although we will not use this theorem in its full strength, it will be useful to have some simple criteria for deciding when a Lie algebroid is integrable. We now state, in the form of corollaries, the ones that will be used in this thesis.

Corollary 2.3.7 Let A be a Lie algebroid whose leafs are all 2-connected. Then A is integrable. In particular, any transitive Lie algebroid over a contractible base is integrable.

Corollary 2.3.8 Let A be a Lie algebroid whose isotropy Lie algebras all have trivial center. Then A is integrable. In particular, any Lie algebroid with injective anchor is integrable.

Corollary 2.3.9 let \mathfrak{g} be a Lie algebra and let \mathfrak{g}^* be its dual Lie-Poisson manifold. Then its cotangent Lie algebroid $T^*\mathfrak{g}^*$ is integrable.

Corollary 2.3.10 Let A be a Lie subalgebroid of an integrable Lie algebroid B . Then A is integrable.

Corollary 2.3.11 Let (X, Π) be an integrable Poisson manifold and let $Y \subset X$ be a cosymplectic submanifold. Then the induced Poisson structure on Y is integrable.

2.4 Examples of Lie Groupoids

In this section we will describe several examples of Lie groupoids. In each example, we will indicate the corresponding Lie algebroid, but we will, however, omit the proofs.

Example 2.4.1 (Fundamental Groupoid) *Let X be a manifold and let $\Pi_1(X)$ denote the manifold consisting of all homotopy classes (with fixed end points) of curves in X . Then $\Pi_1(X) \rightrightarrows X$ can be endowed with the structure of a Lie groupoid as follows: let $\gamma : I \rightarrow X$ be a curve in X , and denote by $[\gamma]$ its homotopy class. The source and target maps associate to $[\gamma]$ its end points, i.e., $\mathbf{s}([\gamma]) = \gamma(0)$, and $\mathbf{t}([\gamma]) = \gamma(1)$. If γ_1 and γ_2 are two curves such that $\gamma_1(1) = \gamma_2(0)$ then we define their product by concatenation of curves,*

$$[\gamma_2][\gamma_1] = [\gamma_2 * \gamma_1].$$

The identity element at a point $x \in X$ is the class of the constant path x , and the inverse of $[\gamma]$ is the class of $\bar{\gamma} : I \rightarrow X$, where $\bar{\gamma}(t) = \gamma(1 - t)$.

Note that the orbits of the fundamental groupoid are the connected components of X , while the isotropy group at x is the fundamental group of X with base point x . The Lie algebroid of $\Pi_1(X)$ is the tangent bundle TX . In fact, $\Pi_1(X)$ is the unique \mathbf{s} -simply connected Lie groupoid which has TX as its Lie algebroid.

Example 2.4.2 (Pair Groupoid) *Another groupoid which has the tangent bundle of X as its Lie algebroid is the **pair groupoid** $X \times X \rightrightarrows X$, whose structure is given by:*

$$\begin{aligned} \mathbf{s}(x_1, x_2) &= x_2, \\ \mathbf{t}(x_1, x_2) &= x_1, \\ (x_1, x_2)(x_2, x_3) &= (x_1, x_3), \\ \mathbf{1}_x &= (x, x), \text{ and} \\ (x_1, x_2)^{-1} &= (x_2, x_1). \end{aligned}$$

Example 2.4.3 (Lie Groups) *Every Lie group can be seen as a Lie groupoid over a point, with the obvious structure.*

Example 2.4.4 (Monodromy Groupoid of a Foliation) *Let \mathcal{F} be a regular foliation on X . The **monodromy groupoid** of the foliation is the unique Lie groupoid $\Pi_1(\mathcal{F}) \rightrightarrows X$ whose Lie algebroid is $T\mathcal{F}$, and whose \mathbf{s} -fibers are connected and simply connected. It can be described as the space of homotopy classes of curves contained in the leaves of \mathcal{F} , where we only allow homotopies which are also contained in the leaves. As with the fundamental groupoid, its multiplication is given by concatenation of paths, its identity elements are given by the class of constant paths, and inversion is given by reversing the direction of the curve.*

The orbits of the monodromy groupoid of \mathcal{F} are precisely the leaves of the foliation. The isotropy Lie group of a point x is the fundamental group of the leaf through x , with base point x , i.e., $\pi_1(L_x, x)$.

Example 2.4.5 (Holonomy Groupoid of a Foliation) *If in the previous example, instead of taking the homotopy class of curves, we take its holonomy class, we obtain another Lie groupoid, called the **holonomy groupoid** of the foliation $\text{Hol}(\mathcal{F}) \rightrightarrows X$ whose Lie algebroid is also $T\mathcal{F}$.*

This Lie groupoid has the property that any Lie groupoid \mathcal{G} whose Lie algebroid is $T\mathcal{F}$ fits into the exact sequence of groupoid covering maps

$$\Pi_1(\mathcal{F}) \longrightarrow \mathcal{G} \longrightarrow \text{Hol}(\mathcal{F}),$$

where by a groupoid covering map we mean a Lie groupoid morphism whose restriction to each \mathbf{s} -fiber is a covering map.

Example 2.4.6 (Gauge Groupoid) *Let*

$$\begin{array}{ccc} P & \xrightarrow{\quad} & G \\ \pi \downarrow & & \\ X & & \end{array}$$

*be a principal G -bundle. The **gauge groupoid** of P , $\mathcal{G}(P) \rightrightarrows X$ is defined as*

$$\mathcal{G}(P) = \frac{P \times P}{G}$$

where the quotient refers to the diagonal action of G on $P \times P$, $(p, q) \cdot g = (pg, qg)$. Let us denote by $[p, q]$ the class of (p, q) . Then the structure of $\mathcal{G}(P)$ is given by:

$$\begin{aligned} \mathbf{s}[p, q] &= \pi(q), \\ \mathbf{t}[p, q] &= \pi(p), \\ [p_1, q_1] [q_1, p_2] &= [p_1, p_2], \\ \mathbf{1}_x &= [p, p] \text{ for some } p \in \pi^{-1}(x), \text{ and} \\ [p, q]^{-1} &= [q, p]. \end{aligned}$$

Note that we can identify a point $[p, q] \in \mathcal{G}(P)$ with the G -equivariant diffeomorphism $\phi_{p,q} : \pi^{-1}(\pi(q)) \rightarrow \pi^{-1}(\pi(p))$ which maps q to p . It follows that a section σ of the source map can be identified with an automorphism of the G -bundle which covers $\mathbf{t} \circ \sigma : X \rightarrow X$. A gauge transformation of P is the special case where $\mathbf{t} \circ \sigma = \text{Id}_X$.

The gauge groupoid of P is a transitive Lie groupoid whose isotropy Lie group at any point x is isomorphic to G . Conversely, if $\mathcal{G} \rightrightarrows X$ is a transitive Lie groupoid, then for each $x \in X$,

$$\begin{array}{ccc} \mathbf{s}^{-1}(x) & \xrightarrow{\quad} & \mathcal{G}_x \\ \mathbf{t} \downarrow & & \\ X & & \end{array}$$

is a principal \mathcal{G}_x -bundle whose gauge groupoid is isomorphic to \mathcal{G} , where \mathcal{G}_x denotes the isotropy Lie group of \mathcal{G} at x .

Example 2.4.7 (Group Actions) Let G be a Lie group acting on a manifold X . We define the **transformation groupoid** $\mathcal{G} = G \times X \rightrightarrows X$ to be the Lie groupoid whose structure is given by:

$$\begin{aligned} \mathbf{s}(g, x) &= x, \\ \mathbf{t}(g, x) &= gx, \\ (h, gx)(g, x) &= (hg, x) \\ \mathbf{1}_x &= (e, x), \text{ and} \\ (g, x)^{-1} &= (g^{-1}, gx), \end{aligned}$$

where e is the identity element of G .

The orbits of the transformation groupoid coincide with those of the action. Also, the isotropy groups of \mathcal{G} coincide with those of the action. The Lie algebroid of the transformation groupoid is the transformation Lie algebroid of the infinitesimal \mathfrak{g} -action associated to the G -action.

Example 2.4.8 (Symplectic Groupoids) A **symplectic groupoid** is a pair (\mathcal{G}, ω) , where $\mathcal{G} \rightrightarrows X$ is a Lie groupoid, and ω is a symplectic form on \mathcal{G} which is compatible with the groupoid structure, in the sense that,

$$m^*\omega = \pi_1^*\omega + \pi_2^*\omega, \quad (2.4.1)$$

where $m : \mathcal{G}^{(2)} \rightarrow \mathcal{G}$ denotes the multiplication of \mathcal{G} and π_1 and π_2 denote the projections of $\mathcal{G}^{(2)} \subset \mathcal{G} \times \mathcal{G}$ onto the first and second components. A differential form on \mathcal{G} satisfying (2.4.1) is called a **multiplicative form**. Note that the existence of a multiplicative symplectic form on \mathcal{G} imposes that $\dim \mathcal{G} = 2 \dim X$.

If (\mathcal{G}, ω) is a symplectic groupoid over X , then there exists a unique Poisson structure on X such that \mathbf{s} is a Poisson morphism and \mathbf{t} is an anti-Poisson morphism. Here we are viewing \mathcal{G} as a Poisson manifold with its non-degenerate Poisson structure induced by ω . The Lie algebroid of \mathcal{G} is then canonically isomorphic to T^*X . In particular, all orbits of \mathcal{G} carry a natural symplectic structure.

Example 2.4.9 (Cotangent Bundle of a Lie Group) A particular example of a symplectic groupoid is the cotangent bundle of a Lie group $\mathcal{G} = T^*G \rightrightarrows \mathfrak{g}^*$ equipped with its canonical symplectic structure, seen as a groupoid over \mathfrak{g}^* . If we identify T^*G with $G \times \mathfrak{g}^*$ then the Lie groupoid structure on T^*G is simply that of the transformation groupoid associated to the coadjoint action of G on \mathfrak{g}^* .

Chapter 3

G -Structures

In this section we recall the basic facts from the theory of G -structures that we will use. We refer to [21, 32, 33] for details.

3.1 Definition and First Examples

Denote by

$$\begin{array}{ccc} \mathcal{B}(M) & \xrightarrow{\text{GL}_n} & \\ \pi \downarrow & & \\ M & & \end{array}$$

the bundle of frames on M . This principal bundle carries a canonical 1-form with values in \mathbb{R}^n , denoted by $\theta \in \Omega^1(\mathcal{B}(M); \mathbb{R}^n)$, and which is defined by

$$\theta_p(\xi) := p^{-1}(\pi_*\xi), \quad (\xi \in T_p\mathcal{B}(M)).$$

This form is called the **tautological form** (or **soldering form**) of $\mathcal{B}(M)$. It is a **tensorial form**, i.e., it is horizontal and $\text{GL}(n)$ -equivariant (with respect to the defining action on \mathbb{R}^n). A subspace $H_p \subset T_p\mathcal{B}_G(M)$ is **horizontal** if the restriction $d_p\pi : H_p \rightarrow T_{\pi(p)}M$ is an isomorphism. When we say that θ is horizontal, we mean that $\theta(\xi) = 0$ if and only if ξ is tangent to the fibers of π . Note that a subspace H_p is horizontal if and only if the restriction $\theta : H_p \rightarrow \mathbb{R}^n$ is an isomorphism.

Every diffeomorphism φ between two manifolds M and N lifts to an isomorphism (a GL_n -equivariant diffeomorphism) of the associated frame bundles:

$$\mathcal{B}(\varphi) : \mathcal{B}(M) \rightarrow \mathcal{B}(N).$$

The correspondence which associates to each manifold its frame bundle and to each diffeomorphism its lift is functorial.

Definition 3.1.1 *Let G be a Lie subgroup of GL_n . A G -structure on M is a reduction of the frame bundle $\mathcal{B}(M)$ to a principal G -bundle $\mathcal{B}_G(M)$.*

This means that $\mathcal{B}_G(M) \subset \mathcal{B}(M)$ is a sub bundle such that for any $p \in \mathcal{B}_G(M)$ and $a \in \text{GL}(n)$ we have $pa \in \mathcal{B}_G(M)$ if and only if $a \in G$. Given a G -structure $\mathcal{B}_G(M)$, we will still denote by θ the restriction of the tautological form to $\mathcal{B}_G(M)$.

Remark 3.1.2 *If G is a closed subgroup of GL_n then $\mathcal{B}(M)/G$ is a fiber bundle over M with fiber GL_n/G . Moreover, $\pi_G : \mathcal{B}(M) \rightarrow \mathcal{B}(M)/G$ is a principal G -bundle. In this case, a G -structure on M is the same as a section $\sigma : M \rightarrow \mathcal{B}(M)/G$, where the G -structure associated to the section σ is*

$$\mathcal{B}_G(M) = \pi_G^{-1}(\sigma(M)) \subset \mathcal{B}(M).$$

Under the action of G , the tautological form behaves as

$$(R_a^*\theta)(\xi) = a^{-1} \cdot (\theta(\xi)) \quad (3.1.1)$$

where the action on the right hand side is the natural action of $G \subset \mathrm{GL}_n$ on \mathbb{R}^n . If $A \in \mathfrak{g}$ and $\tilde{A} \in \mathfrak{X}(\mathcal{B}_G(M))$ denotes the fundamental vector field generated by A , then the infinitesimal version of equation (3.1.1) is

$$\mathcal{L}_{\tilde{A}}\theta = -A \cdot \theta. \quad (3.1.2)$$

It then follows from Cartan's magic formula that

$$\iota_{\tilde{A}}d\theta = -A \cdot \theta. \quad (3.1.3)$$

Definition 3.1.3 *Two G -structures $\mathcal{B}_G(M)$ and $\mathcal{B}_G(N)$ are said to be **equiv-
alent** if there exists a diffeomorphism $\varphi : M \rightarrow N$ such that*

$$\mathcal{B}(\varphi)(\mathcal{B}_G(M)) = \mathcal{B}_G(N).$$

A **symmetry** of a G -structure is a self-equivalence map $\varphi : M \rightarrow M$.

In the case where G is a closed subgroup of GL_n we can give another description of this equivalence relation. If we use the diffeomorphism φ to identify M with N , we can also identify $\mathcal{B}(M)/G$ with $\mathcal{B}(N)/G$. Then φ is an equivalence if and only if $\varphi^*\sigma_N = \sigma_M$, where σ_M is the section determining the G -structure over M and σ_N is the section determining the G -structure over N .

For a given manifold M and a Lie subgroup G of GL_n , there may or may not exist a G -structure on M . We now present some examples:

Example 3.1.4 *If $G = \{e\}$ is the identity subgroup, then a G -structure on M is the same as a section of $\mathcal{B}(M)$, i.e., a complete parallelization of TM . Thus, if M is the 2-sphere, there does not exist an $\{e\}$ -structure over it.*

Example 3.1.5 *When $G = O_n$, a G -structure on M is the same as a Riemannian metric on M . To see this, let $\langle \cdot, \cdot \rangle$ denote the euclidean inner product of \mathbb{R}^n . Given an O_n -structure on M , we can define on M the Riemannian metric given by*

$$(v, w)_{T_m M} = \langle p^{-1}v, p^{-1}w \rangle$$

where $v, w \in T_m M$ and $p \in \mathcal{B}_{O_n}(M)$ is a frame over m . Clearly, since any two frames differ by an element of O_n , this definition does not depend on the choice of p . Conversely, given a Riemannian metric on M , the set of frames $p : \mathbb{R}^n \rightarrow T_m M$ for which the canonical base of \mathbb{R}^n is mapped to an orthonormal base of $T_m M$ forms an O_n -reduction of the frame bundle. It follows that any manifold M admits an O_n -structure.

The table below contains some more examples of G -structures. We leave the details of its verification, which is very similar to the O_n case described above, to the reader.

G	Geometric Structure
$GL_n^+(\mathbb{R}) \subset GL_n(\mathbb{R})$	Orientation
$SL_n(\mathbb{R}) \subset GL_n(\mathbb{R})$	Volume Form
$Sp_n(\mathbb{R}) \subset GL_{2n}(\mathbb{R})$	Almost Symplectic Structure
$GL_n(\mathbb{C}) \subset GL_{2n}(\mathbb{R})$	Almost Complex Structure

Remark 3.1.6 *We have defined G -structures in the setting of real differential geometry. We remark that the same definitions could be used to define geometries over other fields. For example, one can define the complex frame bundle, the \mathbb{C}^n -valued tautological form, and so on.*

3.2 The Equivalence Problem

One of the basic problems we will be interested in is deciding if two G -structures are equivalent. The tautological form is the clue to the solution of this equivalence problem. In fact, if $\varphi : M \rightarrow N$ is an equivalence between G -structures $\mathcal{B}_G(M)$ and $\mathcal{B}_G(N)$, then it is clear that $\mathcal{B}(\varphi) : \mathcal{B}_G(M) \rightarrow \mathcal{B}_G(N)$ is a diffeomorphism satisfying $\mathcal{B}(\varphi)^*(\theta_N) = \theta_M$. Moreover, we have:

Proposition 3.2.1 *A diffeomorphism $\psi : \mathcal{B}_G(M) \rightarrow \mathcal{B}_G(N)$ is the lift of an equivalence map if and only if $\psi^*\theta_N = \theta_M$.*

Proof. It is clear that the lift of an equivalence map preserves the tautological forms.

Conversely, let $\psi : \mathcal{B}_G(M) \rightarrow \mathcal{B}_G(N)$ be a diffeomorphism satisfying $\psi^*\theta_N = \theta_M$. First of all, we prove that ψ is a G -bundle isomorphism. Note that since ψ preserves the tautological forms, it follows that ψ_* maps vertical vectors to vertical vectors. Thus, ψ maps fibers to fibers and covers a diffeomorphism $\varphi : M \rightarrow N$. Now, for $a \in G$, denote by $a \cdot \theta_N$ the \mathbb{R}^n -valued form on $\mathcal{B}_G(N)$ defined by $(a \cdot \theta_N)(\xi) = a \cdot (\theta_N(\xi))$. Then

$$\begin{aligned} \psi^*(a^{-1} \cdot \theta_N)(\xi) &= (a^{-1} \cdot \theta_N)(\psi_*\xi) \\ &= a^{-1} \cdot (\theta_N(\psi_*\xi)) \\ &= a^{-1} \cdot (\theta_M(\xi)) \\ &= (a^{-1} \cdot \theta_M)(\xi), \end{aligned}$$

and thus

$$(R_a \circ \psi)^*\theta_N = (\psi \circ R_a)^*\theta_N$$

It follows that ψ is G -equivariant. In fact, let $p, q \in \mathcal{B}_G(M)$ be points on the same fiber of π_M . Thus, $q = pa$ for a unique $a \in G$. Since ψ maps fibers to fibers, it follows that $\psi(p)$ belongs to the same fiber as $\psi(q)$, and thus $\psi(q) = \psi(p)b$ for a unique $b \in G$. Let $v \in \mathbb{R}^n$ and let $\xi_v \in \mathfrak{X}(\mathcal{B}_G(M))$ be a right invariant

vector field such that $(\theta_M)_p(\xi_v) = v$. Then

$$\begin{aligned}
a^{-1}v &= (\theta_M)_{pa}(\xi_v) \\
&= (\theta_N)_{\psi(q)}(\psi_*\xi_v) \\
&= (\theta_N)_{\psi(p)b}(\psi_*\xi_v) \\
&= (R_b)^*((\theta_N)_{\psi(p)}(\psi_*\xi_v)) \\
&= b^{-1}((\theta_N)_{\psi(p)}(\psi_*\xi_v)) \\
&= b^{-1}v.
\end{aligned}$$

Since this is true for all $v \in \mathbb{R}^n$ it follows that $a = b$, i.e., ψ is G -equivariant.

We conclude by proving that ψ coincides with the lift $\mathcal{B}(\varphi)$ of the diffeomorphism $\varphi : M \rightarrow N$ that it covers. Let $\tilde{\psi} : \mathcal{B}_G(M) \rightarrow \mathcal{B}_G(M)$ be the map $\tilde{\psi} = \mathcal{B}(\varphi)^{-1} \circ \psi$. It is clear that

$$\tilde{\psi}^*\theta_M = \theta_M.$$

Thus, by an argument completely analogous to the one just given, it follows that, $\tilde{\psi}$ is a G -bundle isomorphism which covers the identity map. It follows that $\tilde{\psi}$ must be of the form

$$\tilde{\psi} = R_{g(\pi(p))},$$

for some smooth map $g : M \rightarrow G$. But then, at any point $p \in \mathcal{B}_G(M)$, we have

$$\theta_M = R_g^*\theta_M = g^{-1} \cdot \theta_M,$$

from where we conclude that

$$g(\pi(p)) = e \text{ for all } p \in \mathcal{B}_G(M),$$

which proves the proposition. ■

In order to obtain an invariant of equivalence of G -structures, let us choose some horizontal space H_p at $p \in \mathcal{B}_G(M)$. Given $v \in \mathbb{R}^n$ there exists a unique $\xi_v \in H_p$ such that $\theta(\xi_v) = v$, so one defines:

$$\begin{aligned}
c_{H_p} : \wedge^2 \mathbb{R}^n &\rightarrow \mathbb{R}^n, \\
c_{H_p}(v, w) &:= d\theta(\xi_v, \xi_w).
\end{aligned} \tag{3.2.1}$$

This depends on the choice of horizontal space, so it does not define an invariant yet. If H_p and H'_p are two distinct horizontal spaces at $p \in \mathcal{B}_G(M)$, and $\xi_v \in H_p$ and $\xi'_v \in H'_p$ are the unique vectors such that $\theta(\xi_v) = v = \theta(\xi'_v)$ then $\xi_v - \xi'_v$ is a vertical vector, and thus determines an element of \mathfrak{g} , the Lie algebra of G . In this way, we obtain a map, $S_{H_p, H'_p} : \mathbb{R}^n \rightarrow \mathfrak{g}$. We now calculate:

$$\begin{aligned}
c_{H_p}(v, w) - c_{H'_p}(v, w) &= d\theta(\xi_v, \xi_w) - d\theta(\xi'_v, \xi'_w) \\
&= d\theta(\xi_v - \xi'_v, \xi_w) + d\theta(\xi'_v, \xi_w - \xi'_w) \\
&= (\iota_{\widetilde{S_{H_p, H'_p}(v)}} d\theta)(\xi_w) - (\iota_{\widetilde{S_{H_p, H'_p}(w)}} d\theta)(\xi'_v).
\end{aligned}$$

Using equation (3.1.3) we obtain

$$c_{H_p}(v, w) - c_{H'_p}(v, w) = S_{H_p, H'_p}(w)v - S_{H_p, H'_p}(v)w.$$

It follows that

$$c_{H_p} - c_{H'_p} \in \mathcal{A}(\text{Hom}(\mathbb{R}^n, \mathfrak{g})),$$

where \mathcal{A} denotes the anti-symmetrization operator:

$$\begin{aligned} \mathcal{A} : \text{Hom}(\mathbb{R}^n, \mathfrak{g}) &\rightarrow \text{Hom}(\wedge^2 \mathbb{R}^n, \mathbb{R}^n), \\ \mathcal{A}(T)(u, v) &:= T(u)v - T(v)u. \end{aligned}$$

Hence, we can set:

Definition 3.2.2 *Given a G -structure $\mathcal{B}_G(M)$ one defines its **first order structure function**:*

$$c : \mathcal{B}_G(M) \rightarrow \frac{\text{Hom}(\wedge^2 \mathbb{R}^n, \mathbb{R}^n)}{\mathcal{A}(\text{Hom}(\mathbb{R}^n, \mathfrak{g}))}, \quad c(p) := [c_{H_p}].$$

Since an isomorphism $\psi : \mathcal{B}_G(M) \rightarrow \mathcal{B}_G(N)$ maps horizontal spaces to horizontal spaces and it is an equivalence if and only if $\psi^* \theta_N = \theta_M$, we see that

Proposition 3.2.3 *Let $\mathcal{B}_G(M)$ and $\mathcal{B}_G(N)$ be G -structures. If $\phi : M \rightarrow N$ is an equivalence then*

$$c_N \circ \mathcal{B}(\phi) = c_M.$$

3.3 Prolongation

In order to obtain more refined invariants of equivalence of G -structures one needs to look at higher order terms. This process is known as **prolongation** and takes place on the jet bundles $J^k \mathcal{B}_G(M)$.

Let $\pi : E \rightarrow M$ be a fiber bundle. We denote by $\pi^1 : J^1 E \rightarrow M$ its first jet bundle, which has fiber over $m \in M$:

$$(J^1 E)_m = \{j_m^1 s \mid s \text{ a section of } E\}.$$

This bundle can also be described geometrically as:

$$J^1 E = \{H_p : p \in E \text{ and } H_p \subset T_p E \text{ horizontal}\}.$$

If s is a local section of E such that $s(m) = p$ then the correspondence above is given by

$$j_m^1 s \mapsto H_p = d_m s(T_m M).$$

If one defines the projection $\pi_0^1 : J^1 E \rightarrow E$ by $\pi_0^1(H_p) = p$, then $J^1 E$ is an affine bundle over E .

Example 3.3.1 *For a G -structure $\mathcal{B}_G(M)$ the first structure function can also be described as a function $c : J^1 \mathcal{B}_G(M) \rightarrow \text{Hom}(\wedge^2 \mathbb{R}^n, \mathbb{R}^n)$ by formula (3.2.1).*

It is easy to see that in the case of the frame bundle $\pi : \mathcal{B}(M) \rightarrow M$ its first jet bundle $\pi_0^1 : J^1 \mathcal{B}(M) \rightarrow \mathcal{B}(M)$ can be identified with a sub bundle of $\mathcal{B}(\mathcal{B}(M))$: to a horizontal space $H_p \subset T_p \mathcal{B}(M)$ we associate a frame in $\mathcal{B}(M)$ (which we view as an isomorphism $\phi : \mathbb{R}^n \times \mathfrak{gl}(n) \rightarrow T_p \mathcal{B}(M)$):

$$\mathbb{R}^n \times \mathfrak{gl}(n) \ni (v, \rho) \xrightarrow{\phi} (\pi|_{H_p} \circ p)^{-1}(v) + \rho \cdot p \in T_p \mathcal{B}(M).$$

Note that if H_p and H'_p are two horizontal spaces at $p \in \mathcal{B}(M)$, the corresponding frames $\phi, \phi' : \mathbb{R}^n \times \mathfrak{gl}(n) \rightarrow T_p \mathcal{B}(M)$ are related by:

$$\phi'(v, \rho) = \phi(v, \rho) + T(v) \cdot p,$$

for some $T \in \text{Hom}(\mathbb{R}^n, \mathfrak{gl}(n))$. Conversely, given a frame ϕ associated with some horizontal space H_p and $T \in \text{Hom}(\mathbb{R}^n, \mathfrak{gl}(n))$, this formula determines a frame ϕ' which is associated with another horizontal space H'_p . It follows that $J^1 \mathcal{B}(M)$ is a $\text{Hom}(\mathbb{R}^n, \mathfrak{gl}(n))$ -structure, where we view $\text{Hom}(\mathbb{R}^n, \mathfrak{gl}(n)) \subset \text{GL}(\mathbb{R}^n \oplus \mathfrak{gl}(n))$ as the subgroup formed by those transformations:

$$(v, \rho) \mapsto (v, \rho + T(v)), \text{ with } T \in \text{Hom}(\mathbb{R}^n, \mathfrak{gl}(n)).$$

Assume now that $\mathcal{B}_G(M)$ is a G -structure so that $J^1 \mathcal{B}_G(M) \subset J^1 \mathcal{B}(M)$ is a sub-bundle. An argument entirely similar to one just sketched gives:

Proposition 3.3.2 *If $\mathcal{B}_G(M)$ is a G -structure then $J^1 \mathcal{B}_G(M) \rightarrow \mathcal{B}_G(M)$ is a $\text{Hom}(\mathbb{R}^n, \mathfrak{g})$ -structure.*

In order to motivate our next definition we look at a simple example.

Example 3.3.3 *Let us consider the flat G -structure on \mathbb{R}^n :*

$$\mathcal{B}_G(\mathbb{R}^n) := \mathbb{R}^n \times G \subset \mathcal{B}(\mathbb{R}^n) = \mathbb{R}^n \times \text{GL}(n).$$

Given a vector field ξ in \mathbb{R}^n denote by $\phi_\xi^t : \mathbb{R}^n \rightarrow \mathbb{R}^n$ its flow. Observe that ξ is an infinitesimal automorphism of the G -structure $\mathcal{B}_G(\mathbb{R}^n)$ iff ϕ_ξ^t lifts to an automorphism $\mathcal{B}(\phi_\xi^t) : \mathcal{B}_G(\mathbb{R}^n) \rightarrow \mathcal{B}_G(\mathbb{R}^n)$. The lifted flow $\mathcal{B}(\phi_\xi^t)$ is the flow of a lifted vector field on $\mathcal{B}_G(\mathbb{R}^n)$: in coordinates (m^1, \dots, m^n) , so that $\xi = \xi^i \frac{\partial}{\partial m^i}$, the lifted vector field is given by:⁽¹⁾

$$\tilde{\xi} = \frac{\partial \xi^i}{\partial m^j} \frac{\partial}{\partial p_j^i},$$

where (p_j^i) are the associated coordinates in $\mathcal{B}(\mathbb{R}^n)$ so that a frame $p \in \mathcal{B}(\mathbb{R}^n)$ is written as:

$$p = (p_1^i \frac{\partial}{\partial m^i}, \dots, p_n^i \frac{\partial}{\partial m^i}).$$

It follows that ξ is an infinitesimal automorphism iff:

$$\left[\frac{\partial \xi^i}{\partial m^j} \right]_{i,j=1,\dots,n} \in \mathfrak{g} \subset \mathfrak{gl}(n).$$

Let us assume now that the lifted flow fixes $(0, I) \in \mathbb{R}^n \times \text{GL}(n)$, i.e., the lifted vector field $\tilde{\xi}$ vanishes at this point. If we now prolong to the jet bundle $J^1 \mathcal{B}_G(\mathbb{R}^n)$, we obtain a flow which is generated by a vector field:

$$j^1 \tilde{\xi} = \frac{\partial \xi^i}{\partial m^{j_1} \partial m^{j_2}} \frac{\partial}{\partial p_{j_1, j_2}^i},$$

¹We use the convention of summing over repeated indices.

where $(m^i, p_j^i, p_{j_1, j_2}^i)$ are the induced coordinates on the jet bundle. Note that the coefficients $a_{j_1, j_2}^i = \frac{\partial \xi^i}{\partial m^{j_1} \partial m^{j_2}}$ of $j^1 \tilde{\xi}$ satisfy:

$$[a_{j_1, j_2}^i]_{i, j_1=1, \dots, n} \in \mathfrak{g} \subset \mathfrak{gl}(n),$$

and are symmetric in the indices j_1 and j_2 . Hence, we conclude that:

Lemma 3.3.4 *The lifts of the symmetries of the flat G -structure $\mathcal{B}_G(\mathbb{R}^n) = \mathbb{R}^n \times G$, whose lifted flow fixes $(0, I) \in \mathbb{R}^n \times G$, to the jet space $J^1 \mathcal{B}_G(\mathbb{R}^n)$ generate a Lie subgroup $G^{(1)} \subset \text{Hom}(\mathbb{R}^n, \mathfrak{g})$ with Lie algebra:*

$$\mathfrak{g}^{(1)} := \{T \in \text{Hom}(\mathbb{R}^n, \mathfrak{g}) : T(u)v = T(v)u, \forall u, v \in \mathbb{R}^n\}.$$

This motivates the following definition:

Definition 3.3.5 *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a Lie algebra. The **first prolongation** of \mathfrak{g} is the subspace $\mathfrak{g}^{(1)} \subset \text{Hom}(V, \mathfrak{g})$ consisting of those $T : V \rightarrow \mathfrak{g}$ such that*

$$T(v_1)v_2 = T(v_2)v_1, \quad \forall v_1, v_2 \in V.$$

The k -th prolongation of \mathfrak{g} is the subspace $\mathfrak{g}^{(k)} \subset \text{Hom}(V, \mathfrak{g}^{(k-1)})$ defined inductively by

$$\mathfrak{g}^{(k)} = (\mathfrak{g}^{(k-1)})^{(1)}.$$

A Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ is said to be of finite type k if there exists $k \in \mathbb{N}$ such that $\mathfrak{g}^{(k-1)} \neq 0$ and $\mathfrak{g}^{(k)} = 0$.

