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A Note on the Orlik–Solomon Algebra

RAUL CORDOVIL AND GWIHEN ETIENNE

Let $\mathcal{M} = \mathcal{M}(E)$ be a matroid on a linear ordered set E. The Orlik–Solomon $\mathbb{Z}$-algebra $\mathrm{OS}(\mathcal{M})$ of $\mathcal{M}$ is the free exterior $\mathbb{Z}$-algebra on E, modulo the ideal generated by the circuit boundaries. The $\mathbb{Z}$-module $\mathrm{OS}(\mathcal{M})$ has a canonical basis called 'no broken circuit basis' and denoted ***nbc***. Let $e_X = \prod e_i$, $e_i \in X \subset E$. We prove that when e_X is expressed in the ***nbc*** basis, then all the coefficients are 0 or ± 1. We present here an algorithm for computing these coefficients. We prove in appendix a numerical identity involving the dimensions of the algebras of Orlik–Solomon of the minors of a matroid and its dual.

1. INTRODUCTION

Let $\mathcal{A}$ be an arrangement of hyperplanes (i.e., a finite set of codimension 1 vector subspaces) in $\mathbb{C}^d$. The intersection lattice $L(\mathcal{A})$ is the set of all intersections of the hyperplanes of $\mathcal{A}$ partially ordered by reversed inclusion. Consider the smooth manifold $\mathfrak{M}(\mathcal{A}) = \mathbb{C}^d \backslash \{\bigcup H : H \in \mathcal{A}\}$. Peter Orlik and Louis Solomon proved that the de Rham cohomology algebra of $\mathfrak{M}(\mathcal{A})$ can be described entirely in terms of the geometric lattice $L(\mathcal{A})$, see [7, 8]. This algebra has found use in the work of Kazuhiko Aomoto, and Israel M. Gel'fand and coworkers on the systematic study of the general hypergeometric functions, see [9, 10]. We consider here Orlik–Solomon $\mathbb{Z}$-algebras, defined over arbitrary matroids as introduced by Gel'fand and Rybnikov, see [5].

Throughout this note $\mathcal{M} = \mathcal{M}(E)$ denotes a matroid of rank r on the linear ordered set $E = \{e_1 < e_2 < \cdots < e_n\}$. Let $\mathfrak{C} = \mathfrak{C}(\mathcal{M})$ be the set of the circuits of $\mathcal{M}$. When the smallest element α of a circuit C, $|C| > 1$, is deleted, the remaining set, denoted $\mathrm{bc}(C) := C \backslash \alpha$, is called a *broken circuit*. In order to abbreviate the notation, the singleton set $\{x\}$ is denoted by x. Just as an independent set of a matroid is one which does not contain any circuit, an *internal independent set* of the matroid $\mathcal{M}$ is one which does not contain a broken circuit. Let $\mathrm{Inter}^i = \mathrm{Inter}^i(\mathcal{M})$ be the set of the internal independent subsets of cardinal i of $\mathcal{M}$. Every element of Inter^i is supposed to be ordered with the ordering induced by E. Set $\mathrm{Inter}(\mathcal{M}) = \bigcup_{i=0}^{i=r} \mathrm{Inter}^i(\mathcal{M})$. Consider now an independent set X. Let $\mathrm{cl}(X)$ be the closure of X in $\mathcal{M}$. Pick an element $x \in \mathrm{cl}(X) \backslash X$. Let $C(X, x)$ denote the unique circuit of $\mathcal{M}$ contained in $X \cup x$. The element $x \in \mathrm{cl}(X) \backslash X$ is called *externally active* in the independent set X if x is the minimal element of the circuit $C(X, x)$. Let $\mathrm{EA}(X)$ denote the set of externally active elements in X. Note that $X \in \mathrm{Inter}(\mathcal{M})$ iff $\mathrm{EA}(X) = \emptyset$. If $\mathrm{EA}(X) \neq \emptyset$, let $\alpha(X)$ denote the smallest element of $\mathrm{EA}(X)$. If B is a basis of $\mathcal{M}$ we say that an element $x \in B$ is *internally active* in B if x is *externally active* in the basis $B^\star = E \backslash B$ of the orthogonal matroid $\mathcal{M}^*$. Let $\mathrm{IA}(B)$ denote the set of internal active elements in B. We refer to [8] (resp. [11, 12]) as standard sources for arrangements of hyperplanes (resp. matroids).

