Lie bialgebroids and reduction

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Mechanics and Lie algebroids
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A symplectic action of Lie group $G$ on a symplectic manifold $(M, \Omega)$ with equivariant momentum map $J : M \rightarrow g^*$.

- $\mu \in g^*$ a regular value of $J$

\[ \Downarrow \]

$J^{-1}\mu$ is a submanifold

$G_{\mu}$ acts on $J^{-1}(\mu)$

The space of orbits $J^{-1}\mu/G_{\mu}$ is an smooth manifold and the canonical projection $\pi_{\mu} : J^{-1}\mu \rightarrow J^{-1}\mu/G_{\mu}$ defines a principal $G_{\mu}$-bundle.
Theorem (Symplectic reduction)

There exists a unique symplectic 2-form $\Omega_\mu$ on $J^{-1}_\mu/G_\mu$ such that

$$\pi_\mu^* \Omega_\mu = i_\mu^* \Omega$$

where $\pi_\mu : J^{-1}_\mu \to J^{-1}_\mu/G_\mu$ is the projection and $i_\mu : J^{-1}_\mu \to M$ is the canonical inclusion.
Lie bialgebroids

1.1 Some tools in the Lie algebroid Theory
1.2 Lie bialgebroids

Actions and momentum maps for Lie bialgebroids

2.1 Actions of a Lie group on a Lie bialgebroid by complete lifts
2.2 Momentum maps for Lie bialgebroids

Reduction of Lie bialgebroids

3.1 Reduction of Lie bialgebroids by bialgebroid automorphisms
3.2 Reduction of Lie bialgebroids with momentum map
3.3 Examples
1.1 Some tools in Lie algebroid Theory

\( \tau_A : A \to M \) a Lie algebroid over \( M \)

\((\lbrack \cdot, \cdot \rbrack, \rho)\) a Lie algebroid structure on \( A \)

\( \Gamma(A) \equiv C^\infty(M) - \text{modulo of sections of } A \)

\( (A, \lbrack \cdot, \cdot \rbrack, \rho) \) Lie algebroid \( \iff \) \( A^* \) is a linear Poisson manifold

**The differential of the Lie algebroid \( A \)**

\[ d^A : \Gamma(\bigwedge^k A^*) \longrightarrow \Gamma(\bigwedge^{k+1} A^*) \]

**The Lie derivative with respect to \( X \in \Gamma(A) \)**

\[ \mathcal{L}_X^A : \Gamma(\bigwedge^k A^*) \longrightarrow \Gamma(\bigwedge^k A^*) \]

\[ \mathcal{L}_X^A = i_X \circ d^A + d^A \circ i_X \]
1.1 Some tools in Lie algebroids Theory

The Schouten bracket

\[ [\cdot, \cdot] : \Gamma(\wedge^p A) \times \Gamma(\wedge^q A) \to \Gamma(\wedge^{p+q-1} A) \]

- \[ [X, f] = \rho(X)(f) \]
- \[ [P, Q \wedge R] = [P, Q] \wedge R + (-1)^{q(p+1)} Q \wedge [P, R] \]
- \[ [P, Q] = (-1)^{pq+p+q} [Q, P] \]

\( f \in C^\infty(M), \ X \in \Gamma(A), \ P \in \Gamma(\wedge^p A), \ Q \in \Gamma(\wedge^q A), \ R \in \Gamma(\wedge^* A). \)
1.1 Some tools in Lie algebroids Theory

**Vertical and complete lifts**

The complete and vertical lift to $A$ of $f : M \to \mathbb{R}$

- $f^c : A \to \mathbb{R}$, $f^c(a) = \rho(a)(f)$
- $f^v : A \to \mathbb{R}$, $f^v(a) = f(\tau(a))$, $\forall a \in A$.

The complete and vertical lift to $A$ of $X \in \Gamma(A)$

- $X^c \in \mathfrak{X}(A)$
  - $X^c(f \circ \tau) = \rho(X)(f) \circ \tau$, $f \in C^\infty(M)$
  - $X^c(\hat{\alpha}) = L^A_X \alpha$, $\alpha \in \Gamma(A^*)$

$\hat{\alpha} : A \to \mathbb{R}$ the linear function induced by $\alpha$

- $X^v \in \mathfrak{X}(A)$
  - $X^v(a_x) = (X(x))^v_{a_x}$, $a_x \in A_x$

where $^v_a : A_{\tau(a)} \to T_a(A_{\tau(a)})$ is the canonical isomorphism of vector spaces.
1.1 Some tools in Lie algebroids Theory