Similarly, at the group level, one introduces:

Definition 3.3.6 *Let G be a subgroup of $GL(V)$. The **first prolongation** of G is the abelian subgroup $G^{(1)}$ of $GL(V \oplus \mathfrak{g})$ consisting of those transformations of the form:*

$$(v, \rho) \mapsto (v, \rho + T(v)), \quad \text{with } T \in \mathfrak{g}^{(1)}.$$

Similarly, the k -th prolongation of G is the subgroup $G^{(k)}$ of $GL(V \oplus \mathfrak{g} \oplus \mathfrak{g}^{(1)} \oplus \dots \oplus \mathfrak{g}^{(k)})$ defined inductively by

$$G^{(k)} := (G^{(k-1)})^{(1)}.$$

Now, to each G -structure $\mathcal{B}_G(M)$ we can always reduce the structure group of $J^1 \mathcal{B}_G(M)$ to $G^{(1)}$, obtaining a $G^{(1)}$ -structure:

Proposition 3.3.7 (Singer and Sternberg [32]) *Let $\mathcal{B}_G(M)$ be a G -structure over M with first structure function $c : J^1 \mathcal{B}_G(M) \rightarrow \text{Hom}(\wedge^2 \mathbb{R}^n, \mathbb{R}^n)$. Each choice of a complement C to $\mathcal{A}(\text{Hom}(\mathbb{R}^n, \mathfrak{g}))$ in $\text{Hom}(\wedge^2 \mathbb{R}^n, \mathbb{R}^n)$ determines a sub bundle:*

$$\mathcal{B}_G(M)^{(1)} = \{H_p \in J^1 \mathcal{B}_G(M) : c_{H_p} \in C\},$$

which is a reduction of $J^1 \mathcal{B}_G(M)$ with structure group $G^{(1)}$. Different choices of complements determine sub bundles which are related through right translation by an element in $\text{Hom}(\mathbb{R}^n, \mathfrak{g})$.

Remark 3.3.8 *There is in general no canonical choice of complement C and choosing a 'good' complement is part of the work in solving an equivalence problem. When possible, we look for a complement C which is invariant under the action of G . When such a complement exists we say that the group G is **reductive** (WARNING: do not confuse this with the concept of reductive Lie algebras).*

The $G^{(1)}$ -structure $\mathcal{B}_G(M)^{(1)} \rightarrow \mathcal{B}_G(M)$ is called the **first prolongation** of $\mathcal{B}_G(M)$. Similarly, working inductively, one defines the **k -th prolongation** of $\mathcal{B}_G(M)$:

$$\mathcal{B}_G(M)^{(k)} = (\mathcal{B}_G(M)^{(k-1)})^{(1)},$$

which is $G^{(k)}$ -structure over $\mathcal{B}_G(M)^{(k-1)}$.

The relevance of prolongation for the problem of equivalence is justified by the following basic result:

Theorem 3.3.9 (Singer and Sternberg [32]) *Let $\mathcal{B}_G(M)$ and $\mathcal{B}_G(N)$ be G -structures. Then $\mathcal{B}_G(M)$ and $\mathcal{B}_G(N)$ are equivalent if and only if their first prolongations $\mathcal{B}_G(M)^{(1)}$ and $\mathcal{B}_G(N)^{(1)}$ (corresponding to the same choice of complement C) are equivalent $G^{(1)}$ -structures.*

One can now obtain new necessary conditions for equivalence by looking at the structure function of the prolongation $\mathcal{B}_G(M)^{(1)}$ which is a function

$$c^{(1)} : \mathcal{B}_G(M)^{(1)} \rightarrow \frac{\text{Hom}(\wedge^2(\mathbb{R}^n \oplus \mathfrak{g}), \mathbb{R}^n \oplus \mathfrak{g})}{\mathcal{A}(\text{Hom}(\mathbb{R}^n \oplus \mathfrak{g}, \mathfrak{g}^{(1)}))}$$

called the **second order structure function** of $\mathcal{B}_G(M)$. Then one can continue this process by constructing the second prolongation and analyzing its structure function and so on.

Thus, the importance of structures of finite type is that we can reduce the set of necessary conditions for checking that two G -structures are equivalent to a finite amount. In fact, by the method of prolongation, the equivalence problem for finite type G -structures reduces to an equivalence problem for $\{e\}$ -structures (coframes). Moreover, one can show that G -structures of finite type always have finite dimensional symmetry groups (see Section 6.2).

3.4 Second Order Structure Functions

We now describe the second order structure functions. Assume that we have fixed a complement C to $\mathcal{A}(\text{Hom}(\mathbb{R}^n, \mathfrak{g}))$ in $\text{Hom}(\wedge^2 \mathbb{R}^n, \mathbb{R}^n)$. Let $z = H_p \in \mathcal{B}_G^{(1)} = \mathcal{B}_G(M)^{(1)}$ and let \mathcal{H}_z be a horizontal subspace of $T_z \mathcal{B}_G^{(1)}$. Then $c_{\mathcal{H}_z}^{(1)} \in \text{Hom}(\wedge^2(\mathbb{R}^n \oplus \mathfrak{g}), \mathbb{R}^n \oplus \mathfrak{g})$ and we decompose it into three components:

$$\begin{aligned} \text{Hom}(\wedge^2(\mathbb{R}^n \oplus \mathfrak{g}), \mathbb{R}^n \oplus \mathfrak{g}) &= \text{Hom}(\wedge^2 \mathbb{R}, \mathbb{R}^n \oplus \mathfrak{g}) \oplus \text{Hom}(\mathbb{R}^n \otimes \mathfrak{g}, \mathbb{R}^n \oplus \mathfrak{g}) \oplus \\ &\oplus \text{Hom}(\wedge^2 \mathfrak{g}, \mathbb{R}^n \oplus \mathfrak{g}) \end{aligned}$$

Let us describe each of the components of the (representative of the) second order structure function. We denote by u, v elements of \mathbb{R}^n and by A, B elements of \mathfrak{g} :

- The first component of $c_{\mathcal{H}_z}^{(1)}$ includes the structure function of $\mathcal{B}_G(M)$:

$$c_{\mathcal{H}_z}^{(1)}(u, v) = c_{H_p}(u, v) + R_{\mathcal{H}_z}(u, v),$$

for some $R_{\mathcal{H}_z} \in \text{Hom}(\wedge^2 \mathbb{R}^n, \mathfrak{g})$.

- The second component of $c_{\mathcal{H}_z}^{(1)}$ has the form:

$$c_{\mathcal{H}_z}^{(1)}(A, u) = -Au + S_{\mathcal{H}_z}(A, u)$$

for some $S_{\mathcal{H}_z} \in \text{Hom}(\mathbb{R}^n \otimes \mathfrak{g}, \mathfrak{g})$.

- The last component of $c_{\mathcal{H}_z}^{(1)}$ is given by

$$c_{\mathcal{H}_z}^{(1)}(A, B) = -[A, B]_{\mathfrak{g}}.$$

An important special case occurs when $G^{(1)} = \{e\}$. In this case, a $G^{(1)}$ -structure amounts to choosing a horizontal space at each $p \in \mathcal{B}_G(M)$, which in turn is the same as picking a \mathfrak{g} -valued (not necessarily equivariant) form η on \mathcal{B}_G . The pair (θ, η) is a coframe on \mathcal{B}_G . Now, in this case, the projection from $\mathcal{B}_{G^{(1)}}$ onto \mathcal{B}_G is a diffeomorphism, so we may view the second order structure functions as functions on \mathcal{B}_G . If we do this, we obtain the **structure equations** of the pair (θ, η) :

$$\begin{cases} d\theta = c(\theta \wedge \theta) - \eta \wedge \theta \\ d\eta = R(\theta \wedge \theta) + S(\theta \wedge \eta) - \eta \wedge \eta \end{cases} \quad (3.4.1)$$

where $\eta \wedge \theta$ is the \mathbb{R}^n -valued 2-form obtained from the \mathfrak{g} -action on \mathbb{R}^n and $\eta \wedge \eta$ is the \mathfrak{g} -valued 2-form obtained from the Lie bracket on \mathfrak{g} .

If, additionally, the horizontal spaces can be chosen right invariant (so that $R_{a*}H_p = H_{pa}$ for all $a \in G$), we obtain a principal bundle connection on \mathcal{B}_G with connection form η . In this case, we find that:

- S vanishes identically;
- R is the curvature of the connection;

so we see that, in this case, equations (3.4.1) reduce to the usual structure equations for a connection.

In Chapter 5, we will how to use the structure equations (3.4.1) to solve classification problems for G -structures of type 1, i.e., such that $G^{(1)} = \{e\}$.

3.5 Higher Order Structure Functions

In this section, we will deduce the structure equations of a G -structure such that $G \subset \text{GL}_n$ is a Lie group of type k , i.e., such that $G^{(k)} = \{e\}$. We will do this in an inductive manner. We will see in chapter 5 how to use these structure equations to solve classification problems for G -structures of type k .

Let M be a manifold, $\mathcal{B}_G(M)$ be a G -structure over M and $(\mathcal{B}_G(M))^{(k)}$ its k -th prolongation. Then,

$$\begin{array}{c} (\mathcal{B}_G(M))^{(k)} \xrightarrow{\quad} G^{(k)} \\ \pi^{(k)} \downarrow \\ (\mathcal{B}_G(M))^{(k-1)} \end{array}$$

is a $G^{(k)}$ -structure over $(\mathcal{B}_G(M))^{(k-1)}$. Its tautological form $\theta^{(k)}$ takes values in the vector space $V \oplus \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}^{(k-1)}$. It is easy to see that

$$\theta^{(k)} = (\pi^{(k)})^* \theta^{(k-1)} + \eta^{(k-1)}$$

where $\eta^{(k-1)}$ is a $\mathfrak{g}^{(k-1)}$ -valued differential form and $\theta^{(k-1)}$ is the tautological form on $(\mathcal{B}_G(M))^{(k-1)}$, which takes values in $V \oplus \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}^{(k-2)}$. In order to simplify the notation, we will omit the pullback by $\pi^{(k)}$ and write

$$\theta^{(k)} = (\theta^{(k-1)}, \eta^{(k-1)}).$$

From now on, we will use the notation

$$\mathfrak{g}^l = \mathfrak{g} \oplus \mathfrak{g}^{(1)} \oplus \cdots \oplus \mathfrak{g}^{(l)}, \text{ for each } l \geq 1.$$

Remember that points in $(\mathcal{B}_G(M))^{(k)}$ are pairs (p, H_p) where $p \in (\mathcal{B}_G(M))^{(k-1)}$ and H_p is a horizontal subspace of $T_p(\mathcal{B}_G(M))^{(k-1)}$ such that the $k-1$ -th order structure function $c_{H_p}^{(k-1)}$ lies in a fixed subspace

$$C \subset \text{Hom}(\wedge^2(V \oplus \mathfrak{g}^{k-2}), V \oplus \mathfrak{g}^{k-2}),$$

which is complementary to $\mathcal{A}(\text{Hom}(V \oplus \mathfrak{g}^{k-2}, \mathfrak{g}^{(k-1)}))$. The **k -th order structure function** is a map

$$c^{(k)} : (\mathcal{B}_G(M))^{(k)} \rightarrow \frac{\text{Hom}(\wedge^2(V \oplus \mathfrak{g}^{k-1}), V \oplus \mathfrak{g}^{k-1})}{\mathcal{A}(\text{Hom}(V \oplus \mathfrak{g}^{k-1}, \mathfrak{g}^{(k)}))}.$$

Let $z = (p, H_p) \in (\mathcal{B}_G(M))^{(k)}$ be a frame over $p \in (\mathcal{B}_G(M))^{(k-1)}$ and let $\mathcal{H}_z \subset T_z(\mathcal{B}_G(M))^{(k)}$ be a horizontal subspace. We now describe the representative $c_{\mathcal{H}_z}^{(k)}$ of $c^{(k)}(z)$. In complete analogy with the discussion in the previous section, we decompose $c_{\mathcal{H}_z}^{(k)} \in \text{Hom}(\wedge^2(\mathbb{R}^n \oplus \mathfrak{g}^{k-2} \oplus \mathfrak{g}^{(k-1)}), \mathbb{R}^n \oplus \mathfrak{g}^{k-2} \oplus \mathfrak{g}^{(k-1)})$ into three components:

$$\begin{aligned} & \text{Hom}(\wedge^2(\mathbb{R}^n \oplus \mathfrak{g}^{k-2} \oplus \mathfrak{g}^{(k-1)}), \mathbb{R}^n \oplus \mathfrak{g}^{k-2} \oplus \mathfrak{g}^{(k-1)}) = \\ & = \text{Hom}(\wedge^2(\mathbb{R}^n \oplus \mathfrak{g}^{k-2}), \mathbb{R}^n \oplus \mathfrak{g}^{k-2} \oplus \mathfrak{g}^{(k-1)}) \oplus \\ & \quad \oplus \text{Hom}((\mathbb{R}^n \oplus \mathfrak{g}^{k-2}) \otimes \mathfrak{g}^{(k-1)}, \mathbb{R}^n \oplus \mathfrak{g}^{k-2} \oplus \mathfrak{g}^{(k-1)}) \oplus \\ & \quad \oplus \text{Hom}(\wedge^2 \mathfrak{g}^{(k-1)}, \mathbb{R}^n \oplus \mathfrak{g}^{k-2} \oplus \mathfrak{g}^{(k-1)}) \end{aligned}$$

Let us describe each of the components. We denote by u, v elements of $\mathbb{R}^n \oplus \mathfrak{g}^{k-2}$ and by A, B elements of $\mathfrak{g}^{(k-1)}$:

- The first component of $c_{\mathcal{H}_z}^{(k)}$ includes the structure function of $(\mathcal{B}_G(M))^{(k-1)}$:

$$c_{\mathcal{H}_z}^{(k)}(u, v) = c_{H_p}^{(k-1)}(u, v) + R_{\mathcal{H}_z}^{(k-1)}(u, v),$$

for some $R_{\mathcal{H}_z}^{(k-1)} \in \text{Hom}(\wedge^2(\mathbb{R}^n \oplus \mathfrak{g}^{k-2}), \mathfrak{g}^{(k-1)})$.

- The second component of $c_{\mathcal{H}_z}^{(k)}$ has the form:

$$c_{\mathcal{H}_z}^{(k)}(A, u) = -Au + S_{\mathcal{H}_z}^{(k-1)}(A, u)$$

for some $S_{\mathcal{H}_z}^{(k-1)} \in \text{Hom}(\mathbb{R}^n \otimes \mathfrak{g}^{k-2}, \mathfrak{g}^{(k-1)})$. The first term on the right hand side above refers to the action of $A \in \mathfrak{g}^{(k-1)} \subset \mathfrak{gl}(\mathbb{R}^n \oplus \mathfrak{g}^{k-2})$ on $u \in \mathbb{R}^n \oplus \mathfrak{g}^{k-2}$.

- The last component of $c_{\mathcal{H}_z}^{(k)}$ is given by the bracket of $\mathfrak{g}^{(k-1)}$

$$c_{\mathcal{H}_z}^{(k)}(A, B) = -[A, B]_{\mathfrak{g}^{(k-1)}},$$

which vanishes whenever $k \geq 1$.

When $G^{(k)} = \{e\}$, a $G^{(k)}$ -structure amounts to choosing a horizontal space at each $p \in (\mathcal{B}_G(M))^{(k-1)}$, which in turn is the same as picking a $\mathfrak{g}^{(k-1)}$ -valued (not necessarily equivariant) form $\eta^{(k-1)}$ on $(\mathcal{B}_G(M))^{(k-1)}$. The pair $(\theta^{(k-1)}, \eta^{(k-1)})$ is a coframe on $(\mathcal{B}_G(M))^{(k-1)}$. Note that the tautological form on $(\mathcal{B}_G(M))^{(k)}$ is

$$\theta^{(k)} = (\pi^{(k)})^* \theta^{(k-1)} + \eta^{(k-1)}.$$

Now, in this case, the projection from $(\mathcal{B}_G(M))^{(k)}$ onto $(\mathcal{B}_G(M))^{(k-1)}$ is a diffeomorphism, so we may view the k -th order structure functions as functions on $(\mathcal{B}_G(M))^{(k-1)}$. If we do this, we obtain the **structure equations** of the pair $(\theta^{(k-1)}, \eta^{(k-1)})$:

$$\begin{cases} d\theta^{(k-1)} &= c^{(k-1)}(\theta^{(k-1)} \wedge \theta^{(k-1)}) - \eta^{(k-1)} \wedge \theta^{(k-1)} \\ d\eta^{(k-1)} &= R^{(k-1)}(\theta^{(k-1)} \wedge \theta^{(k-1)}) + S^{(k-1)}(\theta^{(k-1)} \wedge \eta^{(k-1)}) \end{cases} \quad (3.5.1)$$

Remark 3.5.1 *We can continue the process of decomposing the structure functions and the structure equations of $(\mathcal{B}_G(M))^{(k)}$. In fact, the tautological form $\theta^{(k-1)}$ of $(\mathcal{B}_G(M))^{(k-1)}$ takes values in $\mathbb{R}^n \oplus \mathfrak{g}^{k-3} \oplus \mathfrak{g}^{(k-2)}$ and can be written as*

$$\theta^{(k-1)} = (\pi^{(k-1)})^* \theta^{(k-2)} + \eta^{(k-2)}$$

where $\pi^{(k-1)}$ is the projection of $(\mathcal{B}_G(M))^{(k-1)}$ onto $(\mathcal{B}_G(M))^{(k-2)}$, $\theta^{(k-2)}$ is the tautological form of $(\mathcal{B}_G(M))^{(k-2)}$ and $\eta^{(k-2)}$ is a $\mathfrak{g}^{(k-2)}$ -valued 1-form.

If we continue this process, we can decompose $\theta^{(k)}$ into

$$\theta^{(k)} = (\theta, \eta^{(0)}, \eta^{(1)}, \dots, \eta^{(k-1)})$$

where $\eta^{(0)} = \eta$, θ is the tautological form on $\mathcal{B}_G(M)$ and each $\eta^{(l)}$ is a $\mathfrak{g}^{(l)}$ -valued differentiable form. In this way, the structure equations (3.5.1) can also be decomposed into $k+1$ equations involving the components of $\theta^{(k)}$.

Example 3.5.2 Let $\mathcal{B}_G(M)$ be a G -structure, where $G \subset \mathrm{GL}_n$ is a Lie group of type 2, i.e., $G^{(2)} = \{e\}$. In this case, the structure equations (3.5.1) take the form

$$\begin{cases} d\theta^{(1)} &= c^{(1)}(\theta^{(1)} \wedge \theta^{(1)}) - \eta^{(1)} \wedge \theta^{(1)} \\ d\eta^{(1)} &= R^{(1)}(\theta^{(1)} \wedge \theta^{(1)}) + S^{(1)}(\theta^{(1)} \wedge \eta^{(1)}) \end{cases} \quad (3.5.2)$$

Now, in order to further decompose these structure equations in terms of $\theta^{(1)} = (\theta, \eta)$, we must analyze each one of its summands. The first term, $c^{(1)}(\theta^{(1)} \wedge \theta^{(1)})$ was described in details in Section 3.4, where we obtained

$$c^{(1)}(\theta^{(1)} \wedge \theta^{(1)}) = (c(\theta \wedge \theta) - \eta \wedge \theta, R(\theta \wedge \theta) + S(\theta \wedge \eta) - \eta \wedge \eta).$$

The second summand of $d\theta^{(1)}$ is given by the action of $\mathfrak{g}^{(1)}$ on $\mathbb{R}^n \oplus \mathfrak{g}$. Recall that $\mathfrak{g}^{(1)} \subset \mathrm{Hom}(\mathbb{R}^n, \mathfrak{g})$ was embedded into $\mathfrak{gl}(\mathbb{R}^n \oplus \mathfrak{g})$ as the subalgebra formed by the transformations:

$$(u, A) \mapsto (0, T(u)), \quad \text{with } T \in \mathfrak{g}^{(1)}.$$

Thus, we may write

$$\eta^{(1)} \wedge \theta^{(1)} = (0, \eta^{(1)} \wedge \theta).$$

Finally, in order to decompose the equation for $d\eta^{(1)}$, we simply write down the components of $R^{(1)} \in \mathrm{Hom}(\wedge^2(\mathbb{R}^n \oplus \mathfrak{g}), \mathfrak{g}^{(1)})$ and $S^{(1)} \in \mathrm{Hom}((\mathbb{R}^n \oplus \mathfrak{g}) \otimes \mathfrak{g}^{(1)}, \mathfrak{g}^{(1)})$, i.e.,

$$\begin{aligned} \mathrm{Hom}(\wedge^2(\mathbb{R}^n \oplus \mathfrak{g}), \mathfrak{g}^{(1)}) &= \mathrm{Hom}(\wedge^2 \mathbb{R}^n, \mathfrak{g}^{(1)}) \oplus \mathrm{Hom}(\mathbb{R}^n \otimes \mathfrak{g}, \mathfrak{g}^{(1)}) \oplus \mathrm{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g}^{(1)}) \\ R^{(1)} &= R_1^{(1)} + R_2^{(1)} + R_3^{(1)} \end{aligned}$$

and

$$\begin{aligned} \mathrm{Hom}((\mathbb{R}^n \oplus \mathfrak{g}) \otimes \mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}) &= \mathrm{Hom}(\mathbb{R}^n \otimes \mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}) \oplus \mathrm{Hom}(\mathfrak{g} \otimes \mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}) \\ S^{(1)} &= S_1^{(1)} + S_2^{(1)}. \end{aligned}$$

The structure equations (3.5.2) become

$$\begin{cases} d\theta &= c(\theta \wedge \theta) - \eta \wedge \theta \\ d\eta &= R(\theta \wedge \theta) + S(\theta \wedge \eta) - \eta \wedge \eta - \eta^{(1)} \wedge \theta \\ d\eta^{(1)} &= R_1^{(1)}(\theta \wedge \theta) + R_2^{(1)}(\theta \wedge \eta) + R_3^{(1)}(\eta \wedge \eta) + \\ &\quad + S_1^{(1)}(\theta \wedge \eta^{(1)}) + S_2^{(1)}(\eta \wedge \eta^{(1)}). \end{cases} \quad (3.5.3)$$

Chapter 4

The Classification Problem for Coframes

In this chapter, we begin by introducing the equivalence problem for coframes, as presented in [33] and [27]. The analysis of necessary conditions to solve it will lead us to Cartan's realization problem. In order to solve this problem, it will be natural to introduce Maurer-Cartan forms on Lie groupoids, and consider their universal properties. We will then be able to classify all solutions of the realization problem. We then turn to the global equivalence problem. Finally, we describe the symmetries of a realization, as stated in the appendix of [5].

4.1 Equivalence of Coframes

A coframe on an n -dimensional manifold M is a set of everywhere linearly independent 1-forms $\{\theta^1, \dots, \theta^n\}$ on M . There are well known topological obstructions for the existence of global coframes on a given manifold, but we shall not deal with them here. We will however present some results about the global equivalence problem.

Let \bar{M} be another n -dimensional manifold and $\{\bar{\theta}^i\}$ a coframe on \bar{M} . Throughout this text we will use unbarred letters to denote objects on M and barred letters to denote objects on \bar{M} . The (local) equivalence problem then asks:

Problem 4.1.1 (Equivalence Problem) *Does there exist a (locally defined) diffeomorphism*

$$\phi : M \rightarrow \bar{M}$$

satisfying

$$\phi^* \bar{\theta}^i = \theta^i ?$$

We now discuss some necessary conditions for equivalence of coframes. As discussed in the example of the introduction, the structure equations (1.1.1)

$$d\theta^k = \sum_{i < j} C_{ij}^k(m) \theta^i \wedge \theta^j$$

and the structure functions $C_{ij}^k \in C^\infty(M)$ play a crucial role in the study of the equivalence problem. For example, since any coframe $\bar{\theta}$ equivalent to θ must

satisfy

$$\bar{C}_{ij}^k(\phi(m)) = C_{ij}^k(m),$$

the structure functions may be used to obtain a set of necessary conditions for solving the equivalence problem and thus are examples of invariant functions. By this we mean:

Definition 4.1.2 *A function $I \in C^\infty(M)$ is called an **invariant function of a coframe** $\{\theta^i\}$ if for any locally defined self equivalence (symmetry) $\phi : M \rightarrow M$ one has*

$$I \circ \phi = I.$$

Now, for any function $f \in C^\infty(M)$ one can define it's coframe derivatives $\frac{\partial f}{\partial \theta^k}$ as being the coefficients of the differential of f when expressed in terms of the coframe $\{\theta^i\}$,

$$df = \sum_k \frac{\partial f}{\partial \theta^k} \theta^k.$$

Using the fact that $d\phi^* = \phi^*d$, it follows that if $I \in C^\infty(M)$ is an invariant function, then so is $\frac{\partial I}{\partial \theta^k}$ for all $1 \leq k \leq n$. It is then natural to consider the following sets

$$\begin{aligned} \mathcal{F}_0 &= \{C_{ij}^k\} \\ \mathcal{F}_1 &= \left\{ C_{ij}^k, \frac{\partial C_{ij}^k}{\partial \theta^l} \right\} \\ &\vdots \\ \mathcal{F}_t &= \left\{ C_{ij}^k, \frac{\partial C_{ij}^k}{\partial \theta^l}, \dots, \frac{\partial^s C_{ij}^k}{\partial \theta^{l_1} \dots \partial \theta^{l_t}} \right\} \\ &\vdots \end{aligned}$$

which give us necessary conditions to solve an equivalence problem. Schematically, we may write

$$\phi^* \bar{\mathcal{F}}_t = \mathcal{F}_t$$

for all $t \geq 0$.

To be able to find effective criteria for equivalence, we must reduce this infinite set of necessary conditions to a finite number. To accomplish this, we first note that the elements in \mathcal{F}_t in general are not functionally independent, meaning that we can find elements f_1, \dots, f_l in \mathcal{F}_t such that there exists a function $H : \mathbb{R}^{l-1} \rightarrow \mathbb{R}$ satisfying

$$f_l = H(f_1, \dots, f_{l-1}).$$

Locally, in a neighborhood of a point $m \in M$, f_1, \dots, f_l are functionally independent if and only if their differentials are linearly independent at m .

Next, we observe that if $\{\theta^i\}$ and $\{\bar{\theta}^i\}$ are equivalent coframes, and $f_l = H(f_1, \dots, f_{l-1})$ then the corresponding elements $\bar{f}_1, \dots, \bar{f}_l$ in $\bar{\mathcal{F}}_s$ with the same indexes must satisfy the same functional relation

$$\bar{f}_l = H(\bar{f}_1, \dots, \bar{f}_{l-1})$$

for the same function H . This shows that we do not need to deal with all the invariant functions in \mathcal{F}_t , but only with those that are functionally independent.

To be able to make this more precise and work with some generality, we will impose a regularity hypothesis. For this, let $\mathcal{C} \subset C^\infty(M)$ be any set of functions on M , and define:

Definition 4.1.3 *The **rank of \mathcal{C} at $m \in M$** , denoted by $r_m(\mathcal{C})$ is the dimension of the vector space spanned by $\{df_m : f \in \mathcal{C}\}$. \mathcal{C} is called **regular at m** if $r_{m'}(\mathcal{C}) = r_m(\mathcal{C})$ for all m' near enough to m in M .*

By the implicit function theorem, if \mathcal{C} is regular of rank k at m then we can find a coordinate system (m_1, \dots, m_n) around m such that

$$m_1 = f^1, \dots, m_k = f^k$$

and every $f \in \mathcal{C}$ can be written as

$$f = f(m_1, \dots, m_k)$$

in a neighborhood of m

Definition 4.1.4 *A coframe θ is called **fully regular at $m \in M$** if the sets \mathcal{F}_t are regular at m for all $t \geq 0$.*

We observe that the set of points $m \in M$ on which a coframe is fully regular forms an open dense set. Now, define the integers

$$k_t(m) = r_m(\mathcal{F}_t).$$

Since $\mathcal{F}_t \subset \mathcal{F}_{t+1}$ for all t it follows that

$$0 \leq k_0(m) \leq k_1(m) \leq \dots \leq k_t(m) \leq \dots \leq n.$$

Finally, it is not hard to show ([27] or [33]) that if $\{\theta^i\}$ is fully regular at m , then there exists an integer s such that

$$k_{s-1}(m) < k_s(m)$$

and

$$k_s(m) = k_{s+1}(m) = k_{s+2}(m) = \dots.$$

We call s the order of θ at m and $k_s(m)$ the rank of θ at m .

For fully regular coframes, we can now reduce our necessary conditions for *local equivalence* to a finite number. Suppose θ is a fully regular coframe at m of order s and rank d at m . Then we can find a set $\{h_1, \dots, h_d\}$ of invariant functions in \mathcal{F}_s that 'generate' \mathcal{F}_t in a neighborhood of m , for all $t \geq 0$. It is clear that if another fully regular coframe $\bar{\theta}$ is locally equivalent to θ at m through $\phi : M \rightarrow \bar{M}$ then necessarily it's order is s and it's rank is d at $\bar{m} = \phi(m)$. Furthermore, the set $\{\bar{h}_1, \dots, \bar{h}_d\}$ of invariant functions in $\bar{\mathcal{F}}_s$ with corresponding index also 'generate' $\bar{\mathcal{F}}_t$ in a neighborhood of \bar{m} , for all $t \geq 0$. Finally, if

$$h_i(m) = \bar{h}_i(\phi(m))$$

then

$$\phi^* \bar{\mathcal{F}}_t = \mathcal{F}_t \tag{4.1.1}$$

for all $t \geq 0$.

We can now summarize all essential data obtained from a fully regular coframe of order s and rank d , in a neighborhood of a point, in the following convenient manner. First of all we have a set of invariant functions which determine a map

$$h : M \rightarrow \mathbb{R}^d$$

$$h(m) = (h_1(m), \dots, h_d(m))$$

Next, since $\{h_i\}$ are independent and generate \mathcal{F}_t for all $t \geq 0$ (in particular \mathcal{F}_0), if we see h_1, \dots, h_d as coordinates on an open set $X \subset \mathbb{R}^d$, then the structure functions may be seen as $C_{ij}^k \in C^\infty(X)$. Finally, differentiating h_a we obtain

$$dh_a = \sum_i F_i^a \theta^i. \quad (4.1.2)$$

Again, for the same reason, F_i^a can be seen as elements of $C^\infty(X)$. These functions are all related through

$$d\theta^k = \sum_{i < j} C_{ij}^k(h) \theta^i \wedge \theta^j \quad (4.1.3)$$

$$dh_a = \sum_i F_i^a(h) \theta^i. \quad (4.1.4)$$

We have thus obtained the initial data of Cartan's realization problem 4.2.1 which we shall study in detail below.

4.2 Cartan's Realization Problem and Global Equivalence

In Cartan's realization problem, one is given a set of functions and wants to determine what are all coframes which have these functions as their structure functions. More precisely,

Problem 4.2.1 (Cartan's Realization Problem) *One is given:*

- an integer $n \in \mathbb{N}$,
- an open set $X \subset \mathbb{R}^d$,
- a set of functions $C_{ij}^k \in C^\infty(X)$ with indexes $1 \leq i, j, k \leq n$,
- and a set of functions $F_i^a \in C^\infty(X)$ with $1 \leq a \leq d$

and asks for the existence of

- an n -dimensional manifold M
- a coframe $\{\theta^i\}$ on M
- a function $h : M \rightarrow X$

satisfying

$$d\theta^k = \sum_{i < j} C_{ij}^k(h(m))\theta^i \wedge \theta^j \quad (4.2.1)$$

$$dh^a = \sum_i F_i^a(h(m))\theta^i. \quad (4.2.2)$$

This problem was proposed and solved by Cartan in [9]. The exposition of the problem that we presented here is inspired on the appendix of [5].

