FLIPPING IN ACYCLIC AND STRONGLY CONNECTED GRAPHS

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Abstract. A flippable edge in an acyclic digraph is an edge whose reorientation leaves the graph acyclic. We characterize the spanning trees T of an undirected graph G such that there exists an acyclic orientation of G whose set of flippable edges is T. In particular for every edge $e \in E(G)$ we give a linear algorithm returning an acyclic orientation and a spanning tree T containing e such that T is the set of flippable edges of the digraph.

After going to oriented matroid theory and dualizing the proofs we obtain similar results for flippable edges in strongly connected digraphs.

1. Introduction

In this paper we study flippable edges in a directed graph (digraph for short). Given a digraph having a property P (our first example is the property of being acyclic), an edge is *flippable* if its flipping preserves the property P. The flipping of an edge $e = (a, b)$ of a digraph G is the operation which gives the digraph, denoted $_{-e}G$, with the same set of vertices and with the same set of directed edges except that the edge (a, b) is replaced by the edge (b, a).

We will also consider a digraph as a graph G together with an orientation $\mathcal{O}$ of its edges. The set of flippable edges of G then depends on the choice of its orientation $\mathcal{O}$. With this vocabulary, we will look for the reorientations of a graph such that the set of flippable edges have some property (like being covering ora even a tree).

As we already mentioned our first example (Section 2) is the study of "flippings which preserves acyclicity". We show that we can construct at least $2m$ (m is the number of edges) orientations of a graph such the set of flippable edges form a tree.

In Section 3, we consider the dual problem of "finding flippable edges in a strongly connected digraph". These results are very close to those of Section 2. This similarity comes from the duality in oriented matroid theory. This leads us to a natural generalization for this theory in Section 4.

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2. Notations and first results

We now fix some notations and definitions of graph theory. We will call $G = G(V, E)$ a graph or a digraph on a set V of vertices and a set E of edges with $|V| = n$ and $|E| = m$. A *simple graph* is a graph without loops or multiple edges. Note that a graph is simple if and only it has no cycle of size smaller than three. In a digraph G, the edges of a cycle C of the underlying graph are divided naturally, after a choice of a direction, into a positive and a negative part C^+ and C^-. The signed set $C = (C^+, C^-)$ is a *signed circuit* of the graph. Since the choice of the direction on the cycle is free, the opposite signed circuit $C' = (C^-, C^+)$ is also obtained. The signed circuit (C^+, C^-) with $e \in C^+$ is the *signed circuit on the cycle C and along e*. If the negative part is empty the signed circuit is *positive*. A positive signed circuit is sometimes called in the graph literature simply a *circuit*; what leads to some ambiguity is that the signed circuits corresponds to the circuits in the oriented matroid literature. An orientation of a graph is *acyclic* if it has no positive circuit. Every acyclic orientation of the graph G determines at least one linear ordering $v_1 < \cdots < v_n$ of the vertex set $V(G) := \{v_1, \ldots, v_n\}$ such that an edge (v_i, v_j) of G has the direction (v_i, v_j) if and only if $v_i < v_j$. An ordering verifying these conditions is called in the literature a *topological ordering* of the vertices relative to the orientation. Such a topological ordering can classically be obtained in linear time by two depth first search algorithm. Reciprocally every linear ordering $v_{1'} < \cdots < v_{n'}$ of $V(G)$ is a topological ordering of an acyclic orientation and determines this orientation. A *totally cyclic orientation* of a graph is an orientation such that every edge belongs to a positive circuit. It is a well known result that a 2-connected digraph is totally cyclic iff it is *strongly connected*, i.e., for any two vertices a and b there exists a directed path from a to b. For a spanning tree T and an edge $e \in E(G) \setminus T$, the unique cycle [resp. circuit along e] of G contained in $T \cup e$ is the *fundamental cycle* [resp. *fundamental circuit*] and is denoted $C(e, T)$.