**Lie algebroid morphism**

A vector bundle morphism

$$\begin{array}{ccc}
A & \xrightarrow{F} & A' \\
\downarrow{\tau_A} & & \downarrow{\tau_{A'}} \\
M & \xrightarrow{f} & M'
\end{array}$$

$$d^A((F, f)^* \alpha') = (F, f)^* (d^{A'} \alpha'), \text{ for } \alpha' \in \Gamma(\wedge^k (A')^*)$$

$$((F, f)^* \alpha')_x(a_1, \ldots, a_k) = \alpha'_{f(x)}(F(a_1), \ldots, F(a_k))$$
1.1 Some tools in Lie algebroids Theory

\[ \tilde{\pi}: A \to A' \text{ vector bundle epimorphism} \]

\[ X \in \Gamma(A) \tilde{\pi}\text{-projectable} \]

\[ \begin{array}{c}
A \xrightarrow{\tilde{\pi}} A' \\
\downarrow \pi \downarrow \quad \downarrow \pi' \downarrow \\
M \xrightarrow{\pi} M'
\end{array} \]

Proposition

Suppose that \((\lbrack \cdot, \cdot \rbrack, \rho)\) is a Lie algebroid structure on \(A\). Then, there is a Lie algebroid structure on \(A'\) such that \(\tilde{\pi}\) is a Lie algebroid epimorphism if and only if the following conditions hold:

1. \(\lbrack X, Y \rbrack\) is a \(\tilde{\pi}\text{-projectable}\) section of \(A\), for all \(X, Y \in \Gamma(A)\) \(\tilde{\pi}\text{-projectable}\) sections of \(A\).
2. \(\lbrack X, Y \rbrack \in \Gamma(\ker \tilde{\pi})\), for all \(X, Y \in \Gamma(A)\) with \(X \in \Gamma(A)\) \(\tilde{\pi}\text{-projectable}\) section of \(A\) and \(Y \in \Gamma(\ker \tilde{\pi})\).


1.2 Lie bialgebroids

\[(A, [[\cdot,\cdot]], \rho)\] Lie algebroid over \(M\)

\[(A^*, [[\cdot,\cdot]^*], \rho^*)\] Lie algebroid over \(M\)

\[\downarrow\]

\[(A, A^*) \text{ is a } \textbf{Lie bialgebroid} \text{ if}\]

\[d^{A^*} [X, Y] = [X, d^{A^*} Y] - [Y, d^{A^*} X], \quad \forall X, Y \in \Gamma(A)\]

(A, A*) a Lie bialgebroid on $M$

$$\{\cdot, \cdot\}_M : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

is a **Poisson bracket** on $M$

$$\{f, h\}_M = \langle d^A f, d^{A^*} h \rangle \quad \forall h \in C^\infty(M)$$
Examples

**Lie bialgebras**

$\mathfrak{g}$ Lie algebra + $\mathfrak{g}^*$ Lie algebra

$$d_\star [\xi_1, \xi_2]_\mathfrak{g} = [\xi_1, d_\star \xi_2]_\mathfrak{g} - [\xi_2, d_\star \xi_1]_\mathfrak{g}$$
1.2 Lie bialgebroids

**Poisson algebroids**

\((A, [\cdot, \cdot], \rho)\) Lie algebroid on \(M\), \(A\) is Poisson if \(\Lambda \in \Gamma(\wedge^2 A)\)

Properties:

- \(\#_\Lambda : A^* \to A, \quad \alpha_x \to (\#_\Lambda)_x(\alpha_x) = (i_\alpha \Lambda)(x)\) homomorphism of vector bundles
- \(([\cdot, \cdot]^*, \rho^*)\) Lie algebroid structure on \(A^*\)

\[
\begin{align*}
\{\alpha, \beta\}^* &= \mathcal{L}_{\#_\Lambda^\alpha \beta} - \mathcal{L}_{\#_\Lambda^\alpha \beta} - d^A(\Lambda(\alpha, \beta)), \quad \rho^*(\alpha) = \rho(\#_\Lambda \alpha) \\
&\downarrow \\
(A, A^*) &\text{ Lie bialgebroid}
\end{align*}
\]

- Poisson bracket on \(M\) : \(\{f, g\} = \Lambda(d^Af, d^Ag)\)
- The linear Poisson structure on \(A\) : \(\Lambda^c\)
1.2 Lie bialgebroids

SYMPLECTIC ALGEBROIDS

$(A, [[\cdot, \cdot]], \rho)$ Lie algebroid on $M$, $A$ is symplectic if closed nondegenerated $\Omega \in \Gamma(\wedge^2 A^*)$