We will also be interested in answering the following questions:

Local Classification Problem What are all germs of coframes which solve the realization problem?

Local Equivalence Problem When are two of these germs of coframes equivalent?

Globalization Problem When can two of these germs be realized by the same global coframe?

There is also a *global classification problem* that we can deal with once we know how to solve Cartan's realization problem. In order to state it, we first make some definitions.

Definition 4.2.2 Let (M_1, θ_1, h_1) and (M_2, θ_2, h_2) be two realizations of (n, X, C_{ij}^k, F_i^a) .

- A **morphism of realizations** is a local diffeomorphism $\phi : M_1 \rightarrow M_2$ such that $\phi^*\theta_2 = \theta_1$ (and in particular, $h_2 \circ \phi = h_1$).
- A **covering of realizations** is a surjective morphism of realizations $\phi : (M_1, \theta_1, h_1) \rightarrow (M_2, \theta_2, h_2)$. In this case, we say that M_1 is a **realization cover** of M_2 .

If (M, θ, h) is a realization, and $\pi : \tilde{M} \rightarrow M$ is a covering map, then $(\tilde{M}, \pi^*\theta, h \circ \pi)$ is also a realization, called the **induced realization**, and the covering map is a morphism of realizations.

The difficulties in stating and solving the global equivalence problem are due to the fact that the invariants we can construct do *not* distinguish between a realization and the induced realization on its universal cover. Thus, in order to deal with global aspects, we make the following definition:

Definition 4.2.3 Let (M_1, θ_1, h) and (M_2, θ_2, h_2) be realization of a Cartan's problem. We will say that M_1 and M_2 are **globally equivalent, up to covering**, if they have a common realization cover M , i.e., if there exists a realization (M, θ, h) and surjective realization morphisms

$$\begin{array}{ccc} & M & \\ \phi_1 \swarrow & & \searrow \phi_2 \\ M_1 & & M_2 \end{array}$$

We then have:

Problem 4.2.4 (Global Classification Problem) *Given a realization problem, what are all of its solutions up to global equivalence, up to covering?*

The remainder of this chapter is devoted to answering the questions proposed in this section, and analyzing its consequences.

4.3 Necessary Conditions for Existence of Solutions

Obvious necessary conditions for solving the realization problem are obtained by using $d^2 = 0$. In fact we have the following proposition.

Proposition 4.3.1 *For a Cartan's realization problem to have a solution it is necessary that $-C_{ij}^k, F_i^a \in C^\infty(\mathbb{R}^d)$ be the structure functions of a Lie algebroid A over X .*

Proof. If one differentiates equations (4.2.1) and (4.2.2), and sets them equal to zero, one obtains

$$\sum_{b=1}^d \left(F_i^b \frac{\partial F_j^a}{\partial x_b} - F_j^b \frac{\partial F_i^a}{\partial x_b} \right) = - \sum_{l=1}^r C_{ij}^l F_l^a \quad (4.3.1)$$

for all $1 \leq i, j \leq r, 1 \leq a \leq d$ and

$$- \sum_{b=1}^d \left(F_j^b \frac{\partial C_{kl}^i}{\partial x_b} + F_k^b \frac{\partial C_{lj}^i}{\partial x_b} + F_l^b \frac{\partial C_{jk}^i}{\partial x_b} \right) = \sum_{m=1}^r (C_{mj}^i C_{kl}^m + C_{mk}^i C_{lj}^m + C_{ml}^i C_{jk}^m) \quad (4.3.2)$$

for all $1 \leq i, j, k, l \leq r$. Note that these are exactly the differential equations (2.1.3) and (2.1.4) that define a Lie algebroid in local coordinates, with a minus sign on the right hand side, thus proving the proposition. ■

Definition 4.3.2 *The Lie algebroid constructed out of the initial data of a Cartan's realization problem will be called the **classifying algebroid** of the problem.*

It turns out that the reciprocal is also true, but before we plunge into its proof, let us present the simplest example, which serves as inspiration for the more general results.

Example 4.3.3 *Suppose $X = \{pt\}$, i.e., $d = 0$. Then C_{ij}^k are all constant and $F_i^a \equiv 0$. The necessary condition for finding a realization is that $-C_{ij}^k$ form the structure constants of a Lie algebra \mathfrak{g} . An obvious realization is to take any Lie group G integrating \mathfrak{g} and consider a basis of right invariant Maurer-Cartan forms ω_{MC} . Moreover, the universal property of Maurer-Cartan forms implies that any other realization is obtained locally by taking pullback of ω_{MC} . This means that locally, this realization problem has only one solution up to equivalence.*

4.4 Maurer-Cartan Forms

To be able to extend the preceding example to the general case, we will generalize the usual Maurer-Cartan equation for Lie algebra valued differential forms to Lie algebroid valued differential forms. We begin by defining Maurer-Cartan forms on Lie groupoids.

Let \mathcal{F} be a foliation on M . Recall that an \mathcal{F} -foliated k -form on M is a section of $\wedge^k(T\mathcal{F})^*$, i.e., it is a k -form on M which is only defined on k -tuples of vector fields which are all tangent to the foliation.

Definition 4.4.1 *An \mathcal{F} -foliated differential 1-form on M with values in a Lie algebroid $A \rightarrow X$ is a bundle map*

$$\begin{array}{ccc} T\mathcal{F} & \xrightarrow{\theta} & A \\ \downarrow & & \downarrow \\ M & \xrightarrow{h} & X \end{array} \quad (4.4.1)$$

which is compatible with the anchors, i.e., such that $\# \circ \theta(\xi) = h_*\xi$ for all $\xi \in \mathfrak{X}(\mathcal{F}) = \Gamma(T\mathcal{F})$.

On every Lie groupoid we may define a canonical \mathfrak{s} -foliated differential 1-form with values in its Lie algebroid (covering the target map)

Definition 4.4.2 *The Maurer-Cartan form of a Lie groupoid \mathcal{G} is the Lie algebroid valued \mathfrak{s} -foliated 1-form*

$$\begin{array}{ccc} T^{\mathfrak{s}}\mathcal{G} & \xrightarrow{\omega_{MC}} & A \\ \downarrow & & \downarrow \\ \mathcal{G} & \xrightarrow{\mathfrak{t}} & X \end{array}$$

defined by

$$\omega_{MC}(\xi) = (dR_{g^{-1}})_g(\xi) \in A_{\mathfrak{t}(g)} \quad (4.4.2)$$

for $\xi \in T^{\mathfrak{s}}\mathcal{G}$.

Let \mathcal{G} be a Lie groupoid. Since right translation by an element $g \in \mathcal{G}$ is a map

$$R_g : \mathfrak{s}^{-1}(\mathfrak{t}(g)) \rightarrow \mathfrak{s}^{-1}(\mathfrak{s}(g)) \quad (4.4.3)$$

it does not make sense to say that a form on \mathcal{G} is right invariant. We must restrict ourselves to \mathfrak{s} -foliated differential forms.

We will say that an \mathfrak{s} -foliated differential form ω on \mathcal{G} is **right invariant** if

$$\omega(\xi) = \omega((R_g)_*(\xi)) \quad (4.4.4)$$

for every ξ tangent to an \mathfrak{s} -fiber and $g \in \mathcal{G}$. Equivalently we will write

$$(R_g)^*\omega = \omega. \quad (4.4.5)$$

Clearly, the Maurer-Cartan form of a Lie groupoid is right invariant.

We now proceed to define the generalized Maurer-Cartan equation for Lie algebroid valued \mathcal{F} -foliated differential forms. Since there is no canonical way of differentiating forms with values in a vector bundle, we introduce an arbitrary connection ∇ on the vector bundle $A \rightarrow X$. Let $\theta \in \Omega_{\text{fol}}^1(M; A)$ be an A -valued foliated form. For $\xi_1, \xi_2 \in \mathfrak{X}(F)$ define $d_{\nabla}\theta(\xi_1, \xi_2)$ to be the section of $\Gamma(h^*A)$ given by

$$d_{\nabla}\theta(\xi_1, \xi_2) = \nabla_{\xi_1}\theta(\xi_2) - \nabla_{\xi_2}\theta(\xi_1) - \theta([\xi_1, \xi_2]) \quad (4.4.6)$$

where we have written ∇ for the pullback connection on h^*A . At this step the reader should be warned that $d_{\nabla}^2 \neq 0$ so that d_{∇} is not an exterior differential operator.

Next, given another A -valued foliated 1-form $\phi : TM \rightarrow A$ over the same base map h , we want to make sense to $[\theta, \phi]$. Before we do so, in order to motivate our definition, let us present an example.

Example 4.4.3 *Recall that if ∇ is a connection on a Lie algebroid $A \rightarrow X$, we can define the torsion of ∇ by*

$$T_{\nabla}(\alpha, \beta) = \nabla_{\#\alpha}\beta - \nabla_{\#\beta}\alpha - [\alpha, \beta].$$

Now let \mathfrak{g} be a Lie algebra seen as a Lie algebroid over a point. Let ∇ be the trivial zero connection on \mathfrak{g} . Then for any $\alpha, \beta \in \mathfrak{g}$ we have

$$[\alpha, \beta]_{\mathfrak{g}} = -T_{\nabla}(\alpha, \beta).$$

It is then natural to define

$$[\theta, \phi]_{\nabla}(\xi_1, \xi_2) = -(T_{\nabla}(\theta(\xi_1), \phi(\xi_2)) + T_{\nabla}(\phi(\xi_1), \theta(\xi_2))) \quad (4.4.7)$$

where T_{∇} is the torsion of the pullback connection ∇ , as described in (2.1.7).

Definition 4.4.4 *The **generalized Maurer-Cartan equation** for foliated Lie algebroid valued differential 1-forms (θ, h) is*

$$d_{\nabla}\theta + \frac{1}{2}[\theta, \theta]_{\nabla} = 0. \quad (4.4.8)$$

Since this equation is just $R_{\theta} \equiv 0$ (equation (2.1.9)) we obtain:

Proposition 4.4.5 *The generalized Maurer-Cartan equation is independent of the choice of connection. Moreover, an A -valued \mathcal{F} -foliated 1-form (θ, h) satisfies the generalized Maurer-Cartan equation if and only if it is a Lie algebroid morphism $T\mathcal{F} \rightarrow A$.*

As should be expected by the reader, we have the following proposition:

Proposition 4.4.6 *The Maurer-Cartan form ω_{MC} of \mathcal{G} satisfies the generalized Maurer-Cartan equation.*

Proof. This proposition is a consequence of the following calculation:

$$\begin{aligned} d_{\nabla}\omega_{MC}(\xi_1, \xi_2) &= \nabla_{\mathbf{t}_*\xi_1}\omega_{MC}(\xi_2) - \nabla_{\mathbf{t}_*\xi_2}\omega_{MC}(\xi_1) - \omega_{MC}([\xi_1, \xi_2]) \\ &= \nabla_{\#\alpha}\beta - \nabla_{\#\beta}\alpha - [\alpha, \beta] \\ &= T_{\nabla}(\alpha, \beta) \\ &= T_{\nabla}(\omega_{MC}(\xi_1), \omega_{MC}(\xi_2)) \end{aligned} \quad (4.4.9)$$

where ξ_1 and ξ_2 are right invariant vector fields on \mathcal{G} and $\alpha, \beta \in \Gamma(A)$ are their generators. ■

4.5 The Local Universal Property

What we will now show is that any 1-form on a differentiable manifold, with values in an integrable Lie algebroid A , satisfying the generalized Maurer-Cartan equation is locally the pullback of the Maurer-Cartan form on a Lie groupoid integrating A . We will need the following lemma.

Lemma 4.5.1 *Let \mathcal{F} be a foliation on a manifold M and let Ω be an \mathcal{F} -foliated 1-form (over h) on M with values in a Lie algebroid $A \rightarrow X$ equipped with an arbitrary connection ∇ . Assume that the distribution $\mathcal{D} = \{\ker \Omega_x : x \in M\} \subset T\mathcal{F} \subset TM$ has constant rank. Then \mathcal{D} is integrable if and only if $d_{\nabla}\Omega(\xi_1, \xi_2) = 0$ whenever $\xi_1, \xi_2 \in \mathcal{D}$.*

Proof. Choose a local basis $\xi_1, \dots, \xi_r \in \mathfrak{X}(\mathcal{F})$ of \mathcal{D} in an open set of M . Then by Frobenius Theorem, \mathcal{D} is integrable if and only if $[\xi_i, \xi_j] \in \text{span}\{\xi_1, \dots, \xi_r\}$ for all $1 \leq i, j \leq r$, which happens if and only if $\Omega([\xi_i, \xi_j]) = 0$ for all $1 \leq i, j \leq r$. Since $\Omega(\xi_i) = 0$ for all $1 \leq i \leq r$ we have

$$d_{\nabla}\Omega(\xi_i, \xi_j) = \bar{\nabla}_{\xi_i}\Omega(\xi_j) - \bar{\nabla}_{\xi_j}\Omega(\xi_i) - \Omega([\xi_i, \xi_j]) = -\Omega([\xi_i, \xi_j]) \quad (4.5.1)$$

from which the result follows. ■

Proposition 4.5.2 *Let θ be a 1-form (over h) on a manifold M , with values in a Lie algebroid A , that satisfies the generalized Maurer-Cartan equation. Let \mathcal{G} be a Lie groupoid integrating A and denote by ω_{MC} its right invariant Maurer-Cartan form. Then for each $m \in M$ and $g \in \mathcal{G}$ such that $h(m) = \mathbf{t}(g)$, there exists a unique locally defined (in a neighborhood of m) diffeomorphism $\phi : M \rightarrow \mathbf{s}^{-1}(\mathbf{s}(g))$ satisfying*

$$\phi(m) = g \quad (4.5.2)$$

and

$$\phi^*\omega_{MC} = \theta. \quad (4.5.3)$$

Remark 4.5.3 *We can summarize the proposition by saying, that at least locally, there is a unique map $\phi : M \rightarrow \mathcal{G}$ which makes the diagram of Lie algebroid morphisms*

$$\begin{array}{ccc}
 TM & \overset{\phi^*}{\dashrightarrow} & T^s\mathcal{G} \\
 \downarrow & \searrow \theta & \swarrow \omega_{MC} \\
 & & A \\
 \downarrow & & \downarrow \phi \\
 M & \overset{\phi}{\dashrightarrow} & \mathcal{G} \\
 \downarrow h & & \downarrow \mathbf{t} \\
 & & X
 \end{array}$$

commute.

Proof. To prove this proposition we will construct the graph of ϕ by integrating a convenient distribution. Uniqueness will then follow from the uniqueness of integral submanifolds. Let $M \times_{h, \mathbf{t}} \mathcal{G} = \{(m, g) \in M \times \mathcal{G} : h(m) = \mathbf{t}(g)\}$

denote the fibered product over X (which is a manifold because \mathbf{t} is a surjective submersion) equipped with the foliation \mathcal{F} given by the fibers of $\mathbf{s} \circ \pi_{\mathcal{G}}$. On $M \times_{h,\mathbf{t}} \mathcal{G}$ consider the A -valued \mathcal{F} -foliated 1-form

$$\Omega = \pi_M^* \theta - \pi_{\mathcal{G}}^* \omega_{\text{MC}}. \quad (4.5.4)$$

Let $\mathcal{D} = \ker \Omega$ denote the associated distribution on $M \times_{h,\mathbf{s}} \mathcal{G}$.

In order to apply the previous lemma, we must first show that \mathcal{D} has constant rank. We will do this by showing that

$$(\mathrm{d}\pi_M)_{(m,g)}|_{\mathcal{D}_{(m,g)}} : \mathcal{D}_{(m,g)} \rightarrow T_m M \quad (4.5.5)$$

is an isomorphism for each $(m, g) \in M \times_{h,\mathbf{t}} \mathcal{G}$. Note that this already implies that if \mathcal{D} is integrable then its leaf through (m, g) is locally the graph of a locally defined diffeomorphism from M to the \mathbf{s} -fiber containing g of \mathcal{G} .

Suppose that $(\mathrm{d}\pi_M)_{(m,g)}(v, w) = 0$ for some $(v, w) \in \mathcal{D}_{(m,g)}$. Since \mathcal{D} is contained in $T\mathcal{F}$, it follows that w is \mathbf{s} -vertical and $\omega_{\text{MC}}(w)$ is simply the right translation of w to $\mathbf{1}_{\mathbf{t}(g)}$ and thus,

$$\begin{aligned} (\mathrm{d}\pi_M)_{(m,g)}(v, w) = 0 &\implies v = 0 \\ &\implies \omega_{\text{MC}}(w) = 0 \quad (= \theta(v)) \\ &\implies w = 0 \\ &\implies (v, w) = 0 \end{aligned}$$

so that $(\mathrm{d}\pi_M)_{(m,g)}$ is injective. Now, if $v \in T_m M$ then $(v, (R_g)_* \theta(v))$ is an element of $\mathcal{D}_{(m,g)}$ so $(\mathrm{d}\pi_M)_{(m,g)}$ is also surjective.

Having accomplished this, we may use the preceding lemma to complete the proof. We compute omitting the pullbacks for simplicity,

$$\begin{aligned} \mathrm{d}_{\nabla} \Omega &= \mathrm{d}_{\nabla} \theta - \mathrm{d}_{\nabla} \omega_{\text{MC}} \\ &= -\frac{1}{2}[\theta, \theta] + \frac{1}{2}[\omega_{\text{MC}}, \omega_{\text{MC}}]. \end{aligned}$$

Replacing $\theta = \Omega + \omega_{\text{MC}}$ we obtain

$$\begin{aligned} \mathrm{d}_{\nabla} \Omega &= -\frac{1}{2}[\Omega + \omega_{\text{MC}}, \Omega + \omega_{\text{MC}}] + \frac{1}{2}[\omega_{\text{MC}}, \omega_{\text{MC}}] \\ &= -\frac{1}{2}[\Omega, \Omega] - \frac{1}{2}[\Omega, \omega_{\text{MC}}] - \frac{1}{2}[\omega_{\text{MC}}, \Omega]. \end{aligned}$$

So $\mathrm{d}_{\nabla} \Omega(\xi_1, \xi_2) = 0$ whenever $\Omega(\xi_1) = 0 = \Omega(\xi_2)$ and \mathcal{D} is integrable which concludes the proof. ■

Remark 4.5.4 *With a slight modification, the proposition above is still valid even when A is not integrable. In fact, since an A -valued Maurer-Cartan form on M*

$$(\theta, h) \in \Omega^1(M, A)$$

is the same as a Lie algebroid morphism

$$\begin{array}{ccc} TM & \xrightarrow{\theta} & A \\ \downarrow & & \downarrow \\ M & \xrightarrow{h} & X \end{array}$$

it follows that $h(M)$ lies in a single orbit of A in X . By restricting h to a small enough neighborhood, we may assume that its image is a contractible open set $U \subset L$ in an orbit L of A in X . Then, the restriction of A to U is integrable [12] and we may proceed as in the proof of the proposition.

As a consequence of the proposition we obtain the following useful corollary:

Corollary 4.5.5 *Let \mathcal{G} be a Lie groupoid with Maurer-Cartan form ω_{MC} . If $\phi : \mathfrak{s}^{-1}(x) \rightarrow \mathfrak{s}^{-1}(y)$ is a symmetry of ω_{MC} (i.e., $\phi^*\omega_{MC} = \omega_{MC}$) then x and y belong to the same orbit of \mathcal{G} and ϕ is locally of the form $\phi = R_g$ for some $g \in \mathcal{G}$.*

Proof. All we must show is that x and y belong to the same orbit of \mathcal{G} , and the corollary then follows from the uniqueness part of Proposition 4.5.2. For this, note that $(\omega_{MC})_{\mathbf{1}_x}$ takes values in A_x . On the other hand, $(\phi^*\omega_{MC})_{\mathbf{1}_x}$ takes value in $A_{\mathfrak{t}(\phi(\mathbf{1}_x))}$. Since $\phi^*\omega_{MC} = \omega_{MC}$ it follows that $x = \mathfrak{t}(\phi(\mathbf{1}_x))$ and thus $\phi(\mathbf{1}_x)$ is an arrow joining y to x . ■

4.6 The Global Universal Property

There is a more conceptual proof of Proposition 4.5.2 that will lead us to a global version of the universal property of Maurer-Cartan forms. Since the proposition above is a local result we may assume for a moment that M is simply connected. The source simply connected Lie groupoid integrating TM is then the pair groupoid $M \times M \rightrightarrows M$.

By Lie II (Proposition 2.3.2), there exists a unique morphism of Lie groupoids

$$\begin{array}{ccc} M \times M & \xrightarrow{H} & \mathcal{G} \\ \Downarrow & & \Downarrow \\ M & \xrightarrow{h} & X \end{array} \quad (4.6.1)$$

integrating θ . However, after fixing a point m_0 in M we may write

$$H(m, m') = \phi(m)\phi(m')^{-1} \quad (4.6.2)$$

where $\phi : M \rightarrow \mathfrak{s}^{-1}(h(m_0)) \subset \mathcal{G}$ is defined by

$$\phi(m) = H(m, m_0). \quad (4.6.3)$$

But then, ϕ satisfies

$$\phi(m_0) = \mathbf{1}_{h(m_0)} \quad (4.6.4)$$

$$\phi^*\omega_{MC} = \theta. \quad (4.6.5)$$

In general, when M is not simply connected, the Lie algebroid morphism θ integrates to a Lie groupoid morphism

$$\begin{array}{ccc} \Pi_1(M) & \xrightarrow{F} & \mathcal{G} \\ \Downarrow & & \Downarrow \\ M & \xrightarrow{h} & X \end{array} \quad (4.6.6)$$

where $\Pi_1(M)$ denotes the fundamental groupoid of M . The problem of determining when θ is globally the pullback of the Maurer-Cartan form on a Lie groupoid \mathcal{G} integrating A is reduced to determining when the morphism F above factors through the groupoid covering projection

$$\begin{aligned} p : \Pi_1(M) &\rightarrow M \times M \\ p([\gamma]) &= (\gamma(1), \gamma(0)), \end{aligned} \tag{4.6.7}$$

as will be shown below.

Theorem 4.6.1 *Let A be an integrable Lie algebroid with source simply connected Lie groupoid $\mathcal{G}(A)$ and $(\theta, h) \in \Omega^1(M, A)$ be an A -valued differential 1-form on M . Then, there exists a globally defined local diffeomorphism*

$$\phi : M \rightarrow \mathfrak{s}^{-1}(h(m_0))$$

satisfying

$$\begin{cases} \phi(m_0) &= \mathbf{1}_{h(m_0)} \\ \phi^* \omega_{MC} &= \theta \end{cases} \tag{4.6.8}$$

if and only if

1. (local obstruction) θ satisfies the generalized Maurer-Cartan equation and
2. (global obstruction) the Lie groupoid morphism F integrating θ is trivial when restricted to the fundamental group at m_0 , that is, the isotropy group at m_0 .

Proof. We begin by proving that both conditions are necessary. It is clear that if $\theta = \phi^* \omega_{MC}$ then θ satisfies the generalized Maurer-Cartan equation. So all we have to prove is that F is trivial on the isotropy group $\pi_1(M, m_0)$. Suppose that ϕ exists. Then the map

$$H : M \times M \rightarrow \mathcal{G} \tag{4.6.9}$$

given by

$$H(m, m') = \phi(m)\phi(m')^{-1}.$$

defines a Lie groupoid morphism over the map $\mathbf{t} \circ \phi$.

It then follows from $\phi^* \omega_{MC} = \theta$ that $\mathbf{t} \circ \phi = h$ and that H integrates θ . To see this, notice that if f and g are maps from M to \mathcal{G} such that $\varphi(m, m') = f(m) \cdot g(m')$ is well defined then

$$(d\varphi)_{(m, m')}(v, w) = (dL_{f(m)})_{g(m')}(dg)_{m'}(w) + (dR_{g(m')})_{f(m)}(df)_m(v) \tag{4.6.10}$$

for $v, w \in T_{(m, m')}(M \times M)$. Thus, in our case we obtain

$$\begin{aligned} (dH)_{(m, m)}(0, v) &= (dR_{\phi(m)^{-1}})_{\phi(m)}(d\phi)_m(v) \\ &= \omega_{MC}(\phi_* v) \\ &= \phi^* \omega_{MC}(v) \\ &= \theta(v). \end{aligned} \tag{4.6.11}$$

where $v \in T_m M$.

Finally, since H integrates θ we obtain the commutative diagram

$$\begin{array}{ccc} \Pi_1(M) & \xrightarrow{F} & \mathcal{G} \\ p \downarrow & \nearrow H & \\ M \times M & & \end{array} \quad (4.6.12)$$

where p denotes the covering projection $p([\gamma]) = (\gamma(1), \gamma(0))$. Thus, F only depends on the endpoints of γ and not on its homotopy class, proving the claim.

Reciprocally, suppose both conditions are satisfied. Then by Proposition 4.5.2 and condition (1), it follows that θ is locally the pullback of ω_{MC} by a map ϕ_{loc} . However, since F only depends on the end points of paths γ and not on their homotopy class, F factors through

$$\begin{array}{ccc} \Pi_1(M) & \xrightarrow{F} & \mathcal{G} \\ p \downarrow & \dashrightarrow H & \\ M \times M & & \end{array} \quad (4.6.13)$$

Thus, by defining

$$\phi(m) = H(m, m_0) \quad (4.6.14)$$

we obtain a global map which, by the uniqueness result in Proposition 4.5.2, restricts to the locally defined maps ϕ_{loc} . It follows that $\phi^* \omega_{\text{MC}} = \theta$ and $\phi(m_0) = \mathbf{1}_{h(m_0)}$ proving the theorem. ■

Remark 4.6.2 *By the results quoted in Section 2.3, we observe that we can express the global obstruction condition of the preceding theorem at the infinitesimal level, i.e., without any mention to the Lie groupoid \mathcal{G} integrating A . In fact, given any curve $\gamma : I \rightarrow M$, the path $\theta \circ \dot{\gamma} : I \rightarrow A$ satisfies*

$$\#(\theta \circ \dot{\gamma}(t)) = \frac{d}{dt}(h \circ \gamma)(t) \text{ for all } t \in I$$

and thus, is an A -path (see Definition 2.3.3). We can then rewrite the condition as (see Section 2.3):

- (Global Obstruction) *For every loop γ in M , homotopic to the constant curve at m , the A -path $\theta \circ \dot{\gamma}$ is A -homotopic to the constant zero A -path at $h(m)$.*

Note also that this last condition can be expressed in terms of a differential equation. It can be stated as [12]:

- (Global Obstruction) *For every homotopically trivial loop γ in (M, m) , there exists a variation $a_\epsilon(t) = a(\epsilon, t)$ of A -paths joining $a_0 = \theta \circ \dot{\gamma}$ to the constant zero path $a_1 = 0_{h(m)}$ such that the solution $b(\epsilon, t)$ of the differential equation*

$$\partial_t b - \partial_\epsilon a = T_\nabla(a, b), \quad b(\epsilon, 0) = 0$$

satisfies $b(\epsilon, 1) = 0$ for all $\epsilon \in I$, where ∇ is an arbitrary connection on A .

4.7 Solving the Local Classification Problem

We now return to Cartan's realization problem. We have seen that the necessary conditions for solving Cartan's problem, obtained by $d^2 = 0$, imply that $-C_{ij}^k$ and F_i^a form the structure functions of a Lie algebroid $A \rightarrow X$ written in some coordinates x_1, \dots, x_d and sections $\alpha_1, \dots, \alpha_n$. Let us suppose for a moment that A is integrable and that \mathcal{G} is a Lie groupoid integrating A . Denote by ω_{MC} the Maurer-Cartan form of \mathcal{G} and by $(\omega_{\text{MC}}^1, \dots, \omega_{\text{MC}}^n)$ its components with respect to the basis $\alpha_1, \dots, \alpha_n$. Then, it is clear that for each $x_0 \in X$, $(\mathbf{s}^{-1}(x_0), \omega_{\text{MC}}^i, \mathbf{t})$ is a realization of Cartan's problem with initial data (n, X, C_{ij}^k, F_i^a) . A similar argument still holds when A is not integrable. In this case, for each $x_0 \in X$ we can find a neighborhood $U \subset L$ of x_0 in the leaf L containing it such that the restriction of A to U is integrable to a Lie groupoid $\mathcal{G} \rightrightarrows U$. The Maurer-Cartan form of \mathcal{G} takes values in $A|_U \hookrightarrow A$ so we can see it as an A -valued Maurer-Cartan form. It is again clear that $(\mathbf{s}^{-1}(x_0), \omega_{\text{MC}}^i, \mathbf{t})$ is a realization of (n, X, C_{ij}^k, F_i^a) .

Notice also that if (M, θ^i, h) is another realization of (n, X, C_{ij}^k, F_i^a) then the A -valued 1-form $\theta \in \Omega^1(M, A)$ defined by

$$\theta = \sum_i \theta^i(\alpha_i \circ h) \quad (4.7.1)$$

satisfies the generalized Maurer-Cartan equation. This is a consequence of the following computation

$$\begin{aligned} d_{\nabla} \theta \left(\frac{\partial}{\partial \theta^i}, \frac{\partial}{\partial \theta^j} \right) &= \bar{\nabla}_{\left(\frac{\partial}{\partial \theta^i} \right)} \theta \left(\frac{\partial}{\partial \theta^j} \right) - \bar{\nabla}_{\left(\frac{\partial}{\partial \theta^j} \right)} \theta \left(\frac{\partial}{\partial \theta^i} \right) - \theta \left(\left[\frac{\partial}{\partial \theta^i}, \frac{\partial}{\partial \theta^j} \right] \right) \\ &= \nabla_{\# \alpha_i} \alpha_j - \nabla_{\# \alpha_j} \alpha_i - [\alpha_i, \alpha_j] \\ &= T_{\nabla}(\alpha_i, \alpha_j) \\ &= T_{\nabla} \left(\theta \left(\frac{\partial}{\partial \theta^i} \right), \theta \left(\frac{\partial}{\partial \theta^j} \right) \right) \end{aligned} \quad (4.7.2)$$

where $\bar{\nabla}$ denotes the pullback connection on h^*A . Thus, if $m_0 \in M$ is such that $h(m_0) = x_0$ then, by the universal property of Maurer-Cartan forms, we can find a neighborhood V of m_0 in M and a unique diffeomorphism $\phi : V \rightarrow \phi(V) \subset \mathbf{s}^{-1}(x_0)$ such that $\phi(m_0) = \mathbf{1}_{m_0}$ and $\phi^* \omega_{\text{MC}} = \theta$. This means that any realization of Cartan's problem is locally equivalent to a neighborhood of the identity of an \mathbf{s} -fiber of \mathcal{G} equipped with the Maurer-Cartan form. We have proved the following existence and uniqueness result which solves the local equivalence and classification problems.

Theorem 4.7.1 *Let (n, X, C_{ij}^k, F_i^a) be the initial data of a Cartan's realization problem. Then for each $x_0 \in X$ there exists a realization (M, θ^i, h) and $m_0 \in M$ satisfying $h(m_0) = x_0$ if and only if*

$$\sum_{b=1}^d \left(F_i^b \frac{\partial F_j^a}{\partial x_b} - F_j^b \frac{\partial F_i^a}{\partial x_b} \right) = - \sum_{l=1}^r C_{ij}^l F_l^a \quad (4.7.3)$$

for all $1 \leq i, j \leq r, 1 \leq a \leq d$ and

$$\sum_{b=1}^d \left(F_j^b \frac{\partial C_{kl}^i}{\partial x_b} + F_k^b \frac{\partial C_{lj}^i}{\partial x_b} + F_l^b \frac{\partial C_{jk}^i}{\partial x_b} \right) = - \sum_{m=1}^r (C_{mj}^i C_{kl}^m + C_{mk}^i C_{lj}^m + C_{ml}^i C_{jk}^m) \quad (4.7.4)$$

for all $1 \leq i, j, k, l \leq r$.