A *cutset* $C^\star$ in a connected graph is a minimal set of edges which disconnects the vertices of the graph in two non-empty parts A and B. A graph is *k-connected* if it has no cutset of size smaller than k. In a digraph the edges of a cutset $C^\star$ are naturally divided into a positive part ${C^\star}^+$ which are the edges from A to B and a negative part ${C^\star}^-$ which are the edges from B to A. The signed set $C^\star = ({C^\star}^+, {C^\star}^-)$ is a *signed cutset* of the digraph. Like for signed circuits, a signed cutset always exists with its opposite which is obtained by inverting the role of A and B. If an edge e is in ${C^\star}^+$, the signed cutset $C^\star$ is called the *signed cutset on the cutset $C^\star$ and along e*. If the negative part of a cocircuit is empty, the cocircuit is *positive*. The circuits and cocircuits of a directed graph G are precisely the circuits and the cocircuits of the oriented graph matroid $M(G)$. For a spanning tree T and an edge $e \in T$, the unique cutset [resp. cocircuit along e] of G contained in $(E(G) \setminus T) \cup e$ is the *fundamental cutset* [resp. *fundamental cocircuit*] and is denoted $C^\star(e, T)$.

A cycle C [resp. circuit $C = (C^+, C^-]$ of G corresponds to a vector in $\mathbb{R}^{E(G)}$ with 1 on C [resp. 1 on C^+ and -1 on C^-] and 0 on $E(G) \setminus C$. Similarly, a cutset $C^\star$ [resp. cocircuit $C^\star = ({C^\star}^+, {C^\star}^-)$] of G corresponds to a vector in $\mathbb{R}^{E(G)}$ with 1 on $C^\star$ [resp. 1 on ${C^\star}^+$ and -1 on ${C^\star}^-$] and 0 on $E(G) \setminus C$.

Let us remark that the graphic oriented matroid $M(G)$, of a connected digraph G, is acyclic if and only if the oriented dual matroid $M^\star(G)$ is totally cyclic, see [1] for details. We recall that, given a graph G, an orientation of G is determined by the graphic oriented matroid determined by G and the orientation of an edge in every connected component.

Lemma 1. An edge $e = (a, b)$ is not flippable in an acyclic digraph G iff there exists a path from a to b in the digraph $G \setminus e$.

Proof. If there exists a path P from a to b in the digraph $G \setminus e$ then the arc (b, a) is not flippable. Reciprocally if there exists a directed circuit C in $_{-e}G$, it necessarily contains the edge $\bar{e} = (b, a)$. The set $C \setminus \bar{e}$ is a path from a to b in G. □

We say that a set of edges $E' \subseteq E(G)$ *preserves the directed connectivity of the digraph* if for any two vertices $a, b \in V(G)$ the following two conditions are equivalent:

(i) There is a directed path from a to b in G;
(ii) There is a directed path from a to b in $G' = (V, E')$.

Proposition 2. In an acyclic digraph G, the set of flippable edges preserves the directed connectivity of G. Consequently, if the digraph G is connected then there are at least $n - 1$ flippable edges and if there is exactly $n - 1$ flippable edges the set is a tree.

Proof. Let a and b be two vertices connected by a directed path in G. Let P^{ab}_{max} be a maximal path, with respect to cardinality, connecting these two vertices. All the edges of P^{ab}_{max} are flippable since otherwise by the previous lemma we could replace an edge of P^{ab}_{max} by a path, which contradicts the maximally of P^{ab}_{max}. If now the graph is connected, the set of flippable edges is spanning which implies that it has at least $n - 1$ edges. □

From the simple preceding results we can see that the set of flippable directed edges determines the directions of all the edges in the digraph. Indeed let e be a non-flippable edge of G. Necessarily from the lemma it belongs to a circuit C of G where it is the only positive edge. From this simple remark, one can also deduce that two different acyclic orientations of the same graph differ at least on one flippable edge.