$b_\Omega : \Gamma(A) \rightarrow \Gamma(A^*)$

$\Lambda(\alpha, \beta) = \Omega(b_\Omega^{-1}(\alpha), b_\Omega^{-1}(\beta))$
2. Actions and momentum maps for Lie bialgebroids

2.1 Actions of a Lie group on a Lie algebroid by complete lifts

- $(A, [\cdot, \cdot], \rho)$ a Lie algebroid over $M$
- $G$ a connected Lie group
- An action of $G$ on $A$:

$g \in G \implies \Phi_g : A \rightarrow A$ is a commutative diagram

$(\Phi_g, \phi_g)$ is a vector bundle automorphism

$\Phi$ is an action by complete lifts

$\exists \psi : \mathfrak{g} \rightarrow \Gamma(A)$ Lie algebra homomorphism

infinitesimal generator of $\xi$ with respect to $\Phi = \xi_A = \psi(\xi)^c$, $\forall \xi \in \mathfrak{g}$
Some consequences:

1. Infinitesimal generator of $\xi$ with respect to the $\phi = \xi_M = \rho(\psi(\xi))$

2. $(\Phi_g, \phi_g)$ is a Lie algebroid automorphism, $\forall g \in G$
2.1 Actions of a Lie group on a Lie algebroid by complete lifts

\[ G \text{ a Lie group, } \cdot : G \times G \to G \text{ the multiplication of } G \]

\[ \Downarrow \]

\[ TG \text{ is also a Lie group} \]

\[ T \cdot : TG \times TG \to TG \text{ is the multiplication of } TG \]

\[ e \text{ the identity element of } TG \]

\[ \Downarrow \]

\[ TG \cong G \times g \]

\[ TG \to G \times g, \quad X_g \in T_g G \to (g, (T_g l_g^{-1})(X_g)) \in G \times g \]

\[ (g, \xi) \cdot (g', \xi') = (gg', \xi' + \text{Ad}_{(g')}^{-1} \xi) \]

\[ (e, 0) \text{ the identity element of } G \times g \]

The Lie algebra of \( TG \cong G \times g \)

\[ T_{(e,0)}(G \times g) \cong g \times g \]

\[ [(\xi, \eta), (\xi', \eta')]_{g \times g} = ([\xi, \xi']_g, [\xi, \eta']_g - [\xi', \eta]_g) \]
2.1 Actions of a Lie group on a Lie algebroid by complete lifts

Theorem

Let \(((\Phi, \phi), \psi)\) be an action of a connected Lie group \(G\) by complete lifts over the Lie algebroid \(\tau: A \to M\). Then, the map \(\Phi^T: (G \times g) \times A \to A\) given by

\[
\Phi^T((g, \xi), a_x) = \Phi_g(a_x) + \Phi_g(\psi(\xi)(x)), \quad \text{for } (g, \eta) \in G \times g
\]

defines an affine action of \(G \times g \cong TG\) on \(A\). Moreover, if \((\xi, \eta) \in g \times g\) then the infinitesimal generator \((\xi, \eta)_A\) of \((\xi, \eta)\) with respect to the action \(\Phi^T\) is

\[
(\xi, \eta)_A = \psi(\xi)^c + \psi(\eta)^v
\]

\[
\Phi_g(\psi(Ad_{g^{-1}}\xi)(x)) = \psi(\xi)(\phi_g(x)),
\]

for \(g \in G, \xi \in g\) and \(x \in M\).
2.2 Momentum maps for Lie bialgebroids

\((A, A^*)\) a Lie bialgebroid over \(M\)

\(((\Phi, \phi), \psi)\) an action of the Lie group \(G\) on \(A\) by complete lifts \(J : M \to \mathfrak{g}^*\) smooth equivariant map with respect to \(\phi\)

\[\text{Coad}_G(J(x)) = J(\phi_g(x)), \quad \forall x \in M, \quad \forall g \in G\]

**Definition 1**

The action \(\Phi\) is said to be **Hamiltonian with momentum map** \(J\) if

\[\psi(\xi) = d^{A^*} \hat{J}_\xi, \quad \text{for } \xi \in \mathfrak{g},\]

\[\hat{J}_\xi : M \to \mathbb{R}, \quad \hat{J}_\xi(x) = \langle J(x), \xi \rangle, \text{for any } x \in M\]

**Consequence:** \(\Phi_g : A \to A\) and \(\Phi_{g^{-1}}^* : A^* \to A^*\) Lie algebroid automorphisms over \(\phi_g : M \to M\)

\[\downarrow\]

\(\Phi_g\) Lie bialgebroid automorphism
2.2 Momentum maps for Lie bialgebroids

$(A, A^*)$ be a Lie bialgebroid over $M$

**Proposition**

Let $\Phi : G \times A \to A$ be a Hamiltonian action of a connected Lie group $G$ over $\phi : G \times M \to M$ with equivariant momentum map $J : M \to g^*$. Then, we have that:

(i) $\phi$ is an standard Poisson action of $G$ on the Poisson manifold $(M, \Pi_M)$ and $J : M \to g^*$ is a momentum map for $\phi$.