Moreover, any realization is locally equivalent to a neighborhood of the identity of an \mathfrak{s} -fiber of a groupoid \mathcal{G} equipped with the Maurer-Cartan form. Two realizations are locally equivalent if and only if they correspond to the same point $x_0 \in X$, in which case they differ by right translation by an element in the isotropy group \mathcal{G}_{x_0} .

4.8 Symmetries of Realizations

In this section we prove a few results about symmetries of a realization. Many of these results can be traced back to Cartan [9]. The present formulation, however is based on [5]. The purpose of this section is to show how the classifying Lie algebroid can be used to give very simple proofs of these facts.

Definition 4.8.1 *Let (M, θ, h) be a realization of a Cartan's problem. A **symmetry** of (M, θ, h) is a diffeomorphism $\phi : M \rightarrow M$ such that $\phi^*\theta = \theta$. An **infinitesimal symmetry** is a vector field $\xi \in \mathfrak{X}(M)$ such that $\mathcal{L}_\xi \theta = 0$.*

We note that it follows from the Local Classification Theorem 4.7.1 that $\phi : M \rightarrow M$ is a symmetry if and only if

$$h \circ \phi = h,$$

and that $\xi \in \mathfrak{X}(M)$ is an infinitesimal symmetry if and only if $h_*(\xi) = 0$.

Proposition 4.8.2 (Theorem A.2 of [5]) *Let (M, θ, h) be a connected and simply connected realization of a Cartan's problem with classifying Lie algebroid $A \rightarrow X$, such that $h(M)$ is contained in a leaf L of A . Then the set $\mathfrak{s} \subset \mathfrak{X}(M)$ of infinitesimal symmetries of the realization is a Lie algebra of dimension $\dim \mathfrak{g} = \dim M - \dim L$.*

Proof. In order to prove the proposition, we will show that \mathfrak{s} is isomorphic to the isotropy Lie algebra \mathfrak{g}_x of A at some (and hence any) point $x \in L$.

To begin with, notice that the coframe θ on M determines a pair of connections, which we denote by ∇ and $\bar{\nabla}$ on TM . The first one is the canonical flat connection associated to the trivialization of $TM = M \times \mathbb{R}^n$ induced by θ . The second connection is defined by

$$\bar{\nabla}_{\xi_1} \xi_2 = \nabla_{\xi_2} \xi_1 + [\xi_1, \xi_2].$$

After long, but straightforward calculations, one can show that

1. the connection $\bar{\nabla}$ is also flat;

2. a vector field ξ preserves θ^i , for all $1 \leq i \leq n$, if and only if ξ is $\bar{\nabla}$ -parallel, i.e., $\bar{\nabla}_{\xi'}\xi = 0$ for all $\xi' \in \mathfrak{X}(M)$.

It follows from the fact that a parallel vector field depends only on its value at one point that \mathfrak{s} is finite dimensional.

Now, since a vector field $\xi \in \mathfrak{X}(M)$ is an infinitesimal symmetry of the realization if and only if it preserves h , i.e., $h_*(\xi) = 0$, we obtain a map

$$\psi : \mathfrak{s} \rightarrow \mathfrak{g}_x, \quad \psi(\xi) = \theta(\xi(p)),$$

where $p \in M$ is any point such that $h(p) = x$.

Now, note that \mathfrak{s} is closed by the bracket of vector fields. In fact, if $\xi_1, \xi_2 \in \mathfrak{s}$, then

$$h_*[\xi_1, \xi_2] = \# \circ \theta([\xi_1, \xi_2]) = [\# \circ \theta(\xi_1), \# \circ \theta(\xi_2)] = [h_*(\xi_1), h_*(\xi_2)] = 0$$

shows that \mathfrak{s} is a Lie algebra.

Moreover, since θ is a Lie algebroid morphism, it follows that ψ is a Lie algebra homomorphism. We shall show that it is an isomorphism. To see that it is injective, we observe that θ is a fiber wise isomorphism and that if a parallel vector field that vanishes at a point, then it vanishes identically. Thus, $\ker \psi = 0$.

Finally, let α be any element in \mathfrak{g}_x . Denote by $\tilde{\alpha} \in \mathfrak{X}(M)$ the vector field on M obtained by $\bar{\nabla}$ -parallel translation along any curve of the vector $\theta^{-1}(\alpha)$. This is well defined because $\bar{\nabla}$ is flat, and M is assumed to be simply connected. But then it is clear that $\psi(\tilde{\alpha}) = \alpha$, thus proving the proposition. ■

Remark 4.8.3 *If M is not simply connected, then the isotropy Lie algebra \mathfrak{g}_x can be identified with the Lie algebra of germs of vector fields on M at a point p such that $h(p) = x$, which preserve the germ of θ at p .*

We can also state the following semi-global symmetry property of a realization:

Proposition 4.8.4 (Theorem A.3 of [5]) *Let (n, X, C_{ij}^k, F_i^a) be the initial data of a realization problem for which $-C_{ij}^k, F_i^a$ are the structure functions of a Lie algebroid $A \rightarrow X$. Let $L \subset X$ be a leaf of A whose isotropy Lie algebra is \mathfrak{g} , and let G be any Lie group with Lie algebra \mathfrak{g} .*

Then over any contractible set $U \subset L$, there exists a principal G -bundle $h : M \rightarrow U$ and a G -invariant coframe θ of M such that (M, θ, h) is a realization of the Cartan's problem. Moreover, this realization is locally unique up to isomorphism.

Proof. This proposition is just an immediate consequence of our classification result, Theorem 4.7.1. In fact, since U is contractible and the restriction of A to U is transitive, it follows that $A|_U$ is integrable. Moreover, for any Lie group G with Lie algebra \mathfrak{g} , there exists a Lie groupoid \mathcal{G} integrating A whose isotropy Lie group is isomorphic to \mathcal{G} . But then, the restriction of the Maurer-Cartan form of \mathcal{G} to any \mathfrak{s} -fiber furnishes the locally unique G -invariant coframe we were seeking for. ■

4.9 The Globalization Problem

We are now able to solve the *globalization problem*. Suppose that we are given two germs of coframes θ_0 and θ_1 which solve the realization problem and we want to know if they are the germs of the same global realization. We have the following result:

Theorem 4.9.1 *Suppose that A is integrable. Then θ_0 and θ_1 are germs of the same global connected realization (M, θ, h) if and only if they correspond to points on X in the same orbit of A .*

Proof. Suppose that two germs of coframes θ_0 and θ_1 correspond to points x_0 and x_1 on the same orbit of A and let \mathcal{G} be a Lie groupoid integrating A . Then the \mathfrak{s} -fiber at x_0 contains a point g with $\mathfrak{t}(g) = x_1$. Thus, θ_0 can be identified with the germ of ω_{MC} at $\mathbf{1}_{x_0}$ and θ_1 can be identified with the germ of ω_{MC} at g . We conclude that θ_0 and θ_1 are both germs of the realization $(\mathfrak{s}^{-1}(x_0), \omega_{\text{MC}}, \mathfrak{t})$.

Conversely, suppose that there exists a connected realization (M, θ, h) such that θ_0 and θ_1 are the germs of θ at points m_0 and m_1 of M , respectively. Let γ be a curve joining m_0 to m_1 and cover it by a finite family U_1, \dots, U_k of open sets of M with the property that the restriction of θ to each U_i is equivalent to the restriction of the Maurer-Cartan form to an open set of some \mathfrak{s} -fiber of \mathcal{G} , i.e., $\theta|_{U_i} = \phi_i^* \omega_{\text{MC}}$ for some diffeomorphism $\phi_i : U_i \rightarrow \phi_i(U_i) \subset \mathfrak{s}^{-1}(x_i)$.

We proceed by induction on the number of open sets needed to join m_0 to m_1 . Suppose that both m_0 and m_1 belong to the same open set U_1 . Then $\phi_1(m_0)$ and $\phi_1(m_1)$ are both on the same \mathfrak{s} -fiber. Thus, $h(m_0) = \mathfrak{t} \circ \phi_1(m_0)$ belongs to the same orbit as $h(m_1) = \mathfrak{t} \circ \phi_1(m_1)$. Now assume that the result is true for $k-1$ open sets. Then any point q in U_{k-1} is mapped by h to the same orbit of $h(m_0)$. Let q be a point in $U_{k-1} \cap U_k$. Then, on one hand, since q and m_1 belong to U_k it follows that h maps them both to the same orbit of $h(m_0)$. On the other hand, by the inductive hypothesis, it follows that h also maps m_0 and q to the same orbit of $h(m_0)$, from where the result follows. ■

Remark 4.9.2 *The proposition above may not be true when A is not integrable. The problem is that in this case, the global object associated to A is a topological groupoid which is smooth only in a neighborhood of the identity section. Thus, if $x, y \in X$ are points in the same orbit which are 'too far away', then we might not be able to find a differentiable realization 'covering' both points at the same time.*

Motivated by this remark, it is natural to consider the problem of existence of realizations (M, θ, h) of a Cartan's problem, such that the image of h is the whole leaf of the classifying Lie algebroid.

Definition 4.9.3 *A realization (M, θ, h) is **full** if h is surjective onto the orbit of A that it 'covers'.*

Before we proceed, let us present an example.

Example 4.9.4 *Assume that the classifying Lie algebroid A is integrable and let \mathcal{G} be any Lie groupoid integrating it. Then the \mathfrak{s} -fibers of \mathcal{G} equipped with the Maurer-Cartan form are full realizations.*

We can state a partial converse to the example, namely, that if there exists a *complete* realization covering an orbit L of the classifying algebroid, then the restriction $A|_L$ is integrable. An analogous situation occurs in [29].

Definition 4.9.5 *A realization (M, θ, h) of a Cartan's problem is **complete** if it is a full realization such that all of its infinitesimal symmetries $\xi \in \mathfrak{X}(M)$ are complete vector fields.*

Proposition 4.9.6 *Let $A \rightarrow X$ be the classifying Lie algebroid of a Cartan's realization problem, and let $L \subset X$ be an orbit of A . Then there exists a complete realization over L if and only if the restriction $A|_L$ is integrable.*

Proof. Assume that $A|_L$ is integrable by $\mathcal{G} \rightrightarrows L$. Then clearly, for any $x \in L$, the realization $(\mathfrak{s}^{-1}(x), \omega_{\text{MC}}, \mathfrak{t})$ is complete.

Conversely, let (M, θ, h) be a complete realization which covers L . By (the proof of) Proposition 4.8.2, the symmetry Lie algebra of θ is the isotropy Lie algebra \mathfrak{g} of A at a point $x \in L$. Now, since (M, θ, h) is complete, we can integrate the infinitesimal \mathfrak{g} -action on M to a \tilde{G} -action, where \tilde{G} denotes the simply connected Lie group with Lie algebra \mathfrak{g} . But then,

$$\begin{array}{c} M \begin{array}{c} \curvearrowright \\ \tilde{G} \end{array} \\ \downarrow h \\ L \end{array}$$

is a principal bundle. Its Atiyah algebroid is isomorphic to $A|_L$. Thus, $A|_L$ can be integrates by the gauge groupoid of $M \rightarrow L$. ■

4.10 The Global Classification Problem

In this section, we solve the global classification problem 4.2.4. Throughout this section, (n, X, C_{ij}^k, F_i^a) is a realization problem with classifying Lie algebroid $A \rightarrow X$, and (M, θ, h) is a realization.

The global equivalence problem for realizations of a Cartan's problem is much more delicate then the local one. The reason is that the classifying Lie algebroid will not distinguish between a realization and its covering.

Example 4.10.1 *In order to illustrate the difficulty, we now explain an example presented in [27] (see also [28]) of a simply connected manifold, equipped with a Lie algebra valued Maurer-Cartan form which cannot be globally embedded into any Lie group, but which is locally equivalent at every point to an open set of a Lie group.*

Let $M = \mathbb{R}^2 - \{(-1, 0)\}$ be the punctured plane equipped with its canonical coframe $\{dx, dy\}$. Let \tilde{M} be its universal covering space. Then \tilde{M} is an helicoid which we can identify with the half plane $\{(r, \varphi) \in \mathbb{R}^2 : r > 0\}$. With this identification, the covering map $\pi : \tilde{M} \rightarrow M$ becomes

$$\pi(r, \varphi) = (r \cos \varphi - 1, r \sin \varphi)$$

Let

$$\theta^1 = \cos \varphi dr - r \sin \varphi d\varphi, \quad \theta^2 = \sin \varphi dr + r \cos \varphi d\varphi$$

be the pullbacks of dx and dy by π . Then $\theta = (\theta^1, \theta^2)$ is an \mathbb{R}^2 -valued Maurer-Cartan form on \tilde{M} , where we view \mathbb{R}^2 as an abelian Lie algebra.

It follows that θ is locally equivalent at every point of \tilde{M} to the Maurer-Cartan form of the abelian Lie group \mathbb{R}^2 . On the other hand, θ cannot be globally equivalent to the Maurer-Cartan form on any open set of $\tilde{\mathbb{R}}^2$. In order to see this, note that θ induces the structure of a local Lie group on \tilde{M} as follows. If we fix the point $(1, 0) \in \tilde{M}$ to be the identity element, we can define a product $w = u \cdot v$ of two points $u, v \in \tilde{M}$ so that it projects onto the ordinary sum, i.e., $\pi(w) = \pi(u) + \pi(v)$. Moreover, this product can be taken to be smooth, provided that at least one of the summands belongs to the same sheet of the covering space as the identity element $(1, 0)$.

This product, however, fails to be associative. In fact, the product $u \cdot v \cdot w$ is only well defined if the triangle with vertices $\pi(u), \pi(u) + \pi(v)$ and $\pi(u) + \pi(v) + \pi(w)$ does not contain the point $(-1, 0)$ in its interior. Otherwise, the products $(u \cdot v) \cdot w$ and $u \cdot (v \cdot w)$ will lie on different sheets of the covering space \tilde{M} . Even if one restricts to the subset of \tilde{M} where the threefold product of any elements is defined, one will face the same problem for defining fourfold products, and so on. It follows that the coframe θ on \tilde{M} cannot be globally equivalent to the Maurer-Cartan form on a subset of the Lie group \mathbb{R}^2 .

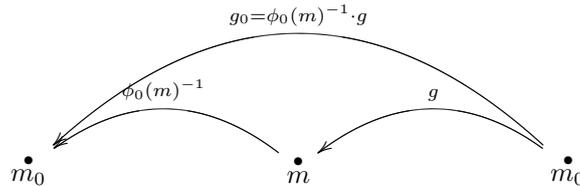
We have the following theorem (compare with Theorem 14.28 of [27]):

Theorem 4.10.2 *Let (M, θ, h) be a full realization of a Cartan problem and suppose that the classifying Lie algebroid $A \rightarrow X$ is integrable. Then M is globally equivalent up to cover to an open set of an \mathfrak{s} -fiber of a groupoid \mathcal{G} integrating A .*

Proof. Let \mathcal{D} be the distribution on $M \times_{h,t} \mathcal{G}$ as in the proof of Theorem 4.5.2 and let N be a maximal integral manifold of \mathcal{D} . Let $\pi_M : M \times_{h,t} \mathcal{G} \rightarrow M$ and $\pi_{\mathcal{G}} : M \times_{h,t} \mathcal{G} \rightarrow \mathcal{G}$ denote the natural projections. Then $\pi_{\mathcal{G}}(N)$ is totally contained in a single \mathfrak{s} -fiber of \mathcal{G} , say, $\mathfrak{s}^{-1}(x)$. Moreover, the restrictions of the projections to N , $\pi_M : N \rightarrow M$ and $\pi_{\mathcal{G}} : N \rightarrow \mathfrak{s}^{-1}(x)$ are local diffeomorphisms.

We claim that N , equipped with the coframe $\pi_M^* \theta = \pi_{\mathcal{G}}^* \omega_{\text{MC}}$, is a common realization cover of M and an open set of an \mathfrak{s} -fiber of \mathcal{G} . To show this, all we must show is that $\pi_M(N) = M$. In fact, if this is true, M will be globally equivalent up to cover to $\pi_{\mathcal{G}}(N) \subset \mathfrak{s}^{-1}(x)$.

To prove this, suppose that $\pi_M(N)$ is a proper submanifold of M and let $m_0 \in M - \pi_M(N)$ be a point in the closure of $\pi_M(N)$ (remember that $\pi_M|_N$ is an open map). Then, by the local universal property of Maurer-Cartan forms, there is an open set U of m_0 in M and a diffeomorphism $\phi_0 : U \rightarrow \phi_0(U) \subset \mathfrak{s}^{-1}(h(m_0))$ such that $\phi_0^* \omega_{\text{MC}} = \theta$ and $\phi_0(m_0) = \mathbf{1}_{h(m_0)}$. It follows that the graph of ϕ_0 is also an integral manifold N_0 of the distribution \mathcal{D} on $M \times_{h,t} \mathcal{G}$ which passes through $(m_0, \mathbf{1}_{h(m_0)})$. Now let $m = \pi_M(m, g)$ be any point in $U \cap \pi_M(N)$ where $(m, g) \in N$. Then $\phi_0(m) \in \mathcal{G}$ is an arrow from $h(m_0)$ to $h(m)$ and g is an arrow from m to $h(p)$ and thus $g_0 = \phi_0(m)^{-1} \cdot g$ is an arrow from x to $h(m_0)$, i.e.,



Now, by virtue of the invariance of the Maurer-Cartan form, the manifold

$$R_{g_0}N_0 = \{(\bar{m}, \bar{g} \cdot g_0) : (\bar{m}, \bar{g}) \in N_0\}$$

is an integral manifold of \mathcal{D} . But then, the point $(m, g) = (m, \phi_0(m) \cdot g_0)$ belongs to $R_{g_0}N_0$ and to N , and thus, by the uniqueness and maximality of N it follows that N contains $R_{g_0}N_0$. But then, $(m_0, \phi_0(m_0) \cdot g_0)$ is a point of N which projects through π_M to m_0 , which contradicts the fact that $m_0 \in M - \pi_M(N)$, proving the theorem. ■

As a consequence of of this theorem, we recover the two main results of the section on global equivalence of [27]. Namely,

Corollary 4.10.3 *1. If θ is a coframe of rank 0 on M , then M is globally equivalent, up to covering, to an open set of a Lie group.*

2. If (M_1, θ_1, h_1) and (M_2, θ_2, h_2) are full realizations of rank n over the same leaf $L \subset X$ of A , then M_1 and M_2 are realization covers of a common realization (L, θ, h) , i.e.,

$$\begin{array}{ccc} M_1 & & M_2 \\ & \searrow h_1 & \swarrow h_2 \\ & L & \end{array}$$

Proof.

1. The first part of the corollary is trivial. The restriction of the classifying Lie algebroid to $h(M) = \{\text{pt}\}$ is a Lie algebra and thus is integrable by Lie group. It follows that M is globally equivalent, up to covering, with an open set of this Lie group.
2. To prove the second statement, let $A \rightarrow X$ be the classifying Lie algebroid of the realization problem and let L be the n -dimensional leaf of A for which $h_1(M_1) = L = h_2(M_2)$. Since A has rank n , it follows that the anchor of the restriction of A to L is injective, and thus $A|_L$ is integrable. Moreover, its Lie groupoid \mathcal{G} is étale, i.e., its source and target maps are local diffeomorphisms.

We can then make L into a realization by equipping it with the pullback of the Maurer-Cartan form by the local inverses of \mathbf{t} . More precisely, let U be an open set in $\mathbf{s}^{-1}(x)$ for which the restriction of \mathbf{t} is one-to-one and let $V = \mathbf{t}(U)$ be the open image of U by \mathbf{t} . Define θ_U to be the coframe on V given by

$$\theta_U = (\mathbf{t}|_U^{-1})^* \omega_{\text{MC}}.$$

Then θ_U is the restriction to V of a globally defined coframe on L . In fact, suppose that \bar{U} is another open set of $\mathbf{s}^{-1}(x)$ for which $\mathbf{t}|_{\bar{U}}$ is one-to-one and such that $\bar{V} = \mathbf{t}(\bar{U})$ intersects V . Denote by $\theta_{\bar{U}}$ the coframe on \bar{V} defined by

$$\theta_{\bar{U}} = (\mathbf{t}|_{\bar{U}}^{-1})^* \omega_{\text{MC}}.$$

We will show that θ_U and $\theta_{\bar{U}}$ coincide on the intersection $V \cap \bar{V}$. After shrinking U and \bar{U} , if necessary, we may assume that $V = \bar{V}$. But then,

since \mathcal{G} is étale, it follows that the isotropy group \mathcal{G}_x is discrete, which implies that U is the right translation of \bar{U} by an element $g \in \mathcal{G}_x$, i.e.,

$$U = R_g(\bar{U}).$$

Thus,

$$\begin{aligned} \theta_U &= (R_g \circ \mathbf{t}|_{\bar{U}}^{-1})^* \omega_{\text{MC}} \\ &= (\mathbf{t}|_{\bar{U}}^{-1})^* (R_g^* \omega_{\text{MC}}) \\ &= (\mathbf{t}|_{\bar{U}}^{-1})^* \omega_{\text{MC}} \\ &= \theta_{\bar{U}}, \end{aligned}$$

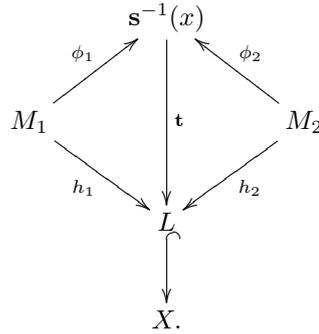
and θ is a well defined global coframe on L .

Now, since both M_1 and M_2 are globally equivalent, up to covering, to open sets in $\mathbf{s}^{-1}(x)$, it follows that the surjective submersions $h_i : M_i \rightarrow L$ are realization covers.

If the reader prefers, there is a direct argument to see that h_i is a realization morphism. In fact, the coframe θ_1 on M_1 is locally the pullback of the Maurer-Cartan form on an \mathbf{s} -fiber of \mathcal{G} by a locally defined diffeomorphism $\phi_1 : W_1 \subset M_1 \rightarrow \mathbf{s}^{-1}(x)$. After shrinking W_1 , we may assume that the restriction of \mathbf{t} to $U_1 = \phi_1(W_1)$ is one-to-one. But then,

$$\begin{aligned} h_1^* \theta &= (\mathbf{t} \circ \phi_1)^* \theta \\ &= \phi_1^* (\mathbf{t}^* \circ (\mathbf{t}|_{U_1}^{-1})^* \omega_{\text{MC}}) \\ &= \phi_1^* \omega_{\text{MC}} \\ &= \theta_1. \end{aligned}$$

Obviously, the very same argument can be given to show that h_2 is also a realization cover. We summarize this proof by the diagram



■

Chapter 5

The Classification Problem for Finite Type G -Structures

This chapter is devoted to the study of realization problems for finite type G -structures. In particular, we shall describe the structure of the classifying Lie algebroid and describe some of its properties.

We begin by explaining the case where $G^{(1)} = \{e\}$. The general case will then follow almost immediately from the fact that the k -th prolongation $\mathcal{B}_{G^{(k)}}$ of a G -structure \mathcal{B}_G coincides with the first prolongation of $\mathcal{B}_{G^{(k-1)}}$.

5.1 The Realization Problem for G -Structures of type 1

Consider a G -structure with G a subgroup of GL_n such that $G^{(1)} = \{e\}$. We saw in chapter 3 that there are structure functions which determine the G -structure up to equivalence. Conversely, suppose that we are given a set of functions and we want to determine all G -structures for which these are the structure functions. Then, by virtue of the structure equations (3.4.1) we must solve the following realization problem:

Problem 5.1.1 (Realization Problem for G -Structures with $G^{(1)} = \{e\}$)
One is given:

1. An open set $X \subset \mathbb{R}^d$,
2. an integer $n \in \mathbb{N}$,
3. a Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}_n$ satisfying $\mathfrak{g}^{(1)} = 0$,
4. a Lie group $G \subset GL(n)$ with Lie algebra \mathfrak{g} , and
5. maps $c : X \rightarrow \text{Hom}(\mathbb{R}^n \wedge \mathbb{R}^n, \mathbb{R}^n)$, $R : X \rightarrow \text{Hom}(\mathbb{R}^n \wedge \mathbb{R}^n, \mathfrak{g})$, $S : X \rightarrow \text{Hom}(\mathbb{R}^n \otimes \mathfrak{g}, \mathfrak{g})$, $\Theta : X \rightarrow \text{Hom}(\mathbb{R}^n, \mathbb{R}^d)$, and $\Phi : X \rightarrow \text{Hom}(\mathfrak{g}, \mathbb{R}^d)$

and asks for the existence of

1. a manifold M^n ,
2. a G -structure $\mathcal{B}_G(M)$ on M with tautological form $\theta \in \Omega^1(\mathcal{B}_G, \mathbb{R}^n)$
3. a vertical one form $\eta \in \Omega^1(\mathcal{B}_G, \mathfrak{g})$ of maximal rank, and
4. a map $h : \mathcal{B}_G \rightarrow X$

such that

$$\begin{cases} d\theta &= c(h)(\theta \wedge \theta) - \eta \wedge \theta \\ d\eta &= R(h)(\theta \wedge \theta) + S(h)(\theta \wedge \eta) - \eta \wedge \eta \\ dh &= \Theta(h) \circ \theta + \Phi(h) \circ \eta. \end{cases} \quad (5.1.1)$$

There is a small variant to this problem that will also be of interest: Instead of taking a Lie group G such that $G^{(1)} = \{e\}$ we take an arbitrary subgroup of GL_n . Then the realization problem above gives us the reductions of the $G^{(1)}$ -structure $\mathcal{B}_G(M)^{(1)}$ over $\mathcal{B}_G(M)$ to the subgroup $\{e\}$. This kind of problem arises, for example, when we want to describe connections on a manifold which preserve a given geometric structure. An example of this is analyzed in detail in Chapter 7.

5.2 Existence of Solutions

In order to solve the realization problem, we first look for necessary conditions for the existence of solutions. For this, note that if $(M, \mathcal{B}_G, \theta, \eta, h)$ is a realization to Cartan's Problem, then, in particular, $(\mathcal{B}_G, (\theta, \eta), h)$ is a solution to the associated realization problem for coframes. This means that the initial data to the realization problem determines a classifying Lie algebroid $A \rightarrow X$ such that

$$\begin{array}{ccc} T\mathcal{B}_G & \xrightarrow{(\theta, \eta)} & A \\ \downarrow & & \downarrow \\ \mathcal{B}_G & \xrightarrow{h} & X \end{array} \quad (5.2.1)$$

is a Lie algebroid morphism.

Next, we determine what extra properties A possesses due to the fact that the coframe it classifies correspond to $\{e\}$ -structures over $\mathcal{B}_G(M)$, i.e., due to the format of the structure equations. First of all, note that as a vector bundle, $A \cong X \times (\mathbb{R}^n \oplus \mathfrak{g})$ and its Lie algebroid structure may be described explicitly (on constant sections) by

$$[(u, \rho), (v, \sigma)](x) = (w, \tau) \quad (5.2.2)$$

where,

$$\begin{aligned} w &= -c(x)(u \wedge v) + \rho \cdot v - \sigma \cdot u \\ \tau &= -R(x)(u \wedge v) - S(x)(u \otimes \sigma - v \otimes \rho) + [\rho, \sigma]_{\mathfrak{g}} \end{aligned} \quad (5.2.3)$$

and

$$\#(u, \rho) = \Theta(x)u + \Phi(x)\rho. \quad (5.2.4)$$

It follows that

$$[(0, \rho), (0, \sigma)]_A = (0, [\rho, \sigma]_{\mathfrak{g}}) \quad (5.2.5)$$

and thus

$$\mathfrak{g} \hookrightarrow \Gamma(A) \quad (5.2.6)$$

is a Lie algebra homomorphism. In this way, we obtain an inner action of \mathfrak{g} on A

$$\psi(\rho)(\alpha) = [(0, \rho), \alpha]_A, \quad (5.2.7)$$

where α is any section of A .

In summary, we have obtained that a necessary conditions for the existence of realizations to Cartan's problem for G -structures satisfying $G^{(1)} = \{e\}$, is that the initial data determines a flat Lie algebroid carrying an inner action of \mathfrak{g} . Moreover, the structure of the Lie algebroid is not completely arbitrary: its structure equations take the specific form (5.2.3) and (5.2.4).

Now we describe to what extent the conditions above are also sufficient (at least locally). What we will show is that if the necessary conditions above are satisfied, then for each x in the base X of the Lie algebroid, we may construct a G -structure on \mathbb{R}^n with a frame p at the origin such that $h(p) = x$ and such that the structure equations are satisfied in a neighborhood of p . More precisely,

Theorem 5.2.1 *Let G be a subgroup of GL_n such that $G^{(1)} = \{e\}$ and let $(n, X, G, (c, b, S), (\Theta, \Phi))$ be the initial data of a Cartan's realization problem for G -structures. Then, for every $x \in X$, there exists a realization $(U, \mathcal{B}_G(U), (\theta, \eta), h)$ defined on a neighborhood U of the origin of \mathbb{R}^n with a frame $p \in \mathcal{B}_G$ satisfying $h(p) = x$ if and only if the bracket (5.2.3) and anchor (5.2.4) determine the structure of a Lie algebroid on $A \cong \mathbb{R}^n \oplus \mathfrak{g} \rightarrow X$.*

Moreover, in this case, \mathfrak{g} acts on A infinitesimally by inner automorphisms, and the action is locally free.

Proof. Suppose that the initial data to the realization problem determines a Lie algebroid $A \rightarrow X$ with structure given by (5.2.3) and (5.2.4). Let $(P, (\theta, \eta), h)$ be a realization to the associated Cartan's problem for coframes, i.e.,

$$\begin{array}{ccc} TP & \xrightarrow{(\theta, \eta)} & A \\ \downarrow & & \downarrow \\ P & \xrightarrow{h} & X \end{array} \quad (5.2.8)$$

is a Lie algebroid morphism. What we will now show through a series of claims is that (after restricting to a small enough open set in P) we may identify P with an open set of a G -structure on a neighborhood of the origin in \mathbb{R}^n .

Claim 1 $\theta = 0$ defines an integrable distribution \mathcal{D} on P .

Since $d\theta = c(\theta \wedge \theta) - \eta \wedge \theta$ it follows that $d\theta(\xi_1, \xi_2) = 0$ whenever $\theta(\xi_1) = 0 = \theta(\xi_2)$. The claim then follows from Frobenius' Theorem.

Claim 2 *Each leaf of \mathcal{D} is locally diffeomorphic to a neighborhood of the identity in G .*

Note that the structure equation for $d\eta$ when restricted to $\theta = 0$ becomes the Maurer-Cartan equation

$$d\eta + \eta \wedge \eta = 0. \quad (5.2.9)$$

It then follows from the universal property of the Maurer-Cartan form on Lie groups that every point on the leaf has a neighborhood that is locally diffeomorphic to a neighborhood of the identity in G . Moreover, viewing η as a map

$$\mathcal{D} \rightarrow X \times \mathfrak{g},$$

and since both of these bundles have the same rank and η is fiberwise surjective, it follows that we may invert η to obtain an infinitesimal action

$$\eta^{-1} : \mathfrak{g} \rightarrow \mathfrak{X}(\mathcal{D}) \subset \mathfrak{X}(P). \quad (5.2.10)$$

Claim 3 θ is equivariant relative to the infinitesimal action of \mathfrak{g} on P .