Theorem 3. *Let T be a spanning tree of a digraph G. Then the following three conditions are equivalent:*

(3.1) *The digraph G is acyclic and T is the set of flippable edges of G.*
(3.2) *For every edge $e \in E(G) \setminus T$, e is the unique positive element in the fundamental circuit $C(e, T)$.*
(3.3) *All the cocircuits $\{C^\star(e, T) : e \in T\}$ are positive.*

Moreover, if G is 2-connected and one of the conditions (3.1)- (3.3) *holds, then T is the set of flippable edges of exactly two opposite acyclic orientations of G.*

Proof. (3.1) $\Longrightarrow$ (3.2) is a consequence of Lemma 1.
(3.2) $\Longrightarrow$ (3.3) is a consequence of the orthogonality of circuits and cocircuits of a digraph: i.e., given a circuit $C = (C^+, C^-)$ and a cocircuit $C^\star = ({C^\star}^+, {C^\star}^-)$ in a digraph such that $|C \cap C^\star| = 2$ then $|(C^+ \cap {C^\star}^+) \cup (C^- \cap {C^\star}^-)| = 1$ and $|(C^+ \cap {C^\star}^-) \cup (C^- \cap {C^\star}^+)| = 1$.
(3.3) $\Longrightarrow$ (3.1). Suppose that G has a directed cycle C. For an edge e of T, let A_e and B_e denote the components of G that are separated by $C^\star(e, T)$. We claim that there exists an edge e of T such that $C \cap A_e$ and $C \cap B_e$ are both non-empty. Since $C^\star(e, T)$ is positive, we see that the acyclicity of G follows from this claim. To prove the claim, pick an edge (a, b) of $C \setminus T$. Let p be the undirected path from a to b that is contained in T, and let e be any edge of p. If we cut the edge e of T, then a and b are separated in T. Therefore, a and b are separated in G by $C^\star(e, T)$, proving the claim, and hence the acyclicity of G. Next, for any $e \notin T$, pick an arbitrary element $a \in C(e, T) \setminus e$. From the orthogonality of the circuit $C(e, T)$ and the cocircuit $C^\star(a, T)$ we conclude that the element e is not flippable. The minimum number of flippable elements in the acyclic graph G is $|T|$, see Proposition 2. So every element of T is flippable.

Suppose now that G is 2-connected and one of the conditions (3.1)- (3.3) holds. From (3.3) we conclude that there are only two possible acyclic orientations of G, necessarily opposite. □

Remark 4. *On the conditions of Proposition 3, it is known that the set of positive cocircuits $\{C^\star(e, T) : e \in T\}$ form a basis of the cocircuit space. We notice however that the existence of a basis of positive cocircuits does not guarantee the fact that the flippable edges form a tree. This can be seen in the following example. Let G be the acyclic digraph on 4 vertices and with 5 edges oriented like in Figure (a). The edges 1, 2, 3, and 4 are flippable and the positive cocircuits $\{2, 4, 5\}$, $\{1, 4, 5\}$ and $\{1, 3, 5\}$ form a basis of the cocircuit space. In the digraph of Figure (b), the flippable edges are 2, 3 and 5 and they form a tree.*

From a classical result of R. W. Shannon [4] (on hyperplane arrangements), there exist at least $2m$ different acyclic orientations having a spanning tree of flippable edges. We present here a linear time algorithm, based on a depth first search, to find for every edge $e \in E(G)$ two acyclic orientations and a spanning tree T corresponding to the edge e.

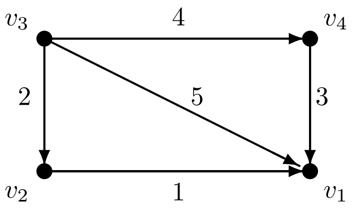

Figure (a)

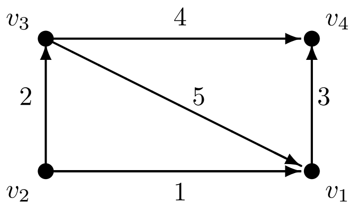

Figure (b)

Theorem 5. *Let G be a 2-connected graph. There exist at least $2m$ acyclic orientations with exactly $n-1$ flippable edges.*

Proof. The proof is based on the following algorithm for numbering the vertices based on a depth first search with postfix numbering starting with a specific edge. Indeed, we can see that each edge e gives two different correct orientations, one for each orientation of e.