(ii) $\Phi^T : TG \times A \to A$ is a Poisson action of the Lie group $TG$ on $A$ and the map $J^T : A \to (g \times g)^* \simeq g^* \times g^*$ given by

$$J^T(a) = ((dJ \circ \rho)(a), J(\tau(a)))$$

is an equivariant momentum map for the action $\Phi^T$. Here, $dJ : TM \to g$ denotes the vector bundle morphism defined by $dJ|_{T_xM} = T_xJ$, for all $x \in M$. 

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Lie bialgebroids and reduction
3. Reduction of Lie bialgebroids

3.1 Reduction of Lie bialgebroids by Lie bialgebroid automorphisms

\((A, A^\ast)\) a Lie bialgebroid, \(\Phi : G \times A \to A\) be an action by Lie bialgebroid automorphisms of a connected Lie group \(G\).

Assume that the action \(\phi\) is free and that \(M/G\) is smooth manifold such that the canonical projection is a surjective submersion

\[\downarrow\]

\[A/G \to M/G\] is a vector bundle

\[\Gamma(A/G) = \{X \in \Gamma(A)/X \in G\text{-invariant}\}\]

\(\pi_A : A \to A/G, \quad \pi_{A^\ast} : A^\ast \to A^\ast/G\) are epimorphisms of vector bundles

\[\{X \in \Gamma(A)/X\text{is } \pi_A\text{-projectable}\} = \{X \in \Gamma(A)/X \text{ is } G\text{-invariant section}\}\]

\[\{\alpha \in \Gamma(A^\ast)/\alpha\text{ is } \pi_{A^\ast}\text{-projectable}\} = \{\alpha \in \Gamma(A^\ast)/\alpha \text{ is } G\text{-invariant section}\}\]

\[\ker((\pi_A|_{Ax})) = \{0\}, \quad \ker((\pi_{A^\ast}|_{A^\ast_x})) = \{0\}\]

\[\downarrow\]

The pair \((A/G, A^\ast/G)\) is a Lie bialgebroid over \(M/G\).
3.2 Reduction of Lie bialgebroids with momentum map

Reduction procedure for Lie bialgebroid analog of the Marsden Weintein reduction for symplectic manifolds

\((A, A^*)\) Lie bialgebroid over \(M \Rightarrow (A_\mu, A^*_\mu)\) Lie bialgebroid over \(J^{-1}(\mu)/G_\mu\)

Proposition

Let \((A, A^*)\) be a Lie bialgebroid over \(M\) and \(\Phi : G \times A \to A\) be a Hamiltonian action of a connected Lie group \(G\) on \(A\) with momentum map \(J : M \to g^*\).

Consider \(\mu \in g^*\) such that \((0, \mu) \in g^* \times g^*\) is a regular value of \(J^T : A \to g^* \times g^*\). Then, we have that:

1. \((J^T)^{-1}(0, \mu)\) is a Lie subalgebroid of \(A\) over \(J^{-1}(\mu)\).
2. The restriction \(\psi_\mu\) of \(\psi\) to the isotropy algebra \(g_\mu\) of \(\mu\) with respect to the coadjoint action takes values in \(\Gamma((J^T)^{-1}(0, \mu))\).
3. The isotropy Lie group \(G_\mu\) of \(\mu\) with respect to the coadjoint action acts on \((J^T)^{-1}(0, \mu)\) by complete lifts with respect to \(\psi_\mu : g_\mu \to \Gamma((J^T)^{-1}(0, \mu))\).
4. The action of \(G_\mu\) on the Lie subalgebroid \((J^T)^{-1}(0, \mu)\) induces an affine action \(\Phi^T_\mu\) of \(TG_\mu\) on this subalgebroid.
suppose

- The action of $G_\mu$ on $J^{-1}(\mu)$ is free
- The space of orbits $J^{-1}(\mu)/G_\mu$ is a smooth manifold such that the projection $\pi_\mu : J^{-1}(\mu) \to J^{-1}(\mu)/G_\mu$ is a surjective submersion