Of course, this will follow again from the structure equation for $d\theta$. Let $\tilde{\rho} = \eta^{-1}(\rho)$ denote the vector field tangent to the leaves of $\theta = 0$ generated by ρ . Then, by Cartan's magic formula, we obtain

$$\begin{aligned} \mathcal{L}_{\tilde{\rho}}\theta &= i_{\tilde{\rho}}d\theta + di_{\tilde{\rho}}\theta \\ &= i_{\tilde{\rho}}d\theta \\ &= i_{\tilde{\rho}}(c(\theta \wedge \theta) - \eta \wedge \theta) \\ &= -\rho \cdot \theta \end{aligned} \quad (5.2.11)$$

proving the claim.

Claim 4 P has (locally) a structure of fiber bundle over \mathbb{R}^n .

Just restrict (θ, η) to a small enough open set (which we continue to denote by P) such that:

1. the leaves of $\theta = 0$ are the fibers of a submersion $P \rightarrow \mathbb{R}^n$ and
2. the restriction of η to each leaf L is the pullback of the Maurer-Cartan form on G by a diffeomorphism $L \rightarrow U \subset G$.

Then $P \cong \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$ is a trivial fiber bundle with fiber U .

Claim 5 We may extend P to a G -structure on \mathbb{R}^n solving the realization problem in a neighborhood of a frame.

Let $\tilde{P} \cong \mathbb{R}^n \times G$ be the trivial principal fiber bundle with fiber G . Note that

$$\begin{array}{ccc} P & \xrightarrow{i} & \tilde{P} \\ & \searrow & \swarrow \\ & \mathbb{R}^n & \end{array} \quad (5.2.12)$$

Now define on \tilde{P} the \mathbb{R}^n valued one form $\tilde{\theta}$ by

$$\tilde{\theta}_{(v,g)}(\xi) := g^{-1} \cdot \theta_{(v,e)}((R_{g^{-1}})_*\xi) \quad (5.2.13)$$

which is well defined by virtue of the equivariance of θ . Then $\tilde{\theta}$ is a non-degenerate tensorial form on \tilde{P} and thus determines an embedding of \tilde{P} into the frame bundle of \mathbb{R}^n . It follows that \tilde{P} is a G -structure \mathcal{B}_G on \mathbb{R}^n with tautological form $\tilde{\theta}$. ■

Remark 5.2.2 Note that the G -structure constructed above is not globally a realization to Cartan's problem. In fact, the existence of the map h and form η such that the structure equations are satisfied only occur in a neighborhood of a frame $p \in \mathcal{B}_G(M)$.

However, if the group G is reductive i.e., if we can choose a G -invariant complement C to $\mathcal{A}(\text{Hom}(\mathbb{R}^n, \mathfrak{g}))$ in $\text{Hom}(\wedge^2 \mathbb{R}^n, \mathbb{R}^n)$, see Chapter 3, or more generally, if the structure function S vanishes, then η will be equivariant (a connection form) and we can also extend it to all of \tilde{P} through

$$\tilde{\eta}_{(v,g)}(\xi) := \text{Ad}_{g^{-1}} \cdot \eta_{(v,e)}((R_{g^{-1}})_* \xi) \quad (5.2.14)$$

obtaining a realization in which every point satisfies the structure equation.

5.3 Geometric Construction of Models

In order to get a better understanding of the realization problem for G -structures, and to obtain semi-global results, we describe in this section what happens in the best possible case, i.e., when the classifying Lie algebroid A is integrable and the infinitesimal action of \mathfrak{g} on A integrates to a free and proper action of G on \mathcal{G} . The realization problem for general finite type G -structures will be left to the next section.

We begin by recalling the properties of actions of Lie groups and Lie algebras on Lie groupoids and Lie algebroids that will be needed. For greater details, we refer the reader to [18] from where this discussion has been extracted.

Definition 5.3.1 1. An *action of a Lie group G on a Lie groupoid \mathcal{G}* is a smooth action $\Psi : G \times \mathcal{G} \rightarrow \mathcal{G}$ such that for each $a \in G$, the map

$$\Psi_a : \mathcal{G} \rightarrow \mathcal{G}, \quad \Psi_a(g) = a \cdot g$$

is an automorphism of the Lie groupoid.

2. An *action of a Lie group G on a Lie algebroid A* is a smooth action $\Psi : G \times A \rightarrow A$ such that for each $a \in G$, the map

$$\Psi_a : A \rightarrow A, \quad \Psi_a(\alpha) = a \cdot \alpha$$

is an automorphism of the Lie algebroid.

If $\Psi : G \times \mathcal{G} \rightarrow \mathcal{G}$ is an action of G on \mathcal{G} , then for each $a \in G$, the map

$$(\Psi_a)_* : A \rightarrow A$$

is a Lie algebroid automorphism. Thus, by differentiating at the identity section of \mathcal{G} , we can go from Lie group actions on groupoids to Lie group actions on algebroids. There is also a way to go from actions on algebroids to actions on groupoids, but we will not need this here.

We now turn to infinitesimal actions of Lie algebras on Lie algebroids.

Definition 5.3.2 1. A *derivation of a Lie algebroid A* is a linear operator $D : \Gamma(A) \rightarrow \Gamma(A)$ such that there exists a vector field $\sigma_D \in \mathfrak{X}(X)$, called the symbol of D , for which

$$D(f\alpha) = fD(\alpha) + \sigma_D(f)\alpha$$

and such that

$$D([\alpha, \beta]) = [D(\alpha), \beta] + [\alpha, D(\beta)]$$

for all sections $\alpha, \beta \in \Gamma(A)$ and for all functions $f \in C^\infty(X)$.

2. An **infinitesimal action of a Lie algebra \mathfrak{g} on a Lie algebroid A** is a Lie algebra homomorphism $\psi : \mathfrak{g} \rightarrow \text{Der}(A)$, where $\text{Der}(A)$ is the space of derivations of A .

If we are given an action $\Psi : G \times A \rightarrow A$ of a Lie group G on a Lie algebroid A , then for $\rho \in \mathfrak{g}$,

$$\psi(\rho)(\alpha) = \left. \frac{d}{dt} \exp(t\rho) \cdot \alpha \right|_{t=0}$$

defines an infinitesimal action of \mathfrak{g} on A . In this case, we say that the G -action on A integrates the infinitesimal \mathfrak{g} -action.

There is also a concept of infinitesimal action of a Lie algebra \mathfrak{g} on a Lie groupoid \mathcal{G} , and all four types of actions stated here are related to each other. However, since this will not be used elsewhere, we will omit its description.

We now return to the realization problem. So $A \cong X \times (\mathbb{R}^n \oplus \mathfrak{g}) \rightarrow X$ is the classifying Lie algebroid for the realization problem of G -structures and $G \subset \text{GL}_n$ is a Lie group satisfying $G^{(1)} = \{e\}$. The inclusion $i : \mathfrak{g} \rightarrow \Gamma(A)$ determines an infinitesimal action

$$\psi(\rho)(\alpha) = [(0, \rho), \alpha]_A$$

by inner derivations. This action induces a Lie algebroid morphism

$$\phi : \mathfrak{g} \times X \rightarrow A, \quad \phi(\rho, x) = i(\rho)(x),$$

from the transformation Lie algebroid $\mathfrak{g} \times X$ to A . Assume that the \mathfrak{g} -action on X is complete, that A is integrable by a Lie groupoid \mathcal{G} and that the map ϕ integrates to a Lie groupoid morphism

$$\Phi : G \times X \rightarrow \mathcal{G},$$

from the transformation groupoid $G \times X$ to \mathcal{G} . We remark that if G is simply connected, then this is always the case. Then we can describe the associated G -action on \mathcal{G} by

$$\Psi : G \times \mathcal{G} \rightarrow \mathcal{G}, \quad \Psi(a, g) = \Phi(a, \mathbf{t}(g)) \cdot g \cdot (\Phi(a, \mathbf{s}(g)))^{-1}.$$

In this case, we obtain a G -action on each \mathbf{s} -fiber of \mathcal{G} , which can be described explicitly by

$$a \cdot g = \Phi(a, \mathbf{t}(g)) \cdot g.$$

Moreover, when the G -action on X is proper and $\ker \Phi = \{e\} \times X$, then the G -action on each \mathbf{s} -fiber of \mathcal{G} is also free and proper. When this happens, we will denote by M_x the manifold obtained by taking the quotient of $\mathbf{s}^{-1}(x)$ by the orbits of the G -action.

Proposition 5.3.3 *The principal G -bundle*

$$\begin{array}{ccc} \mathbf{s}^{-1}(x) & \xrightarrow{\quad} & G \\ \pi \downarrow & & \curvearrowright \\ M_x & & \end{array}$$

is a G -structure over M_x .

Proof. Let ω_{MC} be the restriction to $\mathfrak{s}^{-1}(x)$ of the Maurer-Cartan form and denote by θ its \mathbb{R}^n component. It then follows from the structure equations of the Lie algebroid A and the fact that G -action on A integrates the infinitesimal \mathfrak{g} -action that θ is non-degenerate, horizontal and equivariant. Thus we can find an embedding $\mathfrak{s}^{-1}(x) \rightarrow \mathcal{B}(M_x)$ such that θ is the restriction of the tautological form of $\mathcal{B}(M_x)$ to $\mathfrak{s}^{-1}(x)$, proving the proposition. ■

Now, in order to prove a local classification result, we assume that the 1-form η may be taken to be equivariant (for example if G is reductive). Let $(M, \mathcal{B}_G(M), (\theta, \eta), h)$ be a realization of the Cartan problem, and let $p \in \mathcal{B}_G(M)$ be a frame. Then it follows from the universal property of the Maurer-Cartan form, that there is a neighborhood U of p in $\mathcal{B}_G(M)$ and a diffeomorphism $\phi : U \rightarrow \phi(U) \subset \mathfrak{s}^{-1}(h(p))$ such that $\phi(p) = \mathbf{1}_{h(p)}$ and $\phi^*\omega_{\text{MC}} = (\theta, \eta)$. However, since both (θ, η) and ω_{MC} are G -equivariant, we can extend ϕ to the open set $\pi^{-1}\pi(U) \subset \mathcal{B}_G(M)$ by also requiring it to be equivariant. We have thus proved that:

Theorem 5.3.4 (Local Classification) *Let $A \rightarrow X$ be the classifying Lie algebroid of a realization problem for G -structures, where G is a reductive Lie group satisfying $G^{(1)} = \{e\}$. Suppose that A is integrable and that*

1. *the infinitesimal action of \mathfrak{g} on X integrates to a proper action of G on X ;*
2. *the transformation Lie algebroid $\mathfrak{g} \times X \subset A$ integrates to a Lie subgroupoid of $\mathcal{G}(A)$ which is isomorphic to the transformation groupoid $G \times X \rightrightarrows X$.*

Then any G -structure $\mathcal{B}_G(M) \rightarrow M$ is locally equivalent to $\mathfrak{s}^{-1}(x) \rightarrow M_x$ for some $x \in X$.

Before we proceed, there is one last comment we would like to make. Since we are assuming that G acts freely and properly on \mathcal{G} , and since the induced action on A is by Lie algebroid automorphisms, it follows that $A/G \rightarrow X/G$ is also a Lie algebroid. In fact, it is the Lie algebroid of $\mathcal{G}/G \rightrightarrows X/G$ and its structure is given by

$$[\alpha \cdot G, \beta \cdot G] = [\alpha, \beta] \cdot G, \quad \#(\alpha \cdot G) = \#(\alpha) \cdot G.$$

It follows from the fact that the classifying map $h : \mathcal{B}_G(M) \rightarrow X$ is equivariant, i.e., $h(p \cdot a) = a^{-1} \cdot h(p)$, that there is a one to one correspondence between points of X/G and germs of solutions of the realization problem for G -structures.

Moreover, any realization gives rise to an equivariant morphism of Lie algebroids

$$(\theta, \eta) : T\mathcal{B}_G(M) \rightarrow A.$$

Thus, by passing to the quotient, we obtain a morphism of Lie algebroids

$$\begin{array}{ccc} T\mathcal{B}_G(M)/G & \longrightarrow & A/G \\ \downarrow & & \downarrow \\ M & \longrightarrow & X/G \end{array}$$

where $T\mathcal{B}_G(M)/G$ is the Atiyah algebroid of the principal G -bundle $\mathcal{B}_G(M) \rightarrow M$.

5.4 The Realization Problem for Finite Type G -Structures

We now deal with the realization problem for general finite type G -structures. Its formulation is based on the structure equations (3.5.1) which we deduced in Section 3.5. We will use the notation

$$\mathfrak{g}^l = \mathfrak{g} \oplus \mathfrak{g}^{(1)} \oplus \cdots \oplus \mathfrak{g}^{(l)}$$

introduced in the section referred to above.

Problem 5.4.1 (Realization Problem for G -Structures with $G^{(k)} = \{e\}$)
One is given:

1. An open set $X \subset \mathbb{R}^d$,
2. an integer $n \in \mathbb{N}$,
3. a Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}_n$ satisfying $\mathfrak{g}^{(k)} = 0$,
4. a Lie group $G \subset GL(n)$ with Lie algebra \mathfrak{g} , and
5. maps
 - $c^{(k-1)} : X \rightarrow \text{Hom}(\wedge^2(\mathbb{R}^n \oplus \mathfrak{g}^{k-2} \oplus \mathfrak{g}^{(k-1)}), \mathbb{R}^n \oplus \mathfrak{g}^{k-2})$,
 - $R^{(k-1)} : X \rightarrow \text{Hom}(\wedge^2(\mathbb{R}^n \oplus \mathfrak{g}^{k-2} \oplus \mathfrak{g}^{(k-1)}), \mathfrak{g}^{(k-1)})$,
 - $S^{(k-1)} : X \rightarrow \text{Hom}((\mathbb{R}^n \oplus \mathfrak{g}^{k-2}) \otimes \mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)})$,
 - $\Theta : X \rightarrow \text{Hom}(\mathbb{R}^n \oplus \mathfrak{g}^{k-2}, \mathbb{R}^d)$, and
 - $\Phi : X \rightarrow \text{Hom}(\mathfrak{g}^{(k-1)}, \mathbb{R}^d)$

and asks for the existence of

1. a manifold M^n ,
2. a G -structure $\mathcal{B}_G(M)$ on M with a prolongation $(\mathcal{B}_G(M))^{(k-1)}$, whose tautological form we denote by

$$\theta^{(k-1)} \in \Omega^1((\mathcal{B}_G(M))^{(k-1)}, \mathbb{R}^n \oplus \mathfrak{g}^{k-2} \oplus \mathfrak{g}^{(k-1)})$$

3. a vertical one form $\eta^{(k-1)} \in \Omega^1((\mathcal{B}_G(M))^{(k-1)}, \mathfrak{g}^{(k-1)})$ of maximal rank, and
4. a map $h : (\mathcal{B}_G(M))^{(k-1)} \rightarrow X$

such that

$$\begin{cases} d\theta^{(k-1)} &= c^{(k-1)}(h)(\theta^{(k-1)} \wedge \theta^{(k-1)}) - \eta^{(k-1)} \wedge \theta^{(k-1)} \\ d\eta^{(k-1)} &= R^{(k-1)}(h)(\theta^{(k-1)} \wedge \theta^{(k-1)}) + S^{(k-1)}(h)(\theta^{(k-1)} \wedge \eta^{(k-1)}) \\ dh &= \Theta(h) \circ \theta^{(k-1)} + \Phi(h) \circ \eta^{(k-1)}. \end{cases} \quad (5.4.1)$$

Before we describe the solution to the realization problem for general finite type G -structures, let us look at the particular case when $k = 2$:

Example 5.4.2 ($G^{(2)} = \{e\}$) *Suppose that G is a Lie subgroup of GL_n such that $G^{(2)} = \{e\}$ and let $(n, X, \mathfrak{g}, (c^{(1)}, R^{(1)}, S^{(1)})(\Theta, \Phi))$ be the initial data of a realization problem for G -structures. Instead of looking for solutions of this realization problem, we can pose the question of the existence of:*

- a manifold P of dimension $n + \dim \mathfrak{g}$,
- a $G^{(1)}$ -structure $\mathcal{B}_{G^{(1)}}(P)$ over P with tautological form

$$\theta^{(1)} \in \Omega^1(\mathcal{B}_{G^{(1)}}(P), \mathbb{R}^n \oplus \mathfrak{g})$$

- a vertical one form $\eta^{(1)} \in \Omega^1(\mathcal{B}_{G^{(1)}}(P), \mathfrak{g}^{(1)})$ of maximal rank, and
- a map $h : \mathcal{B}_{G^{(1)}}(P) \rightarrow X$

such that

$$\begin{cases} d\theta^{(1)} &= c^{(1)}(h)(\theta^{(1)} \wedge \theta^{(1)}) - \eta^{(1)} \wedge \theta^{(1)} \\ d\eta^{(1)} &= R^{(1)}(h)(\theta^{(1)} \wedge \theta^{(1)}) + S^{(1)}(h)(\theta^{(1)} \wedge \eta^{(1)}) - \eta^{(1)} \wedge \eta^{(1)} \\ dh &= \Theta(h) \circ \theta^{(1)} + \Phi(h) \circ \eta^{(1)}. \end{cases}$$

We will call this problem the **underlying realization problem for the $G^{(1)}$ -structure**.

Now, suppose that for each $x \in X$ there exists a realization $\mathcal{B}_G(M)$ with a 1-frame $p \in (\mathcal{B}_G(M))^{(1)}$ such that $h(p) = x$. Then it is clear that $(\mathcal{B}_G(M))^{(1)} \rightarrow \mathcal{B}_G(M)$ is a solution of the underlying realization problem for the $G^{(1)}$ -structure. It then follows from Theorem 5.2.1 that $((c^{(1)}, R^{(1)}, S^{(1)})(\Theta, \Phi))$ are the structure functions of a Lie algebroid $A \cong X \times (\mathbb{R}^n \oplus \mathfrak{g} \oplus \mathfrak{g}^{(1)})$ over X . Moreover, A comes equipped with an infinitesimal action of $\mathfrak{g}^{(1)}$ by inner automorphisms.

Conversely, assume that the initial data of the realization problem determines a Lie algebroid structure on $A \cong X \times (\mathbb{R}^n \oplus \mathfrak{g} \oplus \mathfrak{g}^{(1)})$, and let $\mathcal{B}_{G^{(1)}}(P)$ be a solution of the underlying realization problem for the $G^{(1)}$ -structure, which exists by virtue of Theorem 5.2.1. The problem now becomes to determine whether $\mathcal{B}_{G^{(1)}}(P)$ is locally equivalent at each point to the prolongation of a G -structure $\mathcal{B}_G(M)$.

We saw that for an arbitrary G -structure $\mathcal{B}_G(M)$, its structure equations must decompose as (3.5.3)

$$\begin{cases} d\theta &= c(\theta \wedge \theta) - \eta \wedge \theta \\ d\eta &= R(\theta \wedge \theta) + S(\theta \wedge \eta) - \eta \wedge \eta - \eta^{(1)} \wedge \theta \\ d\eta^{(1)} &= R_1^{(1)}(\theta \wedge \theta) + R_2^{(1)}(\theta \wedge \eta) + R_3^{(1)}(\eta \wedge \eta) + \\ &\quad + S_1^{(1)}(\theta \wedge \eta^{(1)}) + S_2^{(1)}(\eta \wedge \eta^{(1)}). \end{cases}$$

It follows that the functions given in the initial data of the realization problem cannot be arbitrary. They must be decomposable in the form described in Example 3.5.2. From now on, we assume that this is the case.

The first two structure equations above imply that the distribution

$$\mathcal{D} = \{\xi \in \mathfrak{X}(\mathcal{B}_{G^{(1)}}(P)) : \theta(\xi) = 0 = \eta(\xi)\}$$

is integrable. Thus, after restricting to an open set if necessary, we may assume that

$$P \cong \mathcal{B}_{G^{(1)}}(P)/\mathcal{D}.$$

The first equation implies that the distribution

$$\mathcal{D}' = \{\xi \in \mathfrak{X}(\mathcal{B}_{G^{(1)}}(P)) : \theta(\xi) = 0\}$$

is also integrable, and again, after restricting to an open set, we may assume that

$$M = \mathcal{B}_{G^{(1)}}(P)/\mathcal{D}'$$

is a manifold. Moreover, since $\mathcal{D} \subset \mathcal{D}'$, it follows that P fibers over M .

Now, let $\sigma : P \rightarrow \mathcal{B}_{G^{(1)}}(P)$ be a section (again, if necessary, restrict to an open set of P), and denote by $\theta' = \sigma^*\theta$ and $\eta' = \sigma^*\eta$ the pullbacks of θ and η by σ . Then θ' and η' form a coframe on P whose structure equations take the form

$$\begin{cases} d\theta' &= c'(\theta' \wedge \theta') - \eta' \wedge \theta' \\ d\eta' &= R'(\theta' \wedge \theta') + S'(\theta' \wedge \eta') - \eta' \wedge \eta' \end{cases} \quad (5.4.2)$$

where c', R' , and S' are the pullbacks of c, R , and S by σ .

By the very same arguments given in the proof of Theorem 5.2.1, it follows that there is an infinitesimal action of \mathfrak{g} on P for which θ is equivariant and horizontal, and thus, at least locally, P can be embedded into a G -structure over M which satisfies the structure equations (5.4.2). But since two G -structures are equivalent if and only if their prolongations are equivalent, it follows that $(\mathcal{B}_G(M))^{(1)}$ is locally equivalent to $\mathcal{B}_{G^{(1)}}(P)$, which proves the existence of a realization.

We can summarize this example by saying that a solution for the realization problem for G -structures of type 2 with initial data $(n, X, \mathfrak{g}, (c^{(1)}, R^{(1)}, S^{(1)})(\Theta, \Phi))$ exists if and only if:

1. The functions $((c^{(1)}, R^{(1)}, S^{(1)})(\Theta, \Phi))$ decompose as if they were the structure functions of the first prolongation of a G -structure, and
2. the functions $((c^{(1)}, R^{(1)}, S^{(1)})(\Theta, \Phi))$ determine the structure of a Lie algebroid on the trivial vector bundle $A \rightarrow X$ with fibers $\mathbb{R}^n \oplus \mathfrak{g} \oplus \mathfrak{g}^{(1)}$.

The solution of the realization problem in the general case of type k G -structures is very similar to the solution of the problem in the case of type 2 G -structures discussed in the preceding example. The functions $(c^{(k-1)}, R^{(k-1)}, S^{(k-1)})$ cannot be completely arbitrary. In fact, there are components of these functions which must be constant. As an illustration, we note that the component of $c^{(k-1)}$ which lies in $\text{Hom}(\mathbb{R}^n \otimes \mathfrak{g}, \mathbb{R}^n)$ must be constant and given by the action of \mathfrak{g} on \mathbb{R}^n and the component in $\text{Hom}(\wedge^2 \mathfrak{g}, \mathfrak{g})$ must also be constant and given by the Lie bracket on \mathfrak{g} . Let us call the functions $(c^{(k-1)}, R^{(k-1)}, S^{(k-1)})$ **admissible** if they decompose as if they were the structure functions on the prolongation of an arbitrary G -structures. We have the following theorem:

Theorem 5.4.3 *Let $(n, X, \mathfrak{g}, (c^{(k-1)}, R^{(k-1)}, S^{(k-1)})(\Theta, \Phi))$ be the initial data of a Cartan's realization problem for a G -structure of type k . Then, for every $x \in X$, there exists a realization $(M, (\mathcal{B}_G(M))^{(k-1)}, (\theta^{(k-1)}, \eta^{(k-1)}), h)$ with a frame $p \in (\mathcal{B}_G(U))^{(k-1)}$ satisfying $h(p) = x$ if and only if*

1. *the functions $(c^{(k-1)}, R^{(k-1)}, S^{(k-1)})$ are admissible, and*
2. *$((c^{(k-1)}, R^{(k-1)}, S^{(k-1)})(\Theta, \Phi))$ are the structure functions of a Lie algebroid on*

$$A \cong X \times (\mathbb{R}^n \oplus \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}^{(k-1)})$$

over X .

Moreover, in this case, $\mathfrak{g}^{(k-1)}$ acts on A infinitesimally by inner automorphisms, and the action is locally free.

Proof. Assume that for each $x \in X$ we can find a solution

$$(M, (\mathcal{B}_G(M))^{(k-1)}, (\theta^{(k-1)}, \eta^{(k-1)}), h)$$

of the realization problem, with a frame $p \in (\mathcal{B}_G(M))^{(k-1)}$ such that $h(p) = x$. Then $(c^{(k-1)}, R^{(k-1)}, S^{(k-1)})$ are the structure functions of the $k-1$ -th prolongation of a G -structure, and thus decompose as such, i.e., are admissible.

Moreover, $(\mathcal{B}_G(M))^{(k-1)}$ is also a realization of the underlying realization problem for the $G^{(k-1)}$ -structure, and thus, by Theorem 5.2.1, it follows that $((c^{(k-1)}, R^{(k-1)}, S^{(k-1)})(\Theta, \Phi))$ are the structure functions of a Lie algebroid $A \rightarrow X$ on which $\mathfrak{g}^{(k-1)}$ acts locally freely by inner automorphisms.

Conversely, assume that $((c^{(k-1)}, R^{(k-1)}, S^{(k-1)})(\Theta, \Phi))$ are the structure functions of a Lie algebroid on

$$A \cong X \times (\mathbb{R}^n \oplus \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}^{(k-1)})$$

over X , and that the functions $(c^{(k-1)}, R^{(k-1)}, S^{(k-1)})$ are admissible. We proceed exactly as in example 5.4.2. Since $A \rightarrow X$ is a Lie algebroid, it follows that for each $x \in X$ we can find a realization $(Q, \mathcal{B}_{G^{(k-1)}}(Q), (\theta^{(k-1)}, \eta^{(k-1)}), h)$ of the underlying realization problem for the $G^{(k-1)}$ -structure such that $h(p_{k-1}) = x$ for some $p_{k-1} \in \mathcal{B}_{G^{(k-1)}}(Q)$.

The one form $\theta^{(k-1)}$ takes values in $\mathbb{R}^n \oplus \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}^{(k-2)}$ and we can decompose it into its components

$$\theta^{(k-1)} = (\theta, \eta, \eta^{(1)}, \dots, \eta^{(k-2)}).$$

Since the functions $c^{(k-1)}, R^{(k-1)}, S^{(k-1)}$ are admissible, the structure equations for $(\theta^{(k-1)}, \eta^{(k-1)})$ decompose into

$$\begin{cases} \theta = c(\theta \wedge \theta) - \eta \wedge \theta \\ \eta = R(\eta \wedge \eta) + S(\theta \wedge \theta) - \eta \wedge \eta - \eta^{(1)} \wedge \theta \\ \vdots \\ \text{higher order equations} \end{cases} \quad (5.4.3)$$

The first two equations implies that the distribution $\mathcal{D} = \ker(\theta, \eta)$ is integrable, while the first equation implies that $\mathcal{D}' = \ker \theta$ is integrable. Thus, after restricting to an open set, we may assume that

$$M = \mathcal{B}_{G^{(k-1)}}(Q)/\mathcal{D}' \text{ and } P = \mathcal{B}_{G^{(k-1)}}(Q)/\mathcal{D}$$

are both manifolds, and that P fibers over M .

Let $\sigma : P \rightarrow \mathcal{B}_{G^{(k-1)}}(Q)$ be a section of the canonical projection $\mathcal{B}_{G^{(k-1)}}(Q) \rightarrow P$ (if a global section does not exist, restrict again to a smaller open set), and denote by θ' and η' the pullbacks of θ and η by σ . Then θ', η' is a coframe on P , which satisfies the structure equations

$$\begin{cases} d\theta' &= c'(\theta' \wedge \theta') - \eta' \wedge \theta' \\ d\eta' &= R'(\theta' \wedge \theta') + S'(\theta' \wedge \eta') - \eta' \wedge \eta' \end{cases}$$

where c', R' , and S' are the pullbacks of c, R , and S by σ .

By the same arguments as those presented in Example 5.4.2 it follows that P can be locally embedded into a G -structure $\mathcal{B}_G(M)$ over M such that $(\mathcal{B}_G(M))^{(k-1)}$ is locally equivalent to $\mathcal{B}_{G^{(k-1)}}(Q)$. Thus, $(M, (\mathcal{B}_G(M))^{(k-1)}, (\theta^{(k-1)}, \eta^{(k-1)}), h)$ is a realization, from where the theorem follows. ■

Chapter 6

Applications

In this chapter we describe several applications of the existence of a classifying algebroid A for a Cartan's problem. On one hand, we will show how to use the classifying Lie algebroid to prove some classical results about the symmetries of finite type G -structures. On the other hand, we will argue that the classifying Lie algebroid of a fixed geometric structure on a manifold M should be seen as a basic invariant of the structure. In particular, we indicate how to define cohomological invariants of a geometric structure out of the Lie algebroid cohomology of A .

In order to do so, we first explain a slight extension of the realization problem which is better suited for these purposes.

6.1 Generalized Realization Problem

In this section we will generalize Cartan's realization problem to the case where X is a manifold. This will be useful both in practical examples and for proving classical results in differential geometry.

Recall that in Cartan's realization problem, one is given as initial data, an integer n , an open subset X of \mathbb{R}^d , and functions $C_{ij}^k, F_i^a \in C^\infty(X)$, where $1 \leq i, j, k \leq n$, and $1 \leq a \leq d$. If x_1, \dots, x_d are coordinates on X , then we can define n vector fields on X by

$$F_i = \sum_a F_i^a \frac{\partial}{\partial x_a}.$$

If (M, θ^i, h) is a realization of the Cartan problem with initial data (n, X, C_{ij}^k, F_i^a) , then we can interpret

$$dh = \sum_i F_i \theta^i$$

as a map $TM \rightarrow TX$. Notice that no explicit reference to the coordinate system x_a on X is made. This suggests that we can generalize the realization problem as follows:

Problem 6.1.1 (Generalized Realization Problem) *Given the following set of data:*

- an integer n ,
- a d -dimensional manifold X ,
- functions $C_{ij}^k \in C^\infty(X)$, with $1 \leq i, j, k \leq n$, and
- n vector fields $F_i \in \mathfrak{X}(X)$,

does there exist

- an n -dimensional manifold M ,
- a coframe θ^i on M , and
- a smooth map $h : M \rightarrow X$

satisfying

$$d\theta^k = \sum_{i < j} C_{ij}^k(h) \theta^i \wedge \theta^j \quad (6.1.1)$$

$$dh = \sum_i F_i(h) \theta^i. \quad (6.1.2)$$

The solution to this problem is completely analogous to the solution of the original realization problem. Necessary conditions for the existence of solutions are obtained by imposing that $d^2 = 0$, which yields

$$F_j C_{kl}^i + F_k C_{lj}^i + F_l C_{jk}^i = - \sum_m (C_{mj}^i C_{kl}^m + C_{mk}^i C_{lj}^m + C_{ml}^i C_{jk}^m) \quad (6.1.3)$$

$$[F_i, F_j] = - \sum_k C_{ij}^k F_k. \quad (6.1.4)$$

Let us denote by $A \rightarrow X$ the trivial bundle over X with fiber \mathbb{R}^n and let $\alpha_1, \dots, \alpha_n$ be a basis of sections of A . Then we can define a bundle map

$$\# : A \rightarrow TX, \quad \#(\alpha_i) = F_i.$$

We can also define a bracket on $\Gamma(A)$ by setting

$$[\alpha_i, \alpha_j] = - \sum_k C_{ij}^k \alpha_k$$

and extending it to arbitrary sections by imposing \mathbb{R} -linearity and the Leibniz identity. It then follows that (6.1.3) is equivalent to the Jacobi identity for the bracket and (6.1.4) is equivalent to $\# : \Gamma(A) \rightarrow \mathfrak{X}(X)$ being a Lie algebra homomorphism. Thus, as necessary conditions for the existence of solutions of the generalized realization problem we obtain that the initial data must determine the structure of a Lie algebroid on the trivial vector bundle $A \cong X \times \mathbb{R}^n$ over X .