Algorithm
Input: A graph G, two vertices a and b forming an edge (and a numbering of the vertices where a is first and b second to determine the search).
Output: A numbering α of the vertices with $\alpha(a) = n$, $\alpha(b) = n-1$ and such that the associated orientation has exactly $n-1$ flippable edges.
Procedure Search(v: vertex)
$S \leftarrow S \setminus v$; For every neighbor v' of v do if $v' \in S$ then Search(v');
$\alpha(v) \leftarrow k$; $k \leftarrow k+1$;
Main
$k \leftarrow 1$; $S \leftarrow V$; Search(a); return α.

The output of the algorithm is a numbering α of the vertices. This numbering induces two orientations: the increasing orientation if the edges $\{a, b\}$ are oriented from a to b if $\alpha(a) < \alpha(b)$ and the decreasing orientation if the edges $\{a, b\}$ are oriented from a to b if $\alpha(a) > \alpha(b)$. In these orientations, the set of flipping edges is exactly the search tree. Since a and b are the two first vertices in the order used for the search they will be numbered n and $n-1$ respectively. When looking at the set of flippable edges they form a tree. □

From the preceding discussion, we deduce the next corollary.

Corollary 6. *Let G be a 2-connected graph. Then the tree T of flippable edges, in one of the $2m$ acyclic orientations obtained using the algorithm, has a unique source or sink, which has also a unique neighbor.* □

Theorem 5 suggests the following open question:

Problem 7. Given a spanning subset $X \subseteq E(G)$ of edges of a connected graph, find a polynomial algorithm to decide if there is an orientation of G whose set of flippable edges is exactly X.

Following definitions of oriented matroids, a 2-connected graph G on n vertices is *simplicial* if every acyclic orientation of G has exactly $n-1$ flippable edges (see [1] for a geometric explanation of this definition).

Proposition 8. In the complete graph K_n a tree is flippable iff it is a path. A 2-connected graph G on n vertices is simplicial iff it is the complete graph K_n.

Proof. Any acyclic orientation of the complete graph K_n, which corresponds to a numbering of the vertices, contains a directed path of length $n-1$. Every other edge outside this path is clearly non-flippable.

For a graph G which is not complete, let a and b be two non-adjacent vertices. Let C be a circuit of G containing a and b, and of minimum length k. Such a circuit exists from the connectivity of G. Let us first remark that k, the length of C, must be at least four. Consider now an acyclic orientation of G defined by an ordering of the vertices such that 1 and k are the neighbors (for consistency) of a, 2 is a, the others vertices of C are numbered in the order with labels $3, 4, \ldots, k-1$ from vertex 1 to vertex k going by b and finally the other vertices outside of C are numbered indifferently from $k+1$ to n. By the minimality of C, in such an acyclic orientation all the edges of C are flippable which with Proposition 2 gives that there are at least n of them. □

Note that the number of flippable trees in the complete graph K_n is $n!$, a small number compared with the total number of trees, the Cayley number n^{n-2}. A graph is *Hamiltonian* if it contains a cycle of length n.

Proposition 9. Let $G = (V, E)$ be a 2-connected non-Hamiltonian graph $G = (V, E)$. There is an acyclic orientation of the edges such that every edge of G is on a positive cocircuit and outside another positive cocircuit.

Proof. By Theorem 5, there exists a tree T and an acyclic orientation of G, such that T is the set of flippable edges of this orientation. By Theorem 3, every edge $e \in E \setminus T$ is the only positive edge in the fundamental circuit $C(e, T)$.

Note that there are exactly $n-1$ positive cocircuits and for every element $x \in T$ there is exactly one positive cocircuit $C^\star_x$ containing x, the fundamental cocircuit $C^\star(x, T)$. Every edge $x \in T$ is then on a positive cocircuit and outside another positive cocircuit.