$\downarrow$

$J^{-1}(\mu)/G_\mu$ is a reduced Poisson manifold

**Theorem**

Let $(A, A^*)$ be a Lie bialgebroid over $M$ and $\Phi : G \times A \to A$ be a Hamiltonian action of a Lie group $G$ on $(A, A^*)$ with momentum map $J : M \to \mathfrak{g}^*$. Assume that $\mu$ is an element of $\mathfrak{g}^*$ such that $(0, \mu)$ is a regular value for $J^T : A \to \mathfrak{g}^* \times \mathfrak{g}^*$ and that the space of orbits $J^{-1}(\mu)/G_\mu$ is a quotient manifold. Then,

(i) The space of orbits $A_\mu$ of the action of $TG_\mu$ on $(J^T)^{-1}(0, \mu)$ is a Lie algebroid over $J^{-1}(\mu)/G_\mu$.

(ii) The dual bundle $A^*_\mu \to J^{-1}(\mu)/G_\mu$ is endowed with a Lie algebroid structure.

(iii) The pair $(A_\mu, A^*_\mu)$ is a Lie bialgebroid over $J^{-1}(\mu)/G_\mu$. 
3.2 Reduction of Lie bialgebroids with momentum map

Sketch of the proof

(i) The Lie algebroid structure over $A_\mu = (J^T)^{-1}(0, \mu)/TG_\mu$

\[ \psi_\mu(g_\mu) \text{ Lie subalgebroid of } (J^T)^{-1}(0, \mu) \text{ over } J^{-1}(\mu) \]

\[ G_\mu \text{ acts by Lie algebroid automorphisms on } (J^T)^{-1}(0, \mu) \text{ and } \psi_\mu(g_\mu) \]

\[ \downarrow \]

induce Lie algebroid structures on $(J^T)^{-1}(0, \mu)/G_\mu$ and $\psi_\mu(g_\mu)/G_\mu$ such that $\psi_\mu(g_\mu)/G_\mu$ is a Lie subalgebroid of $(J^T)^{-1}(0, \mu)/G_\mu$.

\[ \downarrow \]

$\psi_\mu(g_\mu)/G_\mu$ is an ideal of $(J^T)^{-1}(0, \mu)/G_\mu$.

\[ \downarrow \]

$((J^T)^{-1}(0, \mu)/G_\mu)/\psi_\mu(g_\mu)/G_\mu$ Lie algebroid over $J^{-1}(\mu)/G_\mu$.

\[ \downarrow \]

$A_\mu := (J^T)^{-1}(0, \mu)/TG_\mu \cong ((J^T)^{-1}(0, \mu)/G_\mu)/\psi_\mu(g_\mu)/G_\mu$.
3.2 Reduction of Lie bialgebroids with momentum map

Sketch of the proof

\[ A_\mu := (J^T)^{-1}(0, \mu)/TG_\mu \cong ((J^T)^{-1}(0, \mu)/G_\mu)/({\psi_\mu}(g_\mu)/G_\mu) \]

\[ \Gamma(A_\mu) \cong \frac{\Gamma((J^T)^{-1}(0, \mu))^{G_\mu}}{\Gamma(\psi_\mu(g_\mu))^{G_\mu}} \]

\( \Gamma((J^T)^{-1}(0, \mu))^{G_\mu} \) the space of \( G_\mu \)-invariant section on \( (J^T)^{-1}(0, \mu) \)

\( \Gamma(\psi_\mu(g_\mu))^{G_\mu} \) the space of \( G_\mu \)-invariant section on \( \psi_\mu(g_\mu) \)

Lie algebroid structure \( ([\cdot, \cdot]_{A_\mu}, \rho_{A_\mu}) \) is characterized by

\[ [[X], [Y]]_{A_\mu} = [[X, Y]], \]

\[ \rho_{A_\mu}([X]) = T\pi_\mu \circ \rho(X), \quad \text{for } X, Y \in \Gamma((J^T)^{-1}(0, \mu))^{G_\mu}. \]
Sketch of the proof

(ii) The Lie algebroid structure over $A^*_\mu$

To prove that there exists a Lie algebroid structure on $A^*_\mu$, we will show that there exists a linear Poisson structure on $A_\mu$

$\Phi^T$ is a Hamilton action of $TG$ on $(A, \Pi_A)$ with momentum map

$J^T : A \to g^* \times g^*$

$\downarrow$

$\exists$ reduced Poisson structure $\Pi_{A^*_\mu}$ on $(J^T)^{-1}(0, \mu)/(TG)(0, \mu)$