Conversely, if the initial data to the realization problem determines the structure of a Lie algebroid on $A \cong X \times \mathbb{R}^n \rightarrow X$, then (6.1.2) is equivalent to (θ, h) being a Lie algebroid valued one form, and (6.1.1) is equivalent to the Maurer-Cartan equations for this one form. Thus, we may apply the universal property of Maurer-Cartan forms to conclude that realizations exist and that, moreover,

each realization is locally equivalent to a neighborhood of the identity of an \mathfrak{s} -fiber of a local Lie groupoid integrating A , equipped with its Maurer-Cartan form.

The advantage of introducing the generalized realization problem is that now, any fully regular coframe will determine such a problem, and thus, a classifying Lie algebroid. In fact, let θ^i be a fully regular coframe on M , of order s and rank d . Let \mathcal{F}_s denote the set of all coframe derivatives of order up to s of the structure functions C_{ij}^k of the coframe θ^i , i.e.,

$$\mathcal{F}_s = \left\{ C_{ij}^k, \frac{\partial C_{ij}^k}{\partial \theta^l}, \dots, \frac{\partial^s C_{ij}^k}{\partial \theta^{l_1} \dots \partial \theta^{l_s}} \right\}.$$

Let us also denote by $\mathcal{F}_s(M)$, the image of M by all functions in \mathcal{F}_s , i.e.,

$$\mathcal{F}_s(M) = \{I(x) : x \in M \text{ and } I \in \mathcal{F}_s\}.$$

Then, since the coframe is assumed to be regular, $\mathcal{F}_s(M)$ is a d -dimensional immersed submanifold (possibly with self intersection) of the euclidean space \mathbb{R}^N whose coordinates are given by $z = (\dots, z_\sigma, \dots)$ where $\sigma = (i, j, k, l_1, \dots, l_r)$, $0 \leq r \leq s$.

Recall from Section 4.1 that, locally, we can find invariant functions $I_1, \dots, I_d \in \mathcal{F}_s$ which generate \mathcal{F}_t for all $t \geq 0$, in the sense that any $I \in \mathcal{F}_t$ can be written as $I = H(I_1, \dots, I_d)$ for some smooth function $H : \mathbb{R}^d \rightarrow \mathbb{R}$. Thus, the set I_1, \dots, I_d can be regarded as local coordinates on $\mathcal{F}_s(M)$.

In order to set up a generalized realization problem, we let $X = \mathcal{F}_s(M)$. Then we already know that the structure functions C_{ij}^k can be seen as functions on X . Now let $h : M \rightarrow X$ be the map

$$h(x) = \left(C_{ij}^k(x), \frac{\partial C_{ij}^k}{\partial \theta^l}(x), \dots, \frac{\partial^s C_{ij}^k}{\partial \theta^{l_1} \dots \partial \theta^{l_s}}(x) \right)$$

and define n vector fields on X by $F_i = dh(\frac{\partial}{\partial \theta^i})$. Then (n, X, C_{ij}^k, F_i) furnish the initial data to a generalized realization problem.

Finally, we note that the classifying Lie algebroid for the realization problem above is transitive. In fact, one could see this directly, by observing that the map h has rank d . We prefer however to give the following argument: We have seen that each point x on the base of the classifying algebroid corresponds to a germ of a coframe. Moreover, if two such germs belong to the same global coframe, then they correspond to points on the same leaf of A in X . However, by construction, each germ of coframe determined by X corresponds to the germ of the coframe θ^i on some point of M , and thus, the classifying Lie algebroid must be transitive.

6.2 Symmetries of Geometric Structures

In this section we will prove some results about the symmetries of *regular* geometric structures. Our purpose is to show how the existence of a classifying Lie algebroid can be used to recover some classical results in differential geometry. We refer the reader to [21] for more results about transformation groups. We

note however that the results presented here are slightly weaker than those of [21] because we must impose a regularity assumption.

We begin by stating the following simple corollary of Proposition 4.8.2:

Corollary 6.2.1 (Theorem I.3.2 of [21]) *Let θ^i be a fully regular coframe on a manifold M . Then its symmetry group S is a Lie group of dimension $\dim S \leq \dim M$. Moreover, the orbits of the S -action on M are closed submanifolds.*

Proof. Since θ^i is a fully regular coframe, it follows that its structure functions and their coframe derivatives of all orders determine a generalized Cartan's problem for which (M, θ, h) is realization, where h denotes the map which associates to each point p of M the value of its derived invariants at p . It then follows from Proposition 4.8.2 that the symmetry Lie algebra \mathfrak{s} of M is a Lie algebra of dimension less than the dimension of M . The symmetry Lie group S is a Lie group whose Lie algebra is isomorphic to \mathfrak{s} , which proves the first part of the corollary.

Now, in order to see that the orbits of S are closed submanifolds of M , we note that $p, q \in M$ belong to the same orbit if and only if $h(p) = h(q)$. Thus, the orbit of S through a point p is $h^{-1}(h(p))$, which is clearly a closed submanifold. ■

Let G be a Lie subgroup of GL_n , and let $\mathcal{B}_G(M)$ be a G -structure over M . Recall that a symmetry of $\mathcal{B}_G(M)$ is a diffeomorphism $\varphi : M \rightarrow M$ whose lift preserves the G -structure.

Definition 6.2.2 *Let $G \subset \mathrm{GL}_n$ be a Lie subgroup of finite type k . A G -structure is called **fully regular** if the tautological form of its k -th prolongation $(\mathcal{B}_G(M))^{(k)}$ (which is a coframe on $(\mathcal{B}_G(M))^{(k-1)}$) is fully regular.*

Since the symmetries of a G -structure coincide with those of its prolongation (see Theorem 3.3.9), we obtain:

Corollary 6.2.3 (Theorem I.5.1 of [21]) *Let $G \subset \mathrm{GL}_n$ be a Lie subgroup of type k , and let $\mathcal{B}_G(M)$ be a fully regular G -structure over M . Then the symmetry group S of $\mathcal{B}_G(M)$ is a Lie group of dimension*

$$\dim S \leq \dim M + \dim \mathfrak{g}^{k-1},$$

where \mathfrak{g} denotes the Lie algebra of G and $\mathfrak{g}^{k-1} = \mathfrak{g} \oplus \mathfrak{g}^{(1)} \oplus \dots \oplus \mathfrak{g}^{(k-1)}$.

Finally, we describe a slight generalization of the notion of G -structures equipped with connections for which our results are still valid. In what follows, G is an arbitrary Lie group, $H \subset G$ is a closed subgroup of G and

$$\begin{array}{ccc} P & \xrightarrow{H} & \\ \pi \downarrow & & \\ M & & \end{array}$$

is a principle H -bundle over a manifold M whose dimension is equal to $\dim G/H$.

Definition 6.2.4 *A **Cartan connection** on the principal H -bundle P is a one form $\omega \in \Omega^1(P; \mathfrak{g})$ such that*

1. $\omega(\tilde{A}) = A$ for all $A \in \mathfrak{h}$, where \tilde{A} denotes the fundamental vector field on P generated by A ,
2. $(R_h)^*\omega = \text{Ad}_{h^{-1}}\omega$ for all $h \in H$, where Ad denotes the adjoint action of H on \mathfrak{g} , and
3. $\omega(\xi) \neq 0$ for every nonzero vector ξ tangent to P .

We shall call the n -tuple $(P, \pi, M, G, H, \omega)$ a **Cartan geometry** on M .

Remark 6.2.5 A Cartan connection is not a connection on the principal bundle P . In fact, by the last condition in the definition, and since $\dim P = \dim \mathfrak{g}$, it follows that (the components of) a Cartan connection is a coframe on P , and thus cannot be a connection form, because it does not vanish on horizontal vectors.

The **curvature** of a Cartan connection is the \mathfrak{g} -valued two form Ω defined by

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega].$$

As we shall see, a Cartan geometry can be thought of as a homogeneous space deformed by curvature.

We now present a couple of examples of Cartan connections. We refer the reader to [21] and [30] for other examples of Cartan connections, which include projective and conformal geometry.

Example 6.2.6 (G-Structures with Connections) Let $H \subset \text{GL}_n$ be a closed subgroup, $\mathcal{B}_H(M)$ an H -structure over M with tautological form θ , and $\eta \in \Omega^1(\mathcal{B}_H(M), \mathfrak{h})$ a connection form. Then the $\mathbb{R}^n \oplus \mathfrak{h}$ -valued one form $\omega = (\theta, \eta)$ is a Cartan connection on $P = \mathcal{B}_H(M)$. Here, G is the semi-direct product Lie group $\mathbb{R}^n \rtimes H$.

The curvature Ω of this Cartan geometry takes values in $\mathbb{R}^n \oplus \mathfrak{h}$, and its components are the torsion and the curvature of the connection η .

Example 6.2.7 (Homogenous Spaces) Let $M = G/H$ be a homogeneous space. The Maurer-Cartan form ω_{MC} of G is a Cartan connection on the H bundle $P = G$ over M . The curvature Ω of this Cartan geometry vanishes identically.

Conversely, if the curvature of a Cartan geometry on M vanishes, then it is a simple consequence of the universal property of the Maurer-Cartan form on Lie groups that every point of M has a neighborhood which is diffeomorphic to a neighborhood of eH in the homogeneous space G/H .

Definition 6.2.8 A **symmetry** of a Cartan geometry on M is an automorphism ϕ of the H -bundle P which preserves the Cartan connection, i.e., $\phi^*\omega = \omega$.

Again, we shall call a Cartan geometry **fully regular** if its Cartan connection, seen as a coframe on P is fully regular. We then obtain:

Corollary 6.2.9 (Theorem IV.3.1 of [21]) The symmetry group S of a fully regular Cartan geometry $(P, \pi, M, G, H, \omega)$ is a Lie group of dimension $\leq \dim P$. Moreover, its orbits in P are closed submanifolds.

6.3 Cohomological Invariants of Geometric Structures

In this section, we argue that the classifying Lie algebroid A of a fixed geometric structure should be seen as basic invariant of global equivalence up to covering of the structure. Even though the isomorphism class of the classifying Lie algebroid does not distinguish coframes which are globally equivalent, up to covering, we show how to use its Lie algebroid cohomology to obtain invariants of global isomorphism of coframes.

Let θ be a fully regular coframe on a manifold M and let $A \rightarrow X$ be the classifying algebroid of the associated generalized realization problem. Thus, there is a map $h : M \rightarrow X$ such that (M, θ, h) can be seen as a realization.

Recall that we denote by $\mathcal{F}_s(\theta)$ the set of all structure functions of θ and their coframe derivatives up to order s . The key to what follows is that if $\bar{\theta}$ is a coframe on another manifold \bar{M} , and $\pi : \bar{M} \rightarrow M$ is a surjective local diffeomorphism which preserves the coframes, then

$$\mathcal{F}_s(\bar{\theta}) = \pi^* \mathcal{F}_s(\theta), \text{ for all } s \geq 0.$$

Thus, in particular, $(\bar{M}, \bar{\theta}, h \circ \pi)$ is a realization of the generalized Cartan's problem determined by (M, θ) , and π is a realization cover. It is then clear that:

Proposition 6.3.1 *Let θ be a fully regular coframe on M and let $\bar{\theta}$ be an arbitrary coframe on \bar{M} . If (M, θ) and $(\bar{M}, \bar{\theta})$ are globally equivalent, up to covering, then*

1. $\bar{\theta}$ is a fully regular coframe on \bar{M} , and
2. the classifying Lie algebroid of $\bar{\theta}$ is isomorphic to the classifying Lie algebroid of θ .

Remark 6.3.2 *The converse to the proposition above is not true. Example 4.10.1 shows that two coframes (M, θ) and $(\bar{M}, \bar{\theta})$ can have isomorphic classifying Lie algebroid without being globally equivalent, up to covering.*

It follows that the isomorphism class of the classifying Lie algebroid A of a fully regular coframe is an invariant of global equivalence up to covering of the coframe.

The general philosophy we advocate is that a coframe θ on M , viewed as a morphism of Lie algebroids $\theta : TM \rightarrow A$ should relate invariants of A with invariants of the coframe. To illustrate this point of view, we will now describe how to obtain two invariants of coframes by using the Lie algebroid cohomology of A .

Recall from Section 2.1 that on every Lie algebroid there is a differential $d_A : \Gamma(\wedge^\bullet A^*) \rightarrow \Gamma(\wedge^{\bullet+1} A^*)$ which makes $(\Gamma(\wedge^\bullet A^*), d_A)$ into a chain complex whose cohomology $H^\bullet(A)$ is called the Lie algebroid cohomology of A . Explic-

itly,

$$\begin{aligned} d_A \phi(\alpha_0, \dots, \alpha_k) &= \sum_{i=1}^k (-1)^i \#(\alpha_i) \cdot \phi(\alpha_0, \dots, \hat{\alpha}_i, \dots, \alpha_k) + \\ &+ \sum_{0 \leq i < j \leq k} (-1)^{i+j} \phi([\alpha_i, \alpha_j], \alpha_0, \dots, \hat{\alpha}_i, \dots, \hat{\alpha}_j, \dots, \alpha_k). \end{aligned}$$

Recall also that a Lie algebroid morphism $A \rightarrow B$ is a chain map

$$(\Gamma(\wedge^\bullet B^*), d_B) \rightarrow (\Gamma(\wedge^\bullet A^*), d_A)$$

(see Proposition 2.1.11). It follows that a fully regular coframe θ on M induces a map $\theta^* : H^\bullet(A) \rightarrow H_{\text{dR}}^\bullet(M)$ from the Lie algebroid cohomology of the classifying algebroid to the de Rham cohomology of M .

Let us denote by

$$H^*(A) = \bigoplus_{k=1}^n H^k(A),$$

the cohomology ring of A . Then it is clear that:

Proposition 6.3.3 *Let (M, θ) and $(\bar{M}, \bar{\theta})$ be fully regular coframes which are globally equivalent, up to covering. Let (N, η) be a common realization cover, i.e.,*

$$\begin{array}{ccc} & N & \\ \phi \swarrow & & \searrow \bar{\phi} \\ M & & \bar{M} \end{array} .$$

Then the subrings $\phi^ \theta^*(H^*(A))$ and $\bar{\phi}^* \bar{\theta}^*(H^*(A))$ of $H_{\text{dR}}^*(N)$ are isomorphic.*

In particular, if (M, θ) and $(\bar{M}, \bar{\theta})$ are globally equivalent coframes, then the subrings $\theta^(H^*(A)) \subset H_{\text{dR}}^*(M)$ and $\bar{\theta}^*(H^*(A)) \subset H_{\text{dR}}^*(\bar{M})$ are isomorphic.*

Remark 6.3.4 *We can describe the subring $\theta^*(H^*(A))$ in terms of the foliation on M given by the orbits of the infinitesimal action of the symmetry Lie algebra, or equivalently, by the fibers of the map $h : M \rightarrow X$.*

*In fact, let us call a k -form ϕ on M **basic** if it is the pullback by θ of a section φ of $\wedge^k A^*$, i.e.,*

$$\phi = \theta^* \varphi.$$

Then, by definition, every element in $\theta^(H^*(A))$ has a representative which is basic.*

If we write ϕ in terms of the coframe θ

$$\phi = \sum_{i_1 < \dots < i_k} f_{i_1, \dots, i_k} \theta^{i_1} \wedge \dots \wedge \theta^{i_k},$$

then ϕ is basic if and only if the functions $f_{i_1, \dots, i_k} \in C^\infty(M)$ are of the form

$$f_{i_1, \dots, i_k} = a_{i_1, \dots, i_k} \circ h,$$

*where $a_{i_1, \dots, i_k} \in C^\infty(X)$. In other words, the function f_{i_1, \dots, i_k} must be constant along the fibers of h , i.e., they are **basic functions** with respect to the foliation*

on M given by the fibers of h , or equivalently, the orbits of the infinitesimal action of the symmetry Lie algebra of θ .

The one forms θ^k are basic, and since the structure functions of θ can be written as functions on X , the two forms $d\theta^k$ are also basic. Moreover, since the coframe derivatives of h can also be written as functions on X , it follows from the chain rule that the coframe derivatives of a basic function are again basic. Thus, the set of basic forms on M constitute a subcomplex of $(\Omega^\bullet(M), d)$.

Let us call a function $f \in C^\infty(M)$ **1-basic** if its coframe derivatives are basic functions. We define also the set of **l -basic functions** inductively as the set of functions on M whose coframe derivatives are $l-1$ -basic. It is then natural to define the set of **l -basic k -forms**, $\Omega^{l,k}(M, \theta)$ as the set of k -forms on M whose coefficients, when written in terms of the coframe θ , are l -basic. The exterior differential of forms on M then induce a differential

$$\delta : \Omega^{l,k}(M, \theta) \rightarrow \Omega^{l-1,k+1}(M, \theta)$$

which makes $(\Omega^{\bullet,\bullet}, \delta)$ into a bigraded complex. If we denote the cohomology of this complex by $H^{l,k}(M, \theta)$, then it is clear that $\theta^*(H^k(A))$ is isomorphic to $H^{0,k}(M, \theta)$.

Another way to obtain cohomological invariants of coframes is by looking at the characteristic classes of A . Let us recall the simplest of these, namely, the **modular class** of A . We refer the reader to [16] and [22] for more details and examples. For other characteristic classes of Lie algebroids, we refer to [17], [11], and [14].

Definition 6.3.5 *A representation of a Lie algebroid $A \rightarrow X$ on a vector bundle $E \rightarrow X$ is an \mathbb{R} -bilinear map*

$$\nabla : \Gamma(A) \times \Gamma(E) \rightarrow \Gamma(E)$$

which satisfies

- $\nabla_{f\alpha}s = f\nabla_\alpha s$,
- $\nabla_\alpha fs = f\nabla_\alpha s + \#\alpha(f)s$, and
- $\nabla_\alpha \nabla_{\alpha'} s - \nabla_{\alpha'} \nabla_\alpha s - \nabla_{[\alpha, \alpha']} s = 0$,

for all sections $\alpha, \alpha' \in \Gamma(A)$, $s \in \Gamma(E)$ and for all functions $f \in C^\infty(X)$.

When $E = \mathbb{L}$ is an orientable line bundle, and λ is a nowhere vanishing section of \mathbb{L} , the section φ_λ of A^* defined by

$$\varphi_\lambda(\alpha)\lambda = \nabla_\alpha \lambda, \text{ for all } \alpha \in \Gamma(A)$$

is d_A -closed. Moreover, its Lie algebroid cohomology class is independent of the choice of λ . It will be called the **characteristic class** of the representation and denoted by $\text{char}(\nabla)$.

If \mathbb{L} is not orientable, then $\mathbb{L} \otimes \mathbb{L}$ is. It carries a representation $\bar{\nabla}$ induced by ∇ . In this case, we define

$$\text{char}(\nabla) := \frac{1}{2} \text{char}(\bar{\nabla}).$$

Every Lie algebroid has a natural representation on the line bundle

$$\mathbb{L} = \wedge^{\text{top}} A \otimes \wedge^{\text{top}} T^* X$$

defined by

$$\nabla_{\alpha}(\psi \otimes \phi) = [\alpha, \psi] \otimes \phi + \psi \otimes \mathcal{L}_{\#_{\alpha}} \phi,$$

where $[\ , \]$ denotes the Gerstenhaber bracket on $\Gamma(\wedge^{\bullet} A)$. The characteristic class of this representation, denoted by $\text{Mod}(A)$ is called the **modular class** of the Lie algebroid.

Definition 6.3.6 *Let θ be a fully regular coframe on M with classifying Lie algebroid A . The **modular class of the coframe**, denoted by $\text{Mod}(\theta)$, is the cohomology class*

$$\text{Mod}(\theta) = -\theta^* \text{Mod}(A) \in H_{\text{dR}}^1(M).$$

Remark 6.3.7 *Since the modular class of the tangent bundle to a manifold vanishes, the definition above is consistent with the definition in [22] of the modular class of a Lie algebroid morphism.*

It is then clear that:

Proposition 6.3.8 *Let (M, θ) and $(\bar{M}, \bar{\theta})$ be fully regular coframes which are globally equivalent, up to covering. Let (N, η) be a common realization cover, i.e.,*

$$\begin{array}{ccc} & N & \\ \phi \swarrow & & \searrow \bar{\phi} \\ M & & \bar{M} \end{array} .$$

Then

$$\phi^* \text{Mod}(\theta) = \bar{\phi}^* \text{Mod}(\bar{\theta}).$$

In particular, if θ_1 and θ_2 are globally isomorphic fully regular coframes on the same manifold M , then $\text{Mod}(\theta_1) = \text{Mod}(\theta_2)$.

Chapter 7

Examples

This chapter is dedicated to a few examples which will illustrate the results we have obtained through out the thesis. We will exhibit the classifying Lie algebroid for several examples of geometric structures related to torsion-free connections on G -structures. We will begin by considering an arbitrary torsion free connections, on an also arbitrary G -structure. It turns out, however, that the moduli space of all (germs of) torsion free connections on a G -structure will in general be an infinite dimensional space, and thus, not treatable by our methods. On the other hand, there are many interesting classes of connections whose moduli space are finite dimensional. This is the case of constant curvature torsion-free connections and locally symmetric torsion-free connections, which are the first examples to be presented below.

Our main example, though, are the special symplectic connections to which we devote a substantial part of this chapter. Although most of the results concerning these connections are not new, our approach to the problems is slightly different to the original one. Our starting point is the construction of the classifying Lie algebroid, from where all other properties are deduced, thus making many constructions natural and less mysterious. Precise references will be given along the way.

7.1 Structure Equations for Torsion Free Connections

Let $G \subset GL(V)$ be a Lie group, where V is an n -dimensional (real) vector space. If the reader prefers, he may fix a basis of V and identify it with \mathbb{R}^n . Let $\mathcal{B}_G(M)$ be a G -structure over M and let η be a connection on $\mathcal{B}_G(M)$. Then (θ, η) is a coframe on $\mathcal{B}_G(M)$, where θ denotes the tautological form of $\mathcal{B}_G(M)$. It's structure equations take the form

$$\begin{cases} d\theta &= -\eta \wedge \theta \\ d\eta &= R(\theta \wedge \theta) - \eta \wedge \eta \end{cases} \quad (7.1.1)$$

where $R : \mathcal{B}_G(M) \rightarrow \wedge^2 V^* \otimes \mathfrak{g}$ is a smooth map.

If we differentiate the first equation and set it equal to 0, i.e., if we impose

that $d^2\theta = 0$ we obtain, for each $p \in \mathcal{B}_G(M)$

$$R(p)(u, v)w + R(p)(v, w)u + R(p)(w, u)v = 0 \text{ for all } u, v, w \in V,$$

which is known as the **first Bianchi identity**. It follows that the map R takes values in the **space of formal curvatures**

$$\mathcal{K}(\mathfrak{g}) = \{R \in \wedge^2 V^* \otimes \mathfrak{g} : R(u, v)w + \text{cycl. perm.} = 0 \text{ for all } u, v, w \in V\},$$

that is

$$R : \mathcal{B}_G(M) \rightarrow \mathcal{K}(\mathfrak{g}).$$

Since $\mathcal{K}(\mathfrak{g})$ is a vector space, the differential of R is a map

$$dR : T\mathcal{B}_G(M) \rightarrow \mathcal{K}(\mathfrak{g}).$$

Using the fact that (θ, η) is a coframe on $\mathcal{B}_G(M)$, we can express dR in terms of its horizontal component $\frac{\partial R}{\partial \theta}$, its vertical component $\frac{\partial R}{\partial \eta}$, and the coframe as

$$dR = \frac{\partial R}{\partial \theta} \circ \theta + \frac{\partial R}{\partial \eta} \circ \eta, \quad (7.1.2)$$

where

$$\frac{\partial R}{\partial \theta} : \mathcal{B}_G(M) \rightarrow V^* \otimes \mathcal{K}(\mathfrak{g}), \text{ and } \frac{\partial R}{\partial \eta} : \mathcal{B}_G(M) \rightarrow \mathfrak{g}^* \otimes \mathcal{K}(\mathfrak{g}).$$

If we now differentiate the equation for $d\eta$ and impose that $d^2\eta = 0$ we obtain, for each $p \in \mathcal{B}_G(M)$ and for all $u, v, w \in V$ and $A \in \mathfrak{g}$

$$\begin{aligned} \frac{\partial R}{\partial \theta}(p)(u)(v, w) + \frac{\partial R}{\partial \theta}(p)(v)(w, u) + \frac{\partial R}{\partial \theta}(p)(w)(u, v) &= 0 \\ \frac{\partial R}{\partial \eta}(p)(A)(u, v) - R(p)(Au, v) - R(p)(u, Av) - [A, R(p)(u, v)]_{\mathfrak{g}} &= 0. \end{aligned}$$

The first equation above, known as the **second Bianchi identity**, says that $\frac{\partial R}{\partial \theta}$ takes values in the space of **formal covariant derivatives**

$$\mathcal{K}^1(\mathfrak{g}) = \{\psi \in V^* \otimes \mathcal{K}(\mathfrak{g}) : \psi(u)(v, w) + \text{cycl. perm} = 0 \text{ for all } u, v, w \in V\},$$

that is,

$$\frac{\partial R}{\partial \theta} : \mathcal{B}_G(M) \rightarrow \mathcal{K}^1(\mathfrak{g}).$$

On the other hand, the second equation expresses $\frac{\partial R}{\partial \eta}$ as a function of R . If we define a map $\Xi : \mathcal{K}(\mathfrak{g}) \rightarrow \mathfrak{g}^* \otimes \mathcal{K}(\mathfrak{g})$ by

$$\Xi(R)(A)(u, v) = R(Au, v) + R(u, Av) + [A, R(u, v)]_{\mathfrak{g}} \quad (7.1.3)$$

then for any connection η on $\mathcal{B}_G(M)$, we have

$$\frac{\partial R}{\partial \eta}(p) = \Xi(R(p)) \text{ for all } p \in \mathcal{B}_G(M).$$

It follows that $\frac{\partial R}{\partial \eta}(p)$ is determined by $R(p)$, and thus is irrelevant for the equivalence problem of connections on a given G -structure.

In order to proceed, we must now differentiate equation (7.1.2) and impose the condition $d^2R = 0$. This will impose relations on the coefficients of

$$d\left(\frac{\partial R}{\partial \theta}\right) = \frac{\partial^2 R}{(\partial \theta)^2} \circ \theta + \frac{\partial^2 R}{\partial \eta \partial \theta} \circ \eta$$

which must be satisfied. Again, the vertical component, $\frac{\partial^2 R}{\partial \eta \partial \theta}$ will be written as a function of R and $\frac{\partial R}{\partial \theta}$, and the relevant invariant function will be $\frac{\partial^2 R}{(\partial \theta)^2}$. This new invariant will have to satisfy a condition which we will call the **third Bianchi identity**.

The final goal of this process of differentiating the structure equations and imposing $d^2 = 0$ is to obtain at the end, a set of invariant functions for which all other invariant functions are related to. Whenever this occurs (if it ever does), we will have $d^2 = 0$ as a formal consequence of the previous equations.

It turns out, however, that for arbitrary torsion free connections on G -structures, this process will never end, i.e., we will have to introduce at each step new invariant functions which are functionally independent to the previous ones. To see this, we note that there exist connections whose covariant derivatives of order k at a point coincide for all k , but which are not equivalent.

We will now describe some examples, where after restricting the possible values of the curvature R of a torsion free connections, we obtain finite dimensional moduli spaces.

7.2 Constant Curvature Torsion-Free Connections

In this section we describe the space of torsion-free connections on a G -structure $\mathcal{B}_G(M)$ for which the curvature map $R : \mathcal{B}_G(M) \rightarrow \mathcal{K}(\mathfrak{g})$ is constant. Let us begin with an example:

Example 7.2.1 (Flat Torsion-Free Connections) *The most simple example that we may consider is that of imposing that the connection η on $\mathcal{B}_G(M)$ is flat, i.e., that the curvature R vanishes identically, $R(p) \equiv 0$.*

If η is a flat torsion-free connection on a G -structure $\mathcal{B}_G(M)$, then it satisfies the structure equations

$$\begin{cases} d\theta &= -\eta \wedge \theta \\ d\eta &= -\eta \wedge \eta \end{cases} \quad (7.2.1)$$

where θ is the tautological form on $\mathcal{B}_G(M)$.

Note that these are the structure equations of the semi-direct product Lie algebra $W = V \rtimes \mathfrak{g}$, whose bracket is given by

$$[(u, A), (v, B)] = (Av - Bu, [A, B]_{\mathfrak{g}}), \text{ for all } u, v \in V, \text{ and } A, B \in \mathfrak{g}.$$

The $V \rtimes \mathfrak{g}$ -valued one form (θ, η) is a Lie algebra valued Maurer-Cartan form on $\mathcal{B}_G(M)$. It follows that $\mathcal{B}_G(M)$ must be locally equivalent to the canonical flat G -structure $\mathcal{B}_G(V) \cong V \times G$ over V .

We now return to the problem of classifying the constant curvature torsion-free connections. We saw that for any torsion-free connection η , we have

$$dR = \frac{\partial R}{\partial \theta} \theta + \Xi(R)\eta.$$

If the curvature is a constant function, then $dR = 0$, and this imposes restrictions on the possible values of R . In fact, we must have $\Xi(R) = 0$, and thus R must take values in vector space $\ker(\Xi) \subset \mathcal{K}(\mathfrak{g})$. We thus obtain:

Theorem 7.2.2 *Let η be a constant curvature torsion-free connection on a G -structure $\mathcal{B}_G(M)$. Then η satisfies the structure equations*

$$\begin{cases} d\theta &= -\eta \wedge \theta \\ d\eta &= R(\theta \wedge \theta) - \eta \wedge \eta \\ dR &= 0 \end{cases} \quad (7.2.2)$$

where $R : \mathcal{B}_G(M) \rightarrow \ker(\Xi)$.

Remark 7.2.3 *From our perspective, the theorem above should be interpreted as saying that the classifying Lie algebroid of constant curvature torsion-free connections is a bundle of Lie algebras over $\ker(\Xi)$, whose fiber over a point $R \in \ker(\Xi)$ is the Lie algebra $V \rtimes \mathfrak{g}$ equipped with the bracket*

$$[(u, A), (v, B)] = (Av - Bu, [A, B]_{\mathfrak{g}} - R(u, v)).$$

Example 7.2.4 (Metrics of Constant Curvature in \mathbb{R}^2) *As a simple example, suppose we want to classify all Riemann metrics of constant curvature in a neighborhood of the origin in \mathbb{R}^2 . The G -structure to be considered is the orthogonal frame bundle of \mathbb{R}^2 ($G = O_2$). It is well known that in this case, the first order structure function vanishes identically and $O_2^{(1)} = \{e\}$. The structure equation has the following simple form:*

$$\begin{cases} d\omega &= -\eta \wedge \omega \\ d\eta &= k\omega \wedge \omega \end{cases} \quad (7.2.3)$$

where η is the connection form corresponding to the Levi-Civita connection and k is the Gaussian curvature of η . If the curvature k is constant, then $dk = 0$. In terms of the canonical base of \mathbb{R}^2 the structure equation becomes

$$\begin{aligned} d\omega^1 &= -\eta \wedge \omega^2 \\ d\omega^2 &= \eta \wedge \omega^1 \\ d\eta &= k\omega^1 \wedge \omega^2 \\ dk &= 0 \end{aligned} \quad (7.2.4)$$

The Gaussian curvature is the only invariant function. It follows that the Lie algebroid we obtain is $A \rightarrow \mathbb{R}$ with a basis of sections $\{e_1, e_2, e_3\}$ and structure

$$\begin{aligned} [e_1, e_2](k) &= -ke_3 \\ [e_1, e_3](k) &= e_2 \\ [e_2, e_3](k) &= -e_1 \\ \#e_i &= 0. \end{aligned} \quad (7.2.5)$$

Thus, the Lie algebroid we obtain is a bundle of Lie algebras with fibers isomorphic to

\mathfrak{sl}_2	if $k < 0$	Hyperbolic Geometry
\mathfrak{se}_2	if $k = 0$	Euclidean Geometry
\mathfrak{so}_3	if $k > 0$	Spherical Geometry

The inner action of \mathfrak{o}_2 on A is the fiberwise action obtained from the adjoint representation. It follows that to each value of $k \in \mathbb{R}$ there corresponds the germ at 0 of a constant curvature metric on \mathbb{R}^2 .