Since G is non-Hamiltonian, for every edge edge $b \notin T$, we know that $T \not\subseteq C(b,T)$. By the orthogonality of circuits and cocircuits, we have that $x \in C(b,T) \iff b \in C^\star(x,T)$. We can deduce then that b is in all the fundamental cocircuits $C^\star(x,T)$ for $x \in C(b,T) \setminus b$ and in none of the fundamental cocircuits $C^\star(x,T)$ for $x \in T \setminus C(b,T)$, which concludes the proof. □

3. FLIPPINGS PRESERVING STRONG CONNECTIVITY

It is well known that a 2-connected digraph G is totally cyclic iff it is strongly connected. Similarly to the acyclic case, "an edge $e \in E(G)$ in a strongly connected digraph G is *flippable*" if the digraph $_{-e}G$ is also strongly connected. A graph is 3-edge connected if it has no cutset of size smaller than three. Note that in the matroid sense, the notions of "strongly connected" and "3-edge connected" are dual, respectively, to the notions of "acyclic" and "simple". In this section, the dual results of the previous section are given without proof. Dualizing Lemma 1 for 2-connected graphs, we obtain the next lemma. (The reader can see that the result also holds for disconnected or 1-connected graphs.)

Lemma 10. An edge $e = (a,b)$ is not flippable in a strongly connected digraph G iff e is the only edge in the negative part of a cutset. □

Dualizing Proposition 2 and Theorem 3 we have:

Proposition 11. In a 3-edge connected and strongly connected digraph G the set of non-flippable edges does not contain a circuit. Consequently, there are at least $m - n + 1$ flippable edges. □

Theorem 12. *Let T be a spanning tree of a 3-connected digraph G. Then the following three conditions are equivalent:*

(12.1) *The directed graph G is strongly connected and $E \setminus T$ is the set of flippable edges of G.*
(12.2) *Every edge $e \in T$ is the unique positive element in the fundamental cutset $C^\star(e,T)$.*
(12.3) *For every edge $e \in E(G) \setminus T$ the fundamental circuit $C(e,T)$ is positive.* □

Proposition 13. A 2-connected graph $G = (V,E)$ admits a strongly connected orientation such that all the edges are flippable iff it is 4-connected.

Proof. This can be deduced almost directly from the theorem of Nash-Williams [3] which says that: "A graph admits a 2 edge-strongly connected orientation iff it is 4 edge-connected." Indeed the difficult part is the if part and in the Nash-Williams's result the edge can even be deleted and then clearly flipped. The only if part is immediate after one notes that if there is a 3-edge cutset then one of the edges of the cutset in any orientation is not flippable. □

The following theorem can be obtained by dualizing Theorem 5.

Theorem 14. *Let $G = (V, E)$ be a 3-connected graph. There exist at least $2m$ strongly connected orientations with exactly $m - n + 1$ flippable edges.* □

4. Concluding remarks: oriented matroids

In oriented matroid theory, the notion of acyclic reorientation is very important and has a geometric flavor. Every acyclic orientation $\mathcal{O}$ of the orientable matroid M fixes a face lattice $\mathcal{L}(\mathcal{O})$, called the *Las Vergnas face lattice* of $\mathcal{O}$, see [1] for details. In particular there is a natural notion of oriented matroid duality. This notion encodes the duality between circuits and oriented cutsets (cocircuits) in digraphs. Most of the results of this paper have a nice geometric interpretation. For example Proposition 9 can be translated into:

Proposition 15. Let G be a 2-connected non-Hamiltonian graph and $M(G)$ be the corresponding graphic matroid. There is an acyclic orientation $\mathcal{O}$ of $M(G)$, such that all the elements of $M(G)$ are over facets of the associated Las Vergnas face lattice $\mathcal{L}(\mathcal{O})$.

To finish the paper we propose a conjecture:

Conjecture 16. Let G be a 2-connected graph and $M(G)$ be the corresponding graphic matroid. The following two conditions are equivalent:

(16.1) The graph G has no triangle.
(16.2) There exists an acyclic orientation $\mathcal{O}$ of $M(G)$ such that all the elements of $M(G)$ are the atoms of the associated Las Vergnas face lattice $\mathcal{L}(\mathcal{O})$.

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