$(TG)(0, \mu) = TG_\mu$

the Poisson bracket $\{\cdot, \cdot\}_{\Pi_{A^*_\mu}}$ is linear

$\downarrow$

Lie algebroid structure $(\llbracket \cdot, \cdot \rrbracket_{A^*_\mu}, \rho_{A^*_\mu})$
Sketch of the proof

Lie algebroid structure \((\mathbb{L}^\cdot, \cdot)_{A_\mu^*}, \rho_{A_\mu^*})\)

\[\alpha_\mu \in \Gamma(A_\mu^*) \Rightarrow \exists \alpha \in \Gamma(A^*) \text{ such that} \]

\[\alpha(\psi(\xi)) = 0 \quad \mathcal{L}^A_{\psi(\xi)} \alpha = 0, \quad \tilde{\iota}_\mu^*(\alpha) = \tilde{\pi}_\mu^*(\alpha_\mu) \text{ for all } \xi \in \mathfrak{g} \]

\[\tilde{\iota}_\mu : (J^T)^{-1}(0, \mu) \to A \text{ is the inclusion,} \]

\[\tilde{\pi}_\mu : (J^T)^{-1}(0, \mu) \to A_\mu \text{ is the canonical projection} \]

\[\tilde{\pi}_\mu^*(\mathbb{L}_A^\alpha, \beta_\mu)_{A_\mu^*} = \tilde{\iota}_\mu^*(\mathbb{L}_A^\alpha, \beta)_{A_\mu^*}, \]

for \(\alpha_\mu, \beta_\mu \in \Gamma(A_\mu^*)\)

\[\rho_{A_\mu^*}(\alpha_\mu)(f_M) \circ \pi_\mu = \rho_*^*(\alpha)(f) \circ \iota_\mu, \]

for \(f_\mu \in C^\infty(J^{-1}(\mu)/G_\mu),\)

\(f : M \to \) is a real function on \(M\) such that \(f_\mu \circ \pi_\mu = f \circ \iota_\mu, \) with \(\iota_\mu : J^{-1}(\mu) \to J^{-1}(\mu)/G_\mu \) the canonical inclusion
(\( A, [\cdot, \cdot], \rho, \Lambda \)) Poisson Lie algebroid over \( M \)
\[ \downarrow \]
(\( A, A^* \)) is a Lie bialgebroid

\( \Phi : G \times A \to A \) Hamiltonian action of a connected Lie group \( G \) over \( A \)
with momentum map \( J : M \to g^* \) and associated Lie algebra morphism
\( \psi : g \to \Gamma(A) \)
\[ \downarrow \]

\[ \psi(\xi) = \mathcal{H}_{J(\xi)}^\Lambda, \quad \text{for all } \xi \in g \]

\( (\Phi_g)_*(\Lambda) = \Lambda, \quad \text{for all } g \in G \)
3.3 Examples: Poisson Lie algebroids

\((0, \mu)\) a regular value of \(J^T : A \to \mathfrak{g}^* \times \mathfrak{g}^*\)

\((A_\mu, A^*_\mu) = ((J^T)^{-1}(0, \mu)/TG_\mu, ((J^T)^{-1}(0, \mu)/TG_\mu)^*)\) is a Lie bialgebroid over \(J^{-1}(\mu)/G_\mu\)

\[\downarrow\]

\(A_\mu\) admits a linear Poisson structure (the reduced Poisson structure on \(\Lambda^c\) with respect to the action \(\Phi^T\) and momentum map \(J^T\))

\[\Lambda_\mu \in \Gamma(\wedge^2 A_\mu)\]

\[\Lambda_\mu(\alpha_\mu, \beta_\mu) \circ \pi_\mu = \Lambda(\alpha, \beta) \circ i_\mu\] for all \(\alpha_\mu, \beta_\mu \in \Gamma(A_\mu)\)

where \(\alpha, \beta\) are \(TG\)-invariant sections of \(A^*\) satisfying

\[\tilde{\pi}_\mu(\alpha_\mu) = \tilde{i}^*_\mu \alpha,\quad \mathcal{L}^A_{\psi(\xi)} \alpha = 0,\quad \alpha(\psi(\xi)) = 0,\]

\[\tilde{\pi}_\mu(\beta_\mu) = i^*_\mu \beta,\quad \mathcal{L}^A_{\psi(\xi)} \beta = 0,\quad \beta(\psi(\xi)) = 0\] for all \(\xi \in \mathfrak{g}\)

\[\tilde{\pi}_\mu : (J^T)^{-1}(0, \mu) \to A_\mu\quad \pi_\mu : J^{-1}(\mu) \to J^{-1}(\mu)/G_\mu\]

\[\tilde{i}_\mu : (J^T)^{-1}(0, \mu) \to A\quad i_\mu : J^{-1}(\mu) \to M\]
3.3 Examples: Poisson Lie algebroids