7.3 Locally Symmetric Torsion-Free Connections

In the same direction as the examples of the preceding section, we now describe the torsion-free connections on a G -structure $\mathcal{B}_G(M)$ for which the covariant derivative of the curvature vanishes.

Definition 7.3.1 *A torsion-free connection η on a G -structure $\mathcal{B}_G(M)$ will be called **locally symmetric** if the covariant derivative of its curvature function vanishes, i.e., $\frac{\partial R}{\partial \theta} \equiv 0$.*

If η is a locally symmetric torsion-free connection, then

$$dR = \Xi(R)\eta.$$

Differentiating this equation yields a restriction on the possible values of R . In fact, if we denote by $\Psi : \mathcal{K}(\mathfrak{g}) \rightarrow \wedge^2 V^* \otimes \mathcal{K}(\mathfrak{g})$ the map

$$\Psi(R)(u, v) = \Xi(R)(R(u, v)),$$

then $d^2R = 0$, applied to two fundamental horizontal vectors implies that R must take values in the zero set of Ψ , i.e.,

$$R : \mathcal{B}_G(M) \rightarrow Z(\Psi) = \{R \in \mathcal{K}(\mathfrak{g}) : \Psi(R) = 0\}.$$

If we now apply $d^2R = 0$ to two fundamental vertical vectors, we see that R must take values in the zero set of the map $\Phi : \mathcal{K}(\mathfrak{g}) \rightarrow \wedge^2 \mathfrak{g}^* \otimes \mathcal{K}(\mathfrak{g})$,

$$\Phi(R)(A, B)(u, v) = \Xi(\Xi(R)(A))(B) - \Xi(\Xi(R)(B))(A) - \Xi(R)([A, B]).$$

After a long, but straightforward calculation, one can show that

$$\Phi(R)(A, B)(u, v) = -2[[A, B], R(u, v)].$$

Finally, applying $d^2R = 0$ to a pair formed by one horizontal and one vertical fundamental vector yields no new restrictions on R . We conclude that:

Theorem 7.3.2 *Let η be a locally symmetric torsion-free connection on a G -structure $\mathcal{B}_G(M)$. Then η satisfies the structure equations*

$$\begin{cases} d\theta &= -\eta \wedge \theta \\ d\eta &= R \circ \theta \wedge \theta - \eta \wedge \eta \\ dR &= \Xi(R)\eta \end{cases} \quad (7.3.1)$$

where $R : \mathcal{B}_G(M) \rightarrow Z(\Psi) \cap \ker \Phi$ and $Z(\Psi)$ is the zero set of the map Ψ .

Remark 7.3.3 *We note that the space $Z(\Psi) \cap \ker \Phi$ is the intersection of the zero set of a quadratic function with a linear subspace, and (at least a priori) is not a manifold. In order to proceed, we must remove all singular points to obtain a classifying Lie algebroid, or equivalently, a generalized realization problem. This restriction comes from the fact that we consider only coframes which are fully regular.*

7.4 Proper Holonomy Groups

In this section we will state some general facts about the holonomy group of a connection on a principal G -bundle. This section is a collection of some results from [31] and [33] to which we refer for greater details.

Let

$$\begin{array}{c} P \quad \curvearrowright G \\ \pi \downarrow \\ M \end{array}$$

be a principal G -bundle over M , and let $\eta \in \Omega^1(P, \mathfrak{g})$ be a connection form. Denote by Γ the horizontal distribution on P

$$\Gamma_p = \{\xi \in T_p P : \eta_p(\xi) = 0\}$$

associated to η . Let Y be a vector field on M . At each $p \in P$ there is a unique vector $\xi_Y \in \Gamma_p$ such that $\pi(\xi_Y) = Y$. A vector field $\xi \in \mathfrak{X}(P)$ which is everywhere tangent to Γ will be called a **horizontal vector field**.

Now let γ be a curve in M . Using the horizontal lifts of vector fields on M to horizontal vector fields on P , one can show that given any point $p \in \pi^{-1}(\gamma(0))$ there is a unique horizontal curve $\tilde{\gamma}_p$ in P such that $\tilde{\gamma}_p(0) = p$ and $\pi(\tilde{\gamma}_p) = \gamma$. At a first glance, one should expect that the curve $\tilde{\gamma}_p$ should be defined only for $0 \leq t < \epsilon$. However, it is not hard to see that it in fact can be extended up to $t = 1$. We define the **parallel translation** along γ to be the map \mathcal{P}_γ which associates to each $p \in \pi^{-1}(\gamma(0))$ the point $\tilde{\gamma}_p(1) \in \pi^{-1}(\gamma(1))$, i.e.,

$$\mathcal{P}_\gamma : \pi^{-1}(\gamma(0)) \rightarrow \pi^{-1}(\gamma(1)), \quad p \mapsto \tilde{\gamma}_p(1).$$

In particular, if γ is a closed curve in M , then $\mathcal{P}_\gamma : \pi^{-1}(\gamma(0)) \rightarrow \pi^{-1}(\gamma(0))$ is a diffeomorphism of the fiber onto itself. Moreover, \mathcal{P}_γ is G -equivariant, i.e., $\mathcal{P}_\gamma(pa) = \mathcal{P}_\gamma(p)a$ for all $a \in G$. The set

$$\text{Hol}_x(\eta) = \{\mathcal{P}_\gamma : \gamma \text{ is a closed curve on } M \text{ with base point } x\}$$

forms a Lie group whose multiplication is inherited from concatenation of paths. It is called the **holonomy group of η at x** .

Suppose that M is connected, let y be another point of M and let γ be a curve joining x to y . Then it is easy to see that

$$\text{Hol}_y(\eta) = \mathcal{P}_\gamma \text{Hol}_x(\eta) \mathcal{P}_\gamma^{-1},$$

and thus both groups are isomorphic.

Now fix a point p in the fiber of P over x . For any $\mathcal{P} \in \text{Hol}_x(\eta)$ we have $\mathcal{P}(p) = pa_{\mathcal{P}}$ for some $a_{\mathcal{P}} \in G$. The map

$$\psi_p : \text{Hol}_x(\eta) \rightarrow G \quad \mathcal{P} \mapsto a_{\mathcal{P}}$$

is an injective group homomorphism. Further more, the conjugacy class of its image in G is independent of the choices of p and x . By abuse of language, we will refer to both $\text{Hol}(\eta) = \psi_p(\text{Hol}_x(\eta))$ and its conjugacy class as the **holonomy group of η** . If any confusion may arise, we will denote the conjugacy class of $\text{Hol}(\eta)$ by $[\text{Hol}(\eta)]$.

The holonomy of a connection is useful when one wants to find reductions of a principal bundle. In fact, one has the following theorem which is proved in [33]:

Theorem 7.4.1 *Let η be a connection on P with holonomy Lie group $\text{Hol}(\eta)$. Then there exists a reduction P' of P to the group $\text{Hol}(\eta)$. Moreover, η restricts to a connection form on P' .*

There is a simple way of calculating the Lie algebra of the holonomy Lie group of a connection. Recall that the curvature of the connection η is the \mathfrak{g} -valued two form Ω defined by

$$\Omega(\xi_1, \xi_2) = d\eta(\xi_1^\Gamma, \xi_2^\Gamma) = d\eta(\xi_1, \xi_2) - [\eta(\xi_1), \eta(\xi_2)], \text{ for all } \xi_1, \xi_2 \in \mathfrak{X}(P),$$

where ξ^Γ denotes the horizontal component of ξ . Now, denote by $\mathfrak{hol}_x(\eta)$ the Lie algebra of $\text{Hol}_x(\eta)$. If we fix a point $p \in \pi^{-1}(x)$, then we can view the holonomy Lie algebra at x as the subalgebra $d_e\psi_p(\mathfrak{hol}_x(\eta))$ of \mathfrak{g} , where e denotes the identity element of $\text{Hol}_x(\eta)$. One of the main results in the theory of holonomy is the relation between the curvature of η and $\mathfrak{hol}_x(\eta)$. It is given by the Ambrose-Singer holonomy theorem [1] which we now state.

Theorem 7.4.2 (Ambrose-Singer Holonomy Theorem) *Fix a point $p \in \pi^{-1}(x)$. Then the holonomy Lie algebra of a connection η at x , $\mathfrak{hol}_x(\eta)$, seen as a subalgebra of \mathfrak{g} is spanned by elements of the form $\Omega_q(\xi_1, \xi_2)$, where q is any point obtained from p by parallel translation, and ξ_1 and ξ_2 are horizontal tangent vectors at q .*

We now turn to the case where $P = \mathcal{B}(M)$ is the frame bundle of M . It was shown in [19] that any closed subgroup of $\text{GL}(V)$ can appear as the holonomy group of some connection. However, if we restrict our attention to torsion-free connections, the problem of determining the closed subgroups that can appear as the holonomy group of some connection becomes non-trivial.

Definition 7.4.3 *A proper subgroup $H \subset \text{GL}_n$ is a **proper holonomy group** if there exists a manifold M and a torsion-free connection η on $\mathcal{B}(M)$ whose holonomy group is (conjugate to) H .*

In [2], it was introduced an algebraic criteria to verify if a Lie subalgebra $\mathfrak{h} \subset \mathfrak{gl}_n$ can be the Lie algebra of a proper holonomy group. Recall that $\mathcal{K}(\mathfrak{h}) \subset \wedge^2 V^* \otimes \mathfrak{h}$ denotes the space of formal curvatures of torsion free connections on G -structures and that $\mathcal{K}^1(\mathfrak{h}) \subset V^* \otimes \mathcal{K}(\mathfrak{h})$ denotes the space of formal covariant derivatives. We denote by

$$\underline{\mathfrak{h}} = \{R(u, v) \in \mathfrak{h} : R \in \mathcal{K}(\mathfrak{h}) \text{ and } u, v \in V\}$$

the subspace of \mathfrak{h} spanned by all possible values of all formal curvatures. This leads us to:

Definition 7.4.4 *A Lie subalgebra $\mathfrak{h} \subset \mathfrak{gl}_n$ is called a **Berger algebra** if $\underline{\mathfrak{h}} = \mathfrak{h}$. A Berger algebra is called **symmetric** if $\mathcal{K}^1(\mathfrak{h}) = 0$ and **non-symmetric** otherwise.*

A simple consequence of the Ambrose-Singer holonomy theorem is that:

Proposition 7.4.5 *If $H \subset \mathrm{GL}_n$ is a proper holonomy group then \mathfrak{h} is a Berger algebra. Moreover, if \mathfrak{h} is a symmetric Berger algebra, then every torsion-free connection with holonomy Lie algebra \mathfrak{h} is locally symmetric.*

The classification of the irreducible (real and complex) Berger algebras was started in [2] and was concluded only in [25]. We refer the reader to [31] for a historical account and a proof of the classification result.

In order to deal with the classification of the possible proper holonomy groups one still has to solve the problem of deciding which of the Berger algebras is in fact the holonomy Lie algebra of a torsion-free connection.

The case of symmetric Berger algebras is easy to solve. Every such algebra is the holonomy Lie algebra of a locally symmetric torsion-free connection. This can be seen as a consequence of the existence of a classifying Lie algebroid for these connections (Section 7.3).

For non-symmetric Berger algebras, the main efforts were divided into two cases: the Riemannian and non-Riemannian cases. It was shown that every Berger subalgebra of $\mathfrak{so}_{p,q}$ is the holonomy Lie algebra of the Levi-Civita connection of a pseudo-Riemannian manifold. The last Berger algebras to be realized as holonomy Lie algebras of a Riemannian manifold were the exceptional Lie algebras \mathfrak{g}_2 and $\mathfrak{spin}(7)$. We now give a very brief idea of the solution presented in [4].

To begin with, we note that if $H \subset G$ is a Lie subgroup, then any H -structure on M also induces a canonical G -structure on M . For this, we take

$$\mathcal{B}_G(M) = \{\mathcal{B}_H(M) \cdot a : a \in G\},$$

which is well defined because $\mathcal{B}_H(M) \subset \mathcal{B}(M)$. When $G = \mathrm{O}_{p,q}$ is the group of linear transformations of V preserving a non-degenerate inner product, the resulting metric on M is called the **underlying pseudo-Riemannian structure induced by H** .

It then follows that the set of metrics on M with holonomy a subgroup $H \subset \mathrm{O}_{p,q}$ coincides with the set of metrics underlying an H -structure $\mathcal{B}_H(M)$ for which the first order structure function vanishes. These, however, can be described as the solution of an exterior differential system. This system turns out to be involutive in the case where H is either G_2 or $\mathrm{Spin}(7)$, and thus Cartan-Kähler theory can be used to prove the existence of solutions. From our perspective, this method can be thought of as an "infinite dimensional version" of Cartan's realization problem.

In the next sections, we give a detailed account of the problem of realizing a Berger subalgebra of $\mathfrak{sp}(V)$ as the holonomy Lie algebra of a symplectic connection on a symplectic manifold (M, ω) , for which our methods do apply.

7.5 Special Symplectic Lie Algebras

In this section we describe the holonomy Lie algebras of torsion-free symplectic connections. We gather here all the algebraic information that will be needed later on.

Let ω be a symplectic form on a manifold M , and let $\mathcal{B}_{\mathrm{Sp}_n}(M)$ be its associated Sp_n -structure over M . A **symplectic connection** on M is a connection

η on this principal bundle. Such connections correspond to linear connections ∇ on TM for which the symplectic form is parallel

$$\nabla\omega = 0.$$

We will sometimes refer to ∇ instead of η as the symplectic connection.

Definition 7.5.1 *Let (V, ω_0) be a symplectic vector space. A proper irreducible Lie subgroup $H \subset \mathrm{Sp}(V, \omega_0)$ is called a **proper symplectic holonomy group** if it is the holonomy group of a torsion-free symplectic connection on some symplectic manifold (M, ω) . Its Lie algebra $\mathfrak{h} \subset \mathfrak{sp}(V, \omega_0)$ will be called a **proper symplectic holonomy algebra**.*

There is a canonical $\mathrm{Sp}(V)$ -equivariant isomorphism $\mathfrak{sp}(V) \cong S^2(V)$ given explicitly by

$$(u \odot v) \cdot w = \omega_0(u, w)v + \omega_0(v, w)u.$$

Moreover, the symmetric bilinear form

$$(u \odot v, w \odot z) = \omega_0(u, w)\omega_0(v, z) + \omega_0(u, z)\omega_0(v, w)$$

is a multiple of the Killing form on $\mathfrak{sp}(V, \omega_0)$. The following lemma was proven in [25]:

Lemma 7.5.2 *Let \mathfrak{h} be a Lie subalgebra of $\mathfrak{sp}(V)$ and consider the \mathfrak{h} -equivariant map $\circ : S^2(V) \cong \mathfrak{sp}(V) \rightarrow \mathfrak{h}$ given by*

$$(u \circ v, T) = \omega_0(Tu, v) \text{ for all } u, v \in V \text{ and } T \in \mathfrak{h}. \quad (7.5.1)$$

If \mathfrak{h} is a proper symplectic holonomy algebra, then

$$(u \circ v)w - (u \circ w)v = 2\omega_0(v, w)u - \omega_0(u, v)w + \omega_0(u, w)v. \quad (7.5.2)$$

This motivates our next definition.

Definition 7.5.3 *A Lie subalgebra $\mathfrak{h} \subset \mathfrak{sp}(V)$ is a **special symplectic Lie algebra** if it satisfies equation (7.5.2).*

Special symplectic Lie algebras are very closely related to certain simple Lie algebras called 2-gradable Lie algebras. This discussion is closely based on [6] to which the reader should refer for greater details.

Let \mathfrak{g} be a complex simple Lie algebra, let α be a long root in a Cartan decomposition of \mathfrak{g} and let $x \neq 0$ be an element of the root space \mathfrak{g}_α . The **root cone of \mathfrak{g}** is the adjoint orbit of x , $\mathrm{Ad}_G(x)$. It is independent of the choice of Cartan decomposition. Any element $y \in \mathrm{Ad}_G(x)$ is called a **maximal root element of \mathfrak{g}** .

Definition 7.5.4 *Let \mathfrak{g} be a simple Lie algebra over \mathbb{R} . We say that \mathfrak{g} is **2-gradable** if it contains a maximal root element of $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes \mathbb{C}$.*

The reason why these Lie algebras are called 2-gradable is that for each one of them, we can find a long root $\alpha_0 \in \Delta$ and a unique element $H_{\alpha_0} \in [\mathfrak{g}_{\alpha_0}, \mathfrak{g}_{-\alpha_0}]$ satisfying $\alpha_0(H_{\alpha_0}) = 2$ such that, if

$$\mathfrak{g}^i = \bigoplus_{\{\beta \in \Delta : \langle \beta, \alpha_0 \rangle = i\}} \mathfrak{g}_\beta$$

for $i \neq 0$ and

$$\mathfrak{g}^0 = \mathfrak{t} \oplus \bigoplus_{\{\beta \in \Delta: \langle \beta, \alpha_0 \rangle = 0\}} \mathfrak{g}_\beta,$$

where $\langle \beta, \alpha_0 \rangle$ is the Cartan number, then,

$$\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2.$$

It follows that

$$\mathfrak{g}^{\pm 2} = \mathfrak{g}_{\pm \alpha_0}$$

and

$$\mathfrak{g}^0 = \mathbb{R}H_{\alpha_0} \oplus \mathfrak{h}$$

where

$$[\mathfrak{h}, \mathfrak{sl}_{\alpha_0}] = 0$$

and

$$\mathfrak{sl}_{\alpha_0} = \text{span} \langle \mathfrak{g}_{\alpha_0}, \mathfrak{g}_{-\alpha_0}, H_{\alpha_0} \rangle$$

is a Lie subalgebra of \mathfrak{g} isomorphic to $\mathfrak{sl}_2(\mathbb{R})$.

If we set

$$\mathfrak{g}^{\text{ev}} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^2$$

and

$$\mathfrak{g}^{\text{odd}} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^1$$

then \mathfrak{g}^{ev} is isomorphic as a Lie algebra to $\mathfrak{sl}_{\alpha_0} \oplus \mathfrak{h}$. Moreover, $\mathfrak{g}^{\text{odd}}$ is isomorphic to $\mathbb{R}^2 \otimes V$ as a \mathfrak{g}^{ev} -module and the action of \mathfrak{h} on V is effective. Thus $\mathfrak{h} \subset \mathfrak{gl}(V)$. The Lie algebra \mathfrak{g} is isomorphic to $\mathfrak{g}^{\text{ev}} \ltimes \mathfrak{g}^{\text{odd}}$.

Now fix an area form $a \in \wedge^2(\mathbb{R})^*$. It induces an $\mathfrak{sl}_2(\mathbb{R})$ -equivariant isomorphism

$$\begin{aligned} S^2(\mathbb{R}^2) &\rightarrow \mathfrak{sl}_2(\mathbb{R}) \\ (ef) \cdot g &= a(e, g)f + a(f, g)e \end{aligned}$$

Under this identification, the Lie algebra structure on $\mathfrak{sl}_2(\mathbb{R}) \cong S^2(\mathbb{R}^2)$ becomes

$$[ef, gh] = a(e, g)fh + a(e, g)fg + a(f, g)eh + a(f, h)eg$$

We are now able to state the relation between 2-gradable simple Lie algebras and special symplectic subalgebras.

Proposition 7.5.5 *Let \mathfrak{g} be a 2-gradable simple Lie algebra*

$$\mathfrak{g} = \mathfrak{g}^{-2} \oplus \cdots \oplus \mathfrak{g}^2 = \mathfrak{g}^{\text{ev}} \oplus \mathfrak{g}^{\text{odd}} \cong (\mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{h}) \oplus (\mathbb{R} \otimes V).$$

Then there exists an \mathfrak{h} -invariant symplectic form $\omega_0 \in \wedge^2 V^$ and a \mathfrak{h} -equivariant product $\circ : S^2(V) \rightarrow \mathfrak{h}$ such that*

$$\begin{aligned} [\cdot, \cdot] : \wedge^2 \mathfrak{g}^{\text{odd}} &\rightarrow \mathfrak{g}^{\text{ev}} \\ [e \otimes u, f \otimes v] &= \omega_0(u, v)ef + a(e, f)u \circ v. \end{aligned} \quad (7.5.3)$$

Moreover, there is a multiple (\cdot, \cdot) of the Killing form such that

$$(T, u \circ v) = \omega_0(Tu, v) = \omega_0(Tv, u)$$

and

$$(u \circ v)w - (u \circ w)v = 2\omega(v, w)u - \omega(u, v)w + \omega(u, w)v$$

for all $T \in \mathfrak{h}$ and $u, v, w \in V$, i.e., \mathfrak{h} is a special symplectic Lie algebra of $\mathfrak{sp}(V, \omega_0)$.

Conversely, if \mathfrak{h} is a special symplectic subalgebra of $\mathfrak{sp}(V, \omega_0)$ then (7.5.3) can be used to define the structure of a 2-gradable simple Lie algebra on

$$\mathfrak{g} = \mathfrak{g}^{-2} \oplus \cdots \oplus \mathfrak{g}^2 = \mathfrak{g}^{\text{ev}} \oplus \mathfrak{g}^{\text{odd}} \cong (\mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{h}) \oplus (\mathbb{R} \otimes V).$$

We remark that if $\mathfrak{h} \subset \mathfrak{sp}(V)$ is a special symplectic Lie algebra with Lie group $H \subset \text{Sp}(V)$ then H is closed and reductive and

$$\mathfrak{h} = \{T \in \mathfrak{sp}(V) : [T, u \circ v] = (Tu) \circ v + u \circ (Tv) \text{ for all } u, v \in V\}.$$

The proposition above gives a one-to-one correspondence between special symplectic subalgebras and 2-gradable simple Lie algebras, which can be used to classify the special symplectic Lie algebras. We note that all the results presented here are still valid if we take V to be a complex vector space. The following table, which was extracted from [6], exhibits all possible (real and complex) special symplectic Lie groups, as well as their representation space V , and their associated 2-gradable simple Lie algebras. We use the notation $\mathbb{F} = \mathbb{R}$ or \mathbb{C} .

Table 1: Real and Complex Special Symplectic Lie Groups

	Type of Δ	G	H	V
(i)	$A_k, k \geq 2$	$SL_{n+2}(\mathbb{F}), n \geq 1$	$GL_n(\mathbb{F})$	$\mathbb{F}^n \oplus (\mathbb{F}^n)^*$
(ii)		$SU(p+1, q+1), p+q \geq 1$	$U(p, q)$	\mathbb{C}^{p+q}
(iii)	$C_k, k \geq 2$	$Sp_{n+1}(\mathbb{F})$	$Sp_n(\mathbb{F})$	\mathbb{F}^{2n}
(iv)	$B_k, D_{k+1}, k \geq 3$	$SO_{n+4}(\mathbb{C}), n \geq 3$	$SL_2(\mathbb{C}) \cdot SO_n(\mathbb{C})$	$\mathbb{C}^2 \otimes \mathbb{C}^n$
(v)		$SO(p+2, q+2), p+q \geq 3$	$SL_2(\mathbb{R}) \cdot SO(p, q)$	$\mathbb{R}^2 \otimes \mathbb{R}^{p+q}$
(vi)		$SO_{n+2}(\mathbb{H}), n \geq 2$	$Sp_1 \cdot SO_n(\mathbb{H})$	\mathbb{H}^n
(vii)	G_2	$G'_2, G_2^{\mathbb{C}}$	$SL_2(\mathbb{F})$	$S^3(\mathbb{F}^2)$
(viii)	F_4	$F_4^{(1)}, F_4^{\mathbb{C}}$	$Sp_3(\mathbb{F})$	$\mathbb{F}^{14} \subset \wedge^3 \mathbb{F}^6$
(ix)	E_6	$E_6^{\mathbb{F}}$	$SL_6(\mathbb{F})$	$\wedge^3 \mathbb{F}^6$
(x)		$E_6^{(2)}$	$SU(1, 5)$	$\mathbb{R}^{20} \subset \wedge^3 \mathbb{C}^6$
(xi)		$E_6^{(3)}$	$SU(3, 3)$	$\mathbb{R}^{20} \subset \wedge^3 \mathbb{C}^6$
(xii)	E_7	$E_7^{\mathbb{C}}$	$Spin(12, \mathbb{C})$	\mathbb{C}^{32}
(xiii)		$E_7^{(5)}$	$Spin(6, 6)$	$\mathbb{R}^{32} \subset \mathbb{C}^{32}$
(xiv)		$E_7^{(6)}$	$Spin(6, \mathbb{H})$	$\mathbb{R}^{32} \subset \mathbb{C}^{32}$
(xv)		$E_7^{(7)}$	$Spin(2, 10)$	$\mathbb{R}^{32} \subset \mathbb{C}^{32}$
(xvi)	E_8	$E_8^{\mathbb{C}}$	$E_7^{\mathbb{C}}$	\mathbb{C}^{56}
(xvii)		$E_8^{(8)}$	$E_7^{(5)}$	\mathbb{R}^{56}
(xviii)		$E_8^{(9)}$	$E_7^{(7)}$	\mathbb{R}^{56}

7.6 Special Symplectic Manifolds

In this section we will define special symplectic manifolds and give a list of all possible geometries that arise. We will do this only for real manifolds, but we note, however, that almost all constructions presented here can be carried out for complex manifolds as well. Next, we give a description in terms of G -structures and derive its structure equations. This will lead us to the classifying Lie algebroid of each special symplectic geometry. Part of this section is based on [6] and [7].

Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be an arbitrary Lie subalgebra. Recall that the space of formal curvature maps (for torsion free connections) is

$$\mathcal{K}(\mathfrak{g}) = \{R \in \wedge^2 V^* \otimes \mathfrak{g} : R(u, v)w + \text{cycl. perm.} = 0, \text{ for all } u, v, w \in V\}$$

and that the space of formal curvature derivatives is

$$\mathcal{K}^1(\mathfrak{g}) = \{\psi \in V^* \otimes \mathcal{K}(\mathfrak{g}) : \psi(u)(v, w) + \text{cycl. perm.} = 0, \text{ for all } u, v, w \in V\}.$$

When (V, ω) is a symplectic vector space of $\dim(V) \geq 4$ and $\mathfrak{h} \subset \mathfrak{sp}(V)$ is a special symplectic Lie algebra, the map

$$\begin{aligned} \mathfrak{h} &\rightarrow \mathcal{K}(\mathfrak{h}) \\ T &\mapsto R_T \end{aligned}$$

given by

$$R_T(x, y) = 2\omega(x, y)T + x \circ (Ty) - y \circ (Tx).$$

is \mathfrak{h} -equivariant and injective. Here \circ is the operation introduced in Lemma 7.5.2.

It follows from the Ambrose-Singer holonomy Theorem 7.4.2 that if the curvature of a connection is contained in \mathfrak{h} , then its holonomy Lie algebra will also be contained in \mathfrak{h} . Let

$$\mathcal{R}_{\mathfrak{h}} = \{R_T : T \in \mathfrak{h}\} \cong \mathfrak{h}$$

and,

$$\mathcal{R}_{\mathfrak{h}}^{(1)} = \{\psi \in V^* \otimes \mathcal{R}_{\mathfrak{h}} : \psi(u)(v, w) + \text{cycl. perm.} = 0, \text{ for all } u, v, w \in V\}.$$

The later is isomorphic to V with explicit isomorphism

$$\begin{aligned} V &\rightarrow \mathcal{R}_{\mathfrak{h}}^{(1)} \\ x &\mapsto \psi_x = R_{x \circ -} \end{aligned}$$

Definition 7.6.1 *A symplectic connection ∇ on (M, ω) is a **special symplectic connection** associated to a special symplectic Lie algebra $\mathfrak{h} \subset \mathfrak{sp}(V)$ if the curvature of ∇ is contained in $\mathcal{R}_{\mathfrak{h}}$. In this case we call (M, ω, ∇) a **special symplectic manifold**.*

We note that there is a map

$$\text{Ric} : \mathcal{K}(\mathfrak{h}) \rightarrow V^* \otimes V^*, \quad \text{Ric}(R)(u, v) = \text{trace}(R(x, \cdot)y).$$

When \mathfrak{h} is a special symplectic Lie algebra, we can decompose $\mathcal{K}(\mathfrak{h})$ into

$$\mathcal{K}(\mathfrak{h}) = \mathcal{R}_{\mathfrak{h}} \oplus \mathcal{W}_{\mathfrak{h}},$$

where $\mathcal{W}_{\mathfrak{h}} = \ker \text{Ric}$ is the kernel of the Ricci map.

We now list the possible special symplectic geometries according to the different possible special symplectic Lie subalgebras \mathfrak{h} :

- I) Bochner-bi-Lagrangian** If H is the Lie group in entry (i) of table 1, M is equipped with two complementary lagrangian distributions which are ∇ -parallel. The condition that the curvature lies in $\mathcal{R}_{\mathfrak{h}}$ is equivalent to the vanishing of the Bochner curvature.
- II) Bochner-Kähler** If H is the Lie group in entry (ii) of the table, then ∇ is the Levi-Civita connection of a (pseudo-) Riemannian Bochner-Kähler metric, i.e., (M, ω, g) is a Kähler manifold for which the Bochner curvature vanishes.
- III) Ricci Type** If H is the Lie group of entry (iii) of the table, then ∇ is a symplectic connection of Ricci type, i.e., for which the Ricci flat component of its curvature vanishes.
- IV) Proper Symplectic Holonomy** If H is one of the Lie groups in entries (iv) to (xviii) of table 1, then $\mathcal{K}(\mathfrak{h}) = \mathcal{R}_{\mathfrak{h}}$ and thus ∇ is a symplectic connections whose holonomy is a proper irreducible subgroup of $\text{Sp}(V)$.

In what follows, we deduce the structure equations of a special symplectic manifold. In order to do so, we will need the following lemma (see [7] for a proof).

Lemma 7.6.2 *Let $\mathfrak{h} \subset \mathfrak{sp}(V)$ be a special symplectic Lie algebra, and let $\dim V \geq 4$. Suppose that $\varphi \in \mathfrak{h}$ is a linear map $\varphi : V \rightarrow V$ satisfying*

$$\varphi(x) \circ y = \varphi(y) \circ x.$$

Then φ is a multiple of the identity map.

The structure of a special symplectic manifold is given by the following theorem. The proof given here is based on [7].

Theorem 7.6.3 *Let (M, ω, ∇) be a special symplectic manifold associated to a special symplectic Lie algebra \mathfrak{h} , with $\dim M \geq 4$. Then there is an H -structure on M compatible with ∇ ,*

$$\begin{array}{ccc} \mathcal{B}_H(M) & \begin{array}{c} \curvearrowright \\ H \end{array} & \\ \pi \downarrow & & \\ M & & \end{array}$$

and maps $\rho : \mathcal{B}_H(M) \rightarrow \mathfrak{h}$, $u : \mathcal{B}_H(M) \rightarrow V$ and $f : \mathcal{B}_H(M) \rightarrow \mathbb{R}$ such that the tautological form $\theta \in \Omega^1(\mathcal{B}_H(M), V)$ and the connection form $\eta \in \Omega^1(\mathcal{B}_H(M), \mathfrak{h})$

satisfy the structure equations:

$$\begin{cases} d\theta &= -\eta \wedge \theta \\ d\eta &= R_\rho(\theta \wedge \theta) - \eta \wedge \eta \\ d\rho &= u \circ \theta - [\eta, \rho] \\ du &= (\rho^2 + f)\theta - \eta u \\ df &= -2\omega(\rho u, \theta) \quad (= -d(\rho, \rho)) \end{cases} \quad (7.6.1)$$

where

$$R_\rho(x, y) = 2\omega(x, y)\rho + x \circ (\rho y) - y \circ (\rho x).$$

Proof. First of all, we note that since the curvature of ∇ is contained in $\mathcal{R}_\mathfrak{h}$ it follows from the Ambrose-Singer holonomy theorem that the holonomy Lie algebra $\mathfrak{hol}(\nabla)$ is also contained in $\mathcal{R}_\mathfrak{h} \cong \mathfrak{h}$. Thus there exists an H -reduction of the frame bundle

$$\begin{array}{ccc} \mathcal{B}_H(M) & \xrightarrow{H} & \\ \pi \downarrow & & \\ M & & \end{array}$$

that is compatible with ∇ , in the sense that the $\mathfrak{gl}(V)$ -valued connection form corresponding to ∇ restricts to an \mathfrak{h} -valued connection form η on $\mathcal{B}_H(M)$.

The assumption that η is torsion free is the same as

$$d\theta = -\eta \wedge \theta.$$

Also by hypothesis, there exists an equivariant map $\rho : \mathcal{B}_H(M) \rightarrow \mathfrak{h}$ such that the curvature of η is R_ρ . Thus,

$$d\eta = R_\rho(\theta \wedge \theta) - \eta \wedge \eta$$

Now, for $x \in V$ and $T \in \mathfrak{h}$, let $\xi_x, \xi_T \in \mathfrak{X}(\mathcal{B}_H(M))$ be vector fields satisfying

$$\theta(\xi_x) = x, \quad \theta(\xi_T) = 0, \quad \eta(\xi_x) = 0, \quad \text{and} \quad \eta(\xi_T) = T.$$

Since ρ is equivariant, it follows that

$$\xi_T(\rho) = -[T, \rho].$$

In fact, fix $p \in \mathcal{B}_H(M)$ and let $\alpha : I \rightarrow \mathfrak{h}$ be the curve

$$\alpha(t) = \rho(R_{\exp(tT)}(p)) = \text{Ad}(\exp(-tT)) \cdot \rho(p).$$

Then, by differentiating at $t = 0$ we obtain

$$d\rho_p(\xi_T(p)) = -[T, \rho(p)].$$

Note that

$$\xi_x(R_\rho) = R_{\xi_x \rho}.$$

It follows that $\xi_x \rho$ represents the covariant derivative of R_ρ , which implies that $\xi_x \rho \in \mathcal{R}_\mathfrak{h}^{(1)}$. Thus, there exists an equivariant map $u : \mathcal{B}_H(M) \rightarrow V$ such that

$$\xi_x \rho = u \circ x.$$

We conclude that

$$d\rho = u \circ \theta - [\eta, \rho]. \quad (7.6.2)$$

Next, since $u : \mathcal{B}_H(M) \rightarrow V$ is equivariant, it follows that

$$\xi_T u = -T u.$$

Now, by differentiating both sides of (7.6.2) we obtain

$$[d\eta, \rho] - [\eta, d\rho] = du \circ \theta + u \circ d\theta.$$

Applying both sides to ξ_x, ξ_y yields

$$(\xi_x u - \rho^2 x) \circ y = (\xi_y u - \rho^2 y) \circ x.$$

It then follows from Lemma 7.6.2 that there exists a function $f : \mathcal{B}_H(M) \rightarrow \mathbb{R}$ such that

$$du = (\rho^2 + f)\theta - \eta u. \quad (7.6.3)$$

We proceed by differentiating both sides of (7.6.3) and imposing $d^2 = 0$. We obtain

$$\begin{aligned} 0 &= (\rho d\rho + (d\rho)\rho) \wedge \theta + (\rho^2 + f)d\theta - (d\eta)u + \eta \wedge du + df \wedge \theta \\ &= \rho(u \circ \theta) \wedge \theta - \rho[\eta, \rho] \wedge \theta + (u \circ \theta)\rho \wedge \theta - [\eta, \rho]\rho \wedge \theta - \rho^2(\eta \wedge \theta) + \\ &\quad - f\eta \wedge \theta - R_\rho(\theta \wedge \theta)u + \eta \wedge (\rho^2 + f)\theta + (\eta \wedge \eta)u - \eta \wedge (\eta u) + df \wedge \theta \end{aligned}$$

If we apply this last equation to (ξ_T, ξ_x) we obtain that

$$\xi_T f = 0.$$

Applying it to (ξ_x, ξ_y) gives us (after a long but straight forward calculation)

$$(\xi_x(f) + 2\omega(\rho u, x))y - (\xi_y(f) + 2\omega(\rho u, y))x = 0.$$

It follows that

$$df = -2\omega(\rho u, \theta). \quad (7.6.4)$$

Finally, after a tedious computation using the structure equation for $d\rho$ and the identities (7.5.1) and (7.5.2), it can be shown that the right hand side of (7.6.4) is equal to $d(\rho, \rho)$. It follows that $d^2 f = 0$ is an identity. ■

We may use the structure equations from the preceding theorem to construct for each special symplectic Lie algebra \mathfrak{h} the classifying Lie algebroid that controls the moduli space of the corresponding geometric structure.

As a vector bundle, A is the trivial bundle over $X \cong \mathfrak{h} \oplus V \oplus \mathbb{R}$ with fiber type $V \oplus \mathfrak{h}$. Its bracket and anchor are defined on constant sections by

$$\begin{aligned} [(x, T), (y, U)](\rho, u, f) &= (Ty - Ux, [T, U] - R_\rho(x, y)) \\ \#(x, T)(\rho, u, f) &= (u \circ x - [T, \rho], (\rho^2 + f)x - Tu, -2\omega(\rho u, x)) \end{aligned}$$

and then extended by imposing linearity and the Leibniz identity.

7.7 Associated Deforming Maps

In this section we will begin to describe the structure of the classifying Lie algebroid $A \rightarrow X$ for special symplectic connections. In particular, we will show that X can be decomposed into saturated subsets over which the restriction of A is isomorphic to a quadratic deformation of a linear Poisson manifold. This will in turn provide a tool for understanding the structure of the classifying Lie algebroid, and thus for obtaining explicit models for special symplectic connections (see Section 7.8). All of the results of this section are present in many of the papers of Schwachhöfer et al. (see for example [10]). Our approach, which is based only on the analysis of the classifying Lie algebroid, differs from that of the reference cited above.

To begin with, note that the last structure equation satisfied by all special symplectic connections,

$$df = -d(\rho, \rho),$$

implies that $F = f + (\rho, \rho)$ is constant. Thus,

$$F_c = \{(\rho, u, f) \in \mathfrak{h} \oplus V \oplus \mathbb{R} : f + (\rho, \rho) = c\} \subset \mathfrak{h} \oplus V \oplus \mathbb{R}$$

is a submanifold saturated by the leaves of A . Let us denote by $A_c \rightarrow F_c$ the restriction of the classifying Lie algebroid A to F_c .

The map $(\rho, u) \mapsto (\rho, u, c - (\rho, \rho))$ is a diffeomorphism of $\mathfrak{h} \oplus V$ to F_c . If we now identify \mathfrak{h} with \mathfrak{h}^* using (\cdot, \cdot) and V with V^* using $\omega(\cdot, \cdot)$ we may see A_c as a Lie algebroid structure on $T^*(\mathfrak{h}^* \oplus V^*)$. We will use the notation $(\rho, u)^*$ to denote $((\rho, \cdot), \omega(u, \cdot)) \in \mathfrak{h}^* \oplus V^*$.

Proposition 7.7.1 *The Lie algebroid $A_c \rightarrow \mathfrak{h}^* \oplus V^*$ is isomorphic to the cotangent Lie algebroid associated to a Poisson structure $\{ \cdot, \cdot \}_c$ on $\mathfrak{h}^* \oplus V^*$.*

Proof. We can use the anchor of A_c to define a Poisson structure on $\mathfrak{h}^* \oplus V^*$. In fact, for $(T, x), (U, y) \in \mathfrak{h} \oplus V$, seen as functions on $\mathfrak{h}^* \oplus V^*$ we define

$$\{(T, x), (U, y)\}_c(\rho, u)^* = \langle \#_{A_c}(T, x)_{(\rho, u)^*}, (U, y) \rangle$$

If we open up this expression we obtain

$$\begin{aligned} \{(T, x), (U, y)\}_c(\rho, u)^* &= \langle \#_{A_c}(T, x)_{(\rho, u)^*}, (U, y) \rangle \\ &= \left\langle (u \circ x - [T, \rho], (\rho^2 - (\rho, \rho) + c)x - Tu)^*, (U, y) \right\rangle \\ &= (u \circ x, U) - ([T, \rho], U) + \omega((\rho^2 - (\rho, \rho) + c)x, y) \\ &\quad - \omega(Tu, y) \\ &= -\omega(u, Ux) + (\rho, [T, U]) + \omega((\rho^2 - (\rho, \rho) + c)x, y) \\ &\quad + \omega(u, Ty) \\ &= (\rho, [T, U]) + \omega(u, Ty - Ux) + \omega((\rho^2 - (\rho, \rho) + c)x, y) \end{aligned}$$

It is straight foreword to verify that the bracket above defines a Poisson structure on $\mathfrak{h}^* \oplus V^*$. In fact, it is a Lie Poisson structure deformed by a quadratic term.

We now calculate the induced Lie algebroid bracket on $T^*(\mathfrak{h}^* \oplus V^*)$. It is obtained by differentiating the Poisson bracket $\{ \cdot, \cdot \}_c$, i.e.,

$$[(T, x), (U, y)]_c(\rho, u)^* = d_{(\rho, u)^*} \{(T, x), (U, y)\}_c$$

Thus, If $(\rho', u')^* \in T_{(\rho, u)^*}(\mathfrak{h}^* \oplus V^*) \cong \mathfrak{h}^* \oplus V^*$ we obtain

$$\begin{aligned} \langle (\rho', u')^*, [(T, x), (U, y)]_c \rangle_{(\rho, u)^*} &= (\rho', [T, U]) + \omega(u', Ty - Ux) + \\ &+ \omega((\rho\rho' + \rho'\rho - 2(\rho', \rho))x, y) \end{aligned} \quad (7.7.1)$$

Finally, to conclude the proposition, we calculate

$$\begin{aligned} \langle (\rho', u')^*, [(T, x), (U, y)]_{Ac} \rangle_{(\rho, u)^*} &= \langle (\rho', u')^*, ([T, U] - R_\rho(x, y), Ty - Ux) \rangle \\ &= (\rho', [T, U]) + \omega(u', Ty - Ux) - (\rho', R_\rho(x, y)) \\ &= (\rho', [T, U]) + \omega(u', Ty - Ux) - (\rho', 2\omega(x, y)\rho + x \circ (\rho y) - y \circ (\rho x)) \\ &= (\rho', [T, U]) + \omega(u', Ty - Ux) - (\rho', \rho)2\omega(x, y) - (\rho', x \circ (\rho y)) + (\rho', y \circ (\rho x)) \\ &= (\rho', [T, U]) + \omega(u', Ty - Ux) + \omega(-2(\rho', \rho)x, y) - \omega(\rho'x, \rho y) + \omega(\rho'y, \rho x) \\ &= (\rho', [T, U]) + \omega(u', Ty - Ux) + \omega((\rho\rho' + \rho'\rho - 2(\rho', \rho))x, y) \end{aligned}$$

■

We observe that the two first terms of the Poisson bracket

$$\{(T, x), (U, y)\}_c(\rho, u)^* = (\rho, [T, U]) + \omega(u, Ty - Ux) + \omega((\rho^2 - (\rho, \rho) + c)x, y)$$

are linear, and correspond to the Lie-Poisson bracket on the dual of the semi-direct product Lie algebra $\mathfrak{h} \ltimes V$, while the last term is a function which is quadratic on ρ .

The map $\phi_c : \mathfrak{h}^* \rightarrow \wedge^2 V^*$ given by

$$\phi_c(\rho^*)(x, y) = \omega((\rho^2 - (\rho, \rho) + c)x, y)$$

is an example of a *deforming map*.

Definition 7.7.2 Let $G \subset \text{GL}(V)$ be a Lie subgroup with Lie algebra \mathfrak{g} . A smooth map $\phi : \mathfrak{g}^* \rightarrow \wedge^2 V^*$ is called a **deforming map** if

1. ϕ is G -equivariant, and
2. for every $\rho \in \mathfrak{g}^*$, the dual map $(d_\rho \phi)^* : \wedge^2 V \rightarrow \mathfrak{g}$ is contained in $\mathcal{K}(\mathfrak{g})$.

Deforming maps were introduced in [10], with the purpose of showing the existence of an infinite series of proper holonomy groups which did not appear in Berger's original list of Berger groups. They provide a recipe for constructing torsion-free connections on G -structures as follows.

Let $W = V \ltimes \mathfrak{g}$ be the semi-direct product Lie algebra, denote by $\text{pr} : W^* \rightarrow \mathfrak{g}^*$ the natural projection and let $\Phi = \phi \circ \text{pr}$. It follows easily that if $\phi : \mathfrak{g}^* \rightarrow \wedge^2 V^*$ is a deforming map, then

$$\{f, g\}_\phi(p) = p([(x, A), (y, B)]) + \Phi(p)(x, y)$$

is a Poisson bracket on W^* , where $df_p = (x, A)$ and $dg_p = (y, B)$. In fact, for a G -equivariant map $\phi : \mathfrak{g}^* \rightarrow \wedge^2 V^*$, the bracket $\{ , \}_\phi$ is a Poisson bracket if

and only if ϕ is a deforming map. We observe that when $\phi = 0$ we recover the linear Poisson bracket $\{\cdot, \cdot\}$ on W^* .

The Lie algebroid $T^*W^* \rightarrow W^*$ of $(W^*, \{\cdot, \cdot\}_\phi)$ is a flat vector bundle on which \mathfrak{g} acts infinitesimally by inner algebroid automorphisms. It follows that we may view it as the classifying Lie algebroid for a class of connections on G -structures. Connections belonging to this class are called **connections induced by the deforming map ϕ** .

We note that we can use the Lie algebroid structure of T^*W^* to write down the structure equations satisfied by a connection induced by a deforming map:

Proposition 7.7.3 *Let $G \subset \text{GL}(V)$ be a Lie subgroup with Lie algebra \mathfrak{g} and let $\mathcal{B}_G(M)$ be a G -structure, whose tautological form we denote by θ . Then, for any connection η on $\mathcal{B}_G(M)$ which is induced by a deforming map $\phi : \mathfrak{g}^* \rightarrow \wedge^2 V^*$, there exist equivariant maps $\rho : \mathcal{B}_G(M) \rightarrow \mathfrak{g}^*$ and $\mu : \mathcal{B}_G(M) \rightarrow V^*$ such that the structure equations*

$$\begin{cases} d\theta &= -\eta \wedge \theta \\ d\eta &= R_\rho \circ \theta \wedge \theta - \eta \wedge \eta \\ d\rho &= j(\mu \otimes \theta) - \eta \cdot \rho \\ d\mu &= \phi(\rho)^\sharp(\theta) - \eta \cdot \mu \end{cases} \quad (7.7.2)$$

are satisfied. Here, the dots denote the natural actions of \mathfrak{g} on \mathfrak{g}^* and V^* , $R_\rho = (d_\rho \phi)^*$, $j : V^* \otimes V \rightarrow \mathfrak{g}^*$ is the natural projection $j(\mu \otimes u)(A) = \mu(Au)$, and $\phi(\rho)^\sharp : V \rightarrow V^*$ is the map $\phi(\rho)^\sharp(u)(v) = \phi(\rho)(u, v)$.

It then follows from the first equation in the proposition above that every connection induced by the deforming map ϕ is torsion-free. The second equation tells us that the connections curvature is $R_\rho = (d_\rho \phi)^*$.

In summary, it is a consequence of Proposition 7.7.1, that every special symplectic connection is a connection induced by one of the deforming maps ϕ_c .

7.8 Moduli, Symmetries and Models of Special Symplectic Connections

In this section, we describe the structure of the cotangent Lie algebroid of a Poisson structure on $\mathfrak{h} \oplus V \oplus \mathbb{R}$ as an extension of the classifying Lie algebroid A , of special symplectic connections. This will allow us to describe the isotropy Lie algebras of this Lie algebroid as an extension of the symmetry Lie algebra of a special symplectic connection by a 1-dimensional Lie algebra. We also describe how to construct explicit examples of special symplectic manifolds and give results about their moduli space.

We begin by recalling some basic properties of extensions of Lie algebroids. These results are standard, and we refer the reader to [11] for proofs.

Definition 7.8.1 *Let A and \tilde{A} be Lie algebroids and let E be a bundle of Lie algebras all over the same base X . We say that \tilde{A} is an **extension of A by E** if there is an exact sequence of Lie algebroids*

$$0 \longrightarrow E \longrightarrow \tilde{A} \xrightarrow{\pi} A \longrightarrow 0.$$

The extension is called **central** if $[s, \tilde{\alpha}] = 0$ for all $s \in \Gamma(E)$ and $\tilde{\alpha} \in \Gamma(\tilde{A})$ (in this case, E must be abelian, i.e., just a vector bundle).

Remark 7.8.2 *There is a more general notion of extension for the Lie algebroids over different bases [3]. However, we will not need this here.*

Every central extension induces a representation of A on E (see Definition 6.3.5). In fact, we have:

Lemma 7.8.3 *If \tilde{A} is a central extension of A by E then, for $\alpha \in \Gamma(A)$ and $s \in \Gamma(E)$,*

$$\nabla_\alpha s = [\tilde{\alpha}, s]$$

defines a representation of A on E , where $\tilde{\alpha} \in \Gamma(\tilde{A})$ is an arbitrary lift of α .

Now, let $\sigma : A \rightarrow \tilde{A}$ be an arbitrary splitting of π . The curvature of σ is the E -valued 2-form $\Omega_\sigma \in \Gamma(\wedge^2 A^* \otimes E)$ defined by

$$\Omega_\sigma(\alpha, \beta) = \sigma([\alpha, \beta]_A) - [\sigma(\alpha), \sigma(\beta)]_{\tilde{A}}.$$

Lemma 7.8.4 *If \tilde{A} is a central extension of A by E , ∇ is the associated representation of A on E , and σ is an arbitrary splitting of $\pi : \tilde{A} \rightarrow A$, then \tilde{A} is isomorphic to the Lie algebroid $A_\sigma = A \oplus E$ with anchor given by $\#_\sigma(\alpha, v) = \#_A(\alpha)$ and bracket given by*

$$[(\alpha, s), (\alpha', s')]_\sigma = ([\alpha, \alpha']_A, \nabla_\alpha s' - \nabla_{\alpha'} s + \Omega_\sigma(\alpha, \alpha')). \quad (7.8.1)$$

Remark 7.8.5 *Given a representation of A on E , we can define a cohomology with coefficients in E , which we denote by $H^\bullet(A; E)$. This is the cohomology of the complex of differential forms on A with values in E , whose differential $d_{A,E} : \Gamma(\wedge^\bullet A^* \otimes E) \rightarrow \Gamma(\wedge^{\bullet+1} A^* \otimes E)$ is given by*

$$\begin{aligned} d_{A,E}\eta(\alpha_0, \dots, \alpha_k) &= \sum_{i=1}^k (-1)^i \nabla_{\alpha_i} \eta(\alpha_0, \dots, \hat{\alpha}_i, \dots, \alpha_k) + \\ &+ \sum_{0 \leq i < j \leq k} (-1)^{i+j} \eta([\alpha_i, \alpha_j], \alpha_0, \dots, \hat{\alpha}_i, \dots, \hat{\alpha}_j, \dots, \alpha_k). \end{aligned}$$

With this differential, the 2-form Ω_σ is a cocycle. In fact, given any 2-form on A with values in a representation E , the bracket (7.8.1) is a Lie bracket if and only if the 2-form is a cocycle.

We return from this short detour to the study of special symplectic connections. The one parameter family of Poisson structures $\{ , \}_c$ on $\mathfrak{h}^* \oplus V^*$ may be put together into a Poisson structure on $\mathfrak{h}^* \oplus V^* \oplus \mathbb{R}$ by letting

$$\{(T, x, t), (U, y, s)\}(\rho^*, u^*, c) = (\rho, [T, U]) + \omega(u, Ty - Ux) + \omega((\rho^2 - (\rho, \rho) + c)x, y).$$

Proposition 7.8.6 *The inclusions*

$$i_c : (\mathfrak{h}^* \oplus V^*, \{ , \}_c) \rightarrow \mathfrak{h}^* \oplus V^* \oplus \{c\} \subset (\mathfrak{h}^* \oplus V^* \oplus \mathbb{R}, \{ , \})$$

of the Poisson structures induced by each of the deforming maps ϕ_c is a Poisson morphism.

Proof. This is just a simple calculation. If $f \in C^\infty(\mathfrak{h}^* \oplus V^* \oplus \mathbb{R})$ is a linear function and if we denote by $f_{\mathfrak{h}^* \oplus V^*}$ and $f_{\mathbb{R}}$ its $\mathfrak{h}^* \oplus V^*$ and \mathbb{R} components, then

$$f \circ i_c(\rho^*, u^*) = f(\rho^*, u^*, c) = f_{\mathfrak{h}^* \oplus V^*}(\rho^*, u^*) + f_{\mathbb{R}}(c).$$

The last term is constant, and thus will not alter the Poisson bracket. It then follows that

$$\begin{aligned} \{(T, x, t), (U, y, s)\} \circ i_c(\rho^*, u^*) &= \{(T, x, t), (U, y, s)\}(\rho^*, u^*, c) \\ &= \{(T, x), (S, y)\}_c(\rho^*, u^*) \\ &= \{(T, x, t) \circ i_c, (U, y, s) \circ i_c\}_c(\rho^*, u^*), \end{aligned}$$

from where the proposition follows. ■

The result to be presented next should be seen as an infinitesimal version of Theorem B of [6]. Throughout what follows, we identify \mathfrak{h} with \mathfrak{h}^* and V with V^* without further notice.

Proposition 7.8.7 *The cotangent Lie algebroid of the Poisson manifold $(\mathfrak{h}^* \oplus V^* \oplus \mathbb{R}, \{, \})$ is a central extension of the classifying Lie algebroid A by a line bundle \mathbb{L} .*

Proof. First of all, note that the natural projection $\pi : T^*(\mathfrak{h}^* \oplus V^* \oplus \mathbb{R}) \rightarrow A$ is a Lie algebroid morphism. In fact, a simple computation shows that

$$\begin{aligned} \langle (\rho', u')^*, \pi([(T, x, t), (U, y, s)]_{T^*(\mathfrak{h}^* \oplus V^* \oplus \mathbb{R})}) \rangle_{(\rho, u, f)^*} \\ = (\rho', [T, U]) + \omega(u', Ty - Ux) + \omega((\rho\rho' + \rho'\rho - 2(\rho', \rho))x, y) \end{aligned}$$

from where it follows that

$$\pi([(T, x, t), (U, y, s)]_{T^*(\mathfrak{h}^* \oplus V^* \oplus \mathbb{R})}) = ([\pi(T, x, t), \pi(U, y, s)]_A)$$

for all $(T, x, t), (U, y, s) \in \mathfrak{h} \oplus V \oplus \mathbb{R}$. Obviously,

$$\mathbb{L}_{(\rho, u, c)} = \ker \pi_{(\rho, u, c)} = \{(0, 0, t) : t \in \mathbb{R}\}$$

and thus \mathbb{L} is a line bundle. The proposition then follows from the fact that

$$[(0, 0, t), (U, y, s)]_{T^*(\mathfrak{h}^* \oplus V^* \oplus \mathbb{R})} = 0$$

which shows that the extension is central. ■

As explained above, the extension induces a representation of A on \mathbb{L} . However, there is nothing new here. This representation is just the canonical one, i.e.,

$$\nabla_\alpha f = \# \alpha(f)$$

where $f \in C^\infty(\mathfrak{h} \oplus V \oplus \mathbb{R}) \cong \Gamma(\mathbb{L})$.

What is interesting is the two cocycle $\Omega_\sigma \in \Gamma(\wedge^2 A^*)$ induced from the obvious splitting

$$\sigma : (T, x) \mapsto (T, x, 0)$$

of π . A straight forward computation shows that

$$\Omega_\sigma((T, x), (U, y)) = \omega(x, y),$$

so that $\Omega_\sigma = \omega(\theta \wedge \theta)$ is given by the symplectic form.

It follows from equation 7.8.1 that the Lie algebroid structure on $T^*(\mathfrak{h}^* \oplus V^* \oplus \mathbb{R})$ can be written as

$$\begin{aligned} [(T, x, t), (U, y, s)](\rho, u, f) &= ([T, U] - R_\rho(x, y), Ty - Ux, -\omega(x, y)) \\ \#(T, x, t)(\rho, u, f) &= (u \circ x - [T, \rho], (\rho^2 + f)x - Tu, -2\omega(\rho u, x)), \end{aligned}$$

or equivalently, its structure equations are given by

$$\begin{cases} d\theta &= -\eta \wedge \theta \\ d\eta &= R_\rho(\theta \wedge \theta) - \eta \wedge \eta \\ d\kappa &= \omega(\theta \wedge \theta) \\ d\rho &= u \circ \theta - [\eta, \rho] \\ du &= (\rho^2 + f)\theta - \eta u \\ df &= -2\omega(\rho u, \theta) \end{cases} \quad (7.8.2)$$

The following consequences are now clear.

Corollary 7.8.8 *The leaves of A in $\mathfrak{h} \oplus V \oplus \mathbb{R}$ coincide with the symplectic leaves of $T^*(\mathfrak{h}^* \oplus V^* \oplus \mathbb{R})$.*

Corollary 7.8.9 *Let \mathfrak{s}_{λ_0} be the isotropy Lie algebra of A at $\lambda_0 = (\rho, u, f) \in \mathfrak{h} \oplus V \oplus \mathbb{R}$, i.e., the symmetry Lie algebra of the corresponding special symplectic connection, and let*

$$\mathfrak{g}_{\lambda_0} = \{\tilde{\alpha} \in \mathfrak{h} \oplus V \oplus \mathbb{R} \subset \mathfrak{g} : \text{ad}_{\tilde{\alpha}}^* \lambda_0 = 0\}$$

be the isotropy Lie algebra of $T^(\mathfrak{h}^* \oplus V^* \oplus \mathbb{R})$. Then*

$$0 \longrightarrow \mathbb{R}\lambda_0 \longrightarrow \mathfrak{g}_{\lambda_0} \longrightarrow \mathfrak{s}_{\lambda_0} \longrightarrow 0$$

is an extension of Lie algebras. In particular,

$$\dim \mathfrak{s}_{\lambda_0} = \dim \mathfrak{g}_{\lambda_0} - 1.$$

The infinitesimal information gathered here also helps in the construction of models of special symplectic manifolds.

As shown in [6], the Poisson structure on $\mathfrak{h}^* \oplus V^* \oplus \mathbb{R}$ is integrable. In fact, let \mathfrak{g} be the 2-gradable simple Lie algebra associated to \mathfrak{h} and let $a \in \wedge^2 \mathbb{R}^*$ be the area form used to identify $\mathfrak{sl}_2(\mathbb{R})$ with $S^2(\mathbb{R}^2)$ (see Section 7.5). We fix a basis e_+, e_- of \mathbb{R}^2 such that $a(e_+, e_-) = 1$. If we identify \mathfrak{g} with \mathfrak{g}^* using the bi-invariant form (\cdot, \cdot) and let Q be the submanifold of \mathfrak{g}^* defined by

$$Q = \left\{ \frac{1}{2}(e_-^2 + fe_+^2) + \rho + (e_+ \otimes u) : f \in \mathbb{R}, \rho \in \mathfrak{h}, \text{ and } u \in V \right\} \subset \mathfrak{g} \cong \mathfrak{g}^*,$$

then we have the following crucial theorem, (for the proof we refer to [6]).

Theorem 7.8.10 *The diffeomorphism $\Phi : Q \rightarrow \mathfrak{h} \oplus V \oplus \mathbb{R}$ defined by*

$$\frac{1}{2}(e_-^2 + fe_+^2) + \rho + (e_+ \otimes u) \mapsto \rho + u + (f + (\rho, \rho))$$

is a Poisson isomorphism. Moreover, Q is a cosymplectic submanifold of \mathfrak{g}^ which has a symplectic groupoid given by*

$$\Sigma(Q) = \{(\lambda, g) \in Q \times G : \text{Ad}_g^* \lambda \in Q\} \rightrightarrows Q$$

where G is a Lie group integrating \mathfrak{g} .

Remark 7.8.11 *The groupoid structure on $\Sigma(Q)$ is the one inherited by the groupoid $T^*G \cong \mathfrak{g}^* \times G$ integrating the linear Poisson manifold \mathfrak{g}^* .*

We can thus identify $\Sigma(\mathfrak{h} \oplus V \oplus \mathbb{R})$ with the symplectic subgroupoid

$$\{(\lambda, g) \in (\mathfrak{h} \oplus V \oplus \mathbb{R}) \times G : \text{Ad}_g^* \lambda \in \mathfrak{h} \oplus V \oplus \mathbb{R}\} \rightrightarrows \mathfrak{h} \oplus V \oplus \mathbb{R}$$

of $\Sigma(\mathfrak{g}^*) = T^*G$.

Now, since the special symplectic Lie group H of \mathfrak{h} coincides with the identity component of (see [6])

$$\text{stab}(\mathfrak{h} \oplus V \oplus \mathbb{R}) = \{g \in G : \text{Ad}_g^*(\mathfrak{h} \oplus V \oplus \mathbb{R}) \subset \mathfrak{h} \oplus V \oplus \mathbb{R}\}$$

it follows that $\Sigma(\mathfrak{h} \oplus V \oplus \mathbb{R})$ is invariant by the free action of H on T^*G through right multiplication. Moreover, the \mathfrak{s} -fibers of $\Sigma(\mathfrak{h} \oplus V \oplus \mathbb{R})$ are also invariant.

On the other hand, the one parameter subgroup $\exp(\mathbb{R}\lambda)$ of G also acts locally freely on $\mathfrak{s}^{-1}(\lambda)$ for each $\lambda \in \mathfrak{h} \oplus V \oplus \mathbb{R}$ and this action commutes with the H action. Thus, by "integrating" the extension

$$0 \longrightarrow \mathbb{L} \longrightarrow T^*(\mathfrak{h}^* \oplus V^* \oplus \mathbb{R}) \longrightarrow A \longrightarrow 0,$$

we obtain:

Theorem 7.8.12 *If $\mathfrak{s}^{-1}(\lambda)/\exp(\mathbb{R}\lambda)$ is smooth then each of its points has a neighborhood which can be embedded into the total space $\mathcal{B}_H(M)$ of an H -structure corresponding to a special symplectic manifold. Moreover, if*

$$M_\lambda = \frac{(\mathfrak{s}^{-1}(\lambda)/\exp(\mathbb{R}\lambda))}{H}$$

is smooth, then it is itself a special symplectic manifold.

Proof. This theorem is a straightforward consequence of Theorem 5.2.1. If $\mathfrak{s}^{-1}(\lambda)/\exp(\mathbb{R}\lambda)$ is smooth then the restriction of A to the orbit of λ is integrable and it has a Lie groupoid whose \mathfrak{s} -fiber coincides with $\mathfrak{s}^{-1}(\lambda)/\exp(\mathbb{R}\lambda)$. ■

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