Let \((A, [\cdot, \cdot], \rho, \Lambda)\) be a Poisson Lie algebroid over \(M\), \(\Phi : G \times A \to A\) be a Hamiltonian action of a connected Lie group \(G\) over \(A\) with momentum map \(J : M \to \mathfrak{g}^*\) and associated Lie algebra morphism \(\psi : \mathfrak{g} \to \Gamma(A)\). If \((0, \mu)\) is a regular value of \(J^T : A \to (\mathfrak{g} \times \mathfrak{g})^*\), then \(A_{\mu} = (J^T)^{-1}(0, \mu) / TG_{\mu}\) admits a Poisson Lie algebroid structure \(\Lambda_{\mu}\) over \(J^{-1}(\mu) / G_{\mu}\).
3.3 Examples: Symplectic Lie algebroids

\((A, [\cdot, \cdot], \rho, \Omega)\) a symplectic Lie algebroid over \(M\)

\(\Lambda\) the corresponding Poisson 2-section

\(\Phi: G \times A \to A\) a Hamiltonian action of a connected Lie group \(G\) over the induced Lie bialgebroid \((A, A^*)\) with momentum map \(J: M \to g^*\) and associated Lie algebra morphism \(\psi: g \to \Gamma(A)\)

\[\downarrow\]

\[i_{\mathcal{H}_\Omega} \Omega = d^A \hat{J}_\xi.\]

\[\downarrow\]

\(\Phi_g^*(\Omega) = \Omega, \quad \forall g \in G,\)
3.3 Examples: Symplectic Lie algebroids

\[ \mu \in \mathfrak{g}^* \text{ such that } (0, \mu) \text{ is a regular value of } J^T : M \to \mathfrak{g}^* \times \mathfrak{g}^* \]

\[ \Downarrow \]

\[ A_\mu \text{ is a Poisson Lie algebroid} \]

\[ i_\mu : (J^T)^{-1}(0, \mu) \to A \]

2-section \( \tilde{\Omega}_\mu = i_\mu^*(\Omega) \) on the Lie subalgebroid \((J^T)^{-1}(0, \mu) \to J^{-1}(\mu)\)

For all \( \tilde{X}_\mu, \tilde{Y}_\mu \in \Gamma((J^T)^{-1}(0, \mu)) \),

\[ \tilde{\Omega}_\mu(\tilde{X}_\mu, \tilde{Y}_\mu) \text{ is a } \pi_\mu \text{-basic function} \]

\[ \Downarrow \]

For all \( X_\mu, Y_\mu \in \Gamma(A_\mu) \Rightarrow \exists \text{ a function } \Omega_\mu(X_\mu, Y_\mu) \text{ on } J^{-1}(\mu)/G_\mu \text{ such that} \]

\[ \Omega_\mu(X_\mu, Y_\mu) \circ \pi_\mu = \tilde{\Omega}_\mu(X_\mu, Y_\mu) \]
Let \((A, [[\cdot, \cdot]], \rho, \Omega)\) be a symplectic Lie algebroid and \(\Phi : G \times A \to A\) be a Hamilton action with momentum map \(J : M \to g^*\) and associated Lie homomorphism \(\psi : g \to \Gamma(A)\). If \(\mu\) is an element of \(g^*\) such that \((0, \mu)\) is a regular of \(J^T\), then \(A_\mu = (J^T)^{-1}(0, \mu)/T_G\mu\) is a symplectic Lie algebroid over \(J^{-1}(\mu)/G_\mu\).
3.3 Examples: Another Example

\((M, \Lambda)\) Poisson manifold \(\quad + \quad (\mathfrak{g}, \mathfrak{g}^*)\) Lie bialgebra

\(\psi : \mathfrak{g}^* \rightarrow \mathcal{X}(M)\) representation

The Lie bialgebroid \(TM \oplus_M M \times \mathfrak{g} \rightarrow M\) over \(M\)

\[
(TM, [\cdot, \cdot], 1_{TM}) \quad + \quad (M \times \mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})
\]

\[
\mathcal{X}(M) \times C^\infty(M, \mathfrak{g}) \rightarrow C^\infty(M, \mathfrak{g}) \quad (X, \xi) \mapsto X(\xi)
\]

\[
C^\infty(M, \mathfrak{g}) \times \mathcal{X}(M) \rightarrow \mathcal{X}(M) \quad (\xi, X) \mapsto 0
\]

\[
\downarrow
\]

\((TM \oplus_M M \times \mathfrak{g}, [\cdot, \cdot], \rho)\) is a Lie algebroid

\[
\rho = pr_1 : TM \oplus_M M \times \mathfrak{g} \rightarrow TM
\]

\[
[(X_1, \xi_1), (X_2, \xi_2)] = ([X_1, X_2], [\xi_1, \xi_2]_{\mathfrak{g}} + X_1(\xi_2) - X_2(\xi_1))
\]
3.3 Examples: Another Example

\[(M, \Lambda) \text{ Poisson manifold } + (\mathfrak{g}, \mathfrak{g}^*) \text{ Lie bialgebra}\]

\[\psi : \mathfrak{g}^* \to \mathfrak{X}(M) \text{ representation}\]

\[(T^* M, [\cdot, \cdot], \#) + (M \times \mathfrak{g}^*, [\cdot, \cdot], \psi)\]

\[\Omega^1(M) \times C^\infty(M, \mathfrak{g}^*) \to C^\infty(M, \mathfrak{g}^*) \quad (\alpha, \eta) \mapsto -\#(\alpha)(\eta) + \text{coad}_{\psi^*}(\alpha)\eta\]

\[D : C^\infty(M, \mathfrak{g}^*) \times \Omega^1(M) \to C^\infty(M, \mathfrak{g}) \quad D(\eta, \alpha)(X) = -(\mathcal{L}_\psi(\eta)\alpha)(X) + \psi(X(\eta))\]

\[\downarrow\]

\[(T^* M \oplus_M M \times \mathfrak{g}^*, [\cdot, \cdot], \rho^*) \text{ is a Lie algebroid}\]

\[\rho^* = T^* M \oplus_M M \times \mathfrak{g}^* \to TM \quad \rho^*(\alpha, \eta) = \#(\alpha) + \psi(\eta)\]

\[\left[ (\alpha_1, \eta_1), (\alpha_2, \eta_2) \right] = \]

\[\left[ [\alpha_1, \alpha_2] \Lambda + \mathcal{L}_\psi(\eta_1)\alpha_2 - \mathcal{L}_\psi(\eta_2)\alpha_1, [\eta_1, \eta_2] \mathfrak{g}^* + \text{coad}_{\psi^*}(\alpha_1)\eta_2 - \text{coad}_{\psi^*}(\alpha_2)(\eta_1) \right]\]
3.3 Examples: Another Example

\((M, \Lambda)\) Poisson manifold \(\quad + \quad (\mathfrak{g}, \mathfrak{g}^*)\) Lie bialgebra

\(\psi : \mathfrak{g}^* \rightarrow \mathfrak{X}(M)\) representation

**The Lie bialgebroid** \(TM \oplus_M M \times \mathfrak{g} \rightarrow M\) over \(M\)

\(\Phi : G \times M \rightarrow M\) Poisson action with momentum map \(J : M \rightarrow \mathfrak{g}^*\)

\[\tilde{\Phi} : G \times TM \oplus_M M \times \mathfrak{g} \rightarrow TM \oplus_M M \times \mathfrak{g}, \quad \tilde{\Phi}(h, X, \xi) = (T\Phi_h(X), -Ad_h\psi^*(d\hat{J}_\xi))\]

\[\tilde{\Psi} : \mathfrak{g} \rightarrow \mathfrak{X}(M) \times C^\infty(M, \mathfrak{g})\quad d_\star \hat{J}_\xi = (\#_\Lambda d\hat{J}_\xi, -\psi^*(d\hat{J}_\xi))\]

\(\downarrow\)

\((\tilde{\Psi}(\xi))^c \equiv \text{Infinitesimal generator of } \xi \text{ with respect } \tilde{\Phi}\)

\((0, 0)\) is a regular value for \(J^T\)

\(\downarrow\)

\((J^T)^{-1}(0, 0) = T(J^{-1}(0)) \oplus_{J^{-1}(0)} J^{-1}(0) \times \mathfrak{g}\)

\((TJ^{-1}(0) \oplus J^{-1}(0) \times \mathfrak{g})/TG \rightarrow J^{-1}(0)/G\) is a Lie bialgebroid
Thanks !!!!