2. *nbc* BASES

The following definition is due to I. M. Gel'fand and G. L. Rybnikov [5]. It is the 'combinatorial analogous' of one proposed in [7].

DEFINITION 2.1 ([5, 7]). The *Orlik–Solomon algebra* of the matroid $\mathcal{M}(E)$ is the $\mathbb{Z}$-algebra $\mathrm{OS} = \mathrm{OS}(\mathcal{M})$ given by the set of generators E, and the relations:

- $e^2 = e \cdot e' + e' \cdot e = 0, \forall e, e' \in E$,
- If e is a loop of $\mathcal{M}$, then we have $e = 0$.
- If $\{e_{i_1}, e_{i_2}, \dots, e_{i_m}\} \in \mathfrak{C}(\mathcal{M})$, $m > 1$, $e_{i_1} < \cdots < e_{i_m}$, then

$$\sum_{j=1}^{j=m} (-1)^{j-1} e_{i_1} \cdot \cdots \cdot \widehat{e}_{i_j} \cdot \cdots \cdot e_{i_m} = 0,$$

where $\widehat{\ }$ indicates an omitted factor.

If $X = \{e_{i_1} < \cdots < e_{i_p}\} \subset E$, set $e_X := e_{i_1} \cdot e_{i_2} \cdot \cdots \cdot e_{i_p}$. Set $e_\emptyset := 1$. We say that e_X is a *strongly decomposable element* of the algebra OS.

REMARK 2.2. In [3] it is shown that the matroid $\mathcal{M}$ cannot be reconstructed from the abstract algebra $\mathrm{OS}(\mathcal{M})$. In other words, when the algebra $\mathrm{OS}(\mathcal{M})$ is determined by an arbitrary basis $\boldsymbol{B}$ and the corresponding structure constants. It is an open question (implicit in the Conjecture 5.4 of [3]) to decide when, given an abstract Orlik–Solomon algebra OS, there is an unique loop free matroid $\mathcal{M}$ such that $\mathrm{OS} = \mathrm{OS}(\mathcal{M})$. In the following, for each abstract Orlik–Solomon algebra OS, we fixe an associated matroid $\mathcal{M}$ such that $\mathrm{OS} = \mathrm{OS}(\mathcal{M})$.

Let $\bigoplus_{e \in E} \mathbb{Z}e$ be the free $\mathbb{Z}$-module, generated by the family of generators $e_1, e_2, \dots, e_n$. Consider the *graded exterior algebra* $\Lambda E = \bigoplus_{i \in \mathbb{N}} \Lambda^i E$ of the module $\bigoplus_{e \in E} \mathbb{Z}e$. Define the graded linear mapping $\partial : \Lambda E \longrightarrow \Lambda E$ as a linear extension of the linear maps:

- $\partial_0 : \mathbb{Z} \to (0)$,
- $\partial_1 : \Lambda^1 E \to \mathbb{Z}$, where $\partial_1(e) = 1, \forall e \in E$,
- $\forall \ell = 2, 3, \dots, n$, the maps $\partial_\ell : \Lambda^\ell E \longrightarrow \Lambda^{\ell-1} E$, where

$$\partial_\ell(e_{i_1} \wedge \cdots \wedge e_{i_\ell}) = \sum_{j=1}^{j=\ell} (-1)^{j-1} e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_j} \wedge \cdots \wedge e_{i_\ell}.$$

Let $\mathcal{I}$ be the two-sided ideal of the exterior algebra ΛE generated by the set $\{\partial(e_C) : C \in \mathfrak{C}(\mathcal{M}), |C| > 1\} \cup \{e : \text{e is a loop of} \mathcal{M}\}$. Note that $\mathrm{OS}(\mathcal{M}) = \Lambda E / \mathcal{I}$. Set $\mathrm{OS}_i(\mathcal{M}) = \Lambda^i E / (\mathcal{I} \cap \Lambda^i E), \forall i \in \mathbb{N}$.

PROPOSITION 2.3. *The grading* $\mathrm{OS}(\mathcal{M}) = \bigoplus_{i \in \mathbb{N}} \mathrm{OS}_i(\mathcal{M})$ *is canonical, i.e., it is independent of the knowledge of the matroid* $\mathcal{M}$.

PROOF. We know that $\mathrm{OS}_i = (0)$, for all $i > r$. If $\mathrm{OS} = \mathrm{OS}_0 = \mathbb{Z}$ (i.e., $r = 0$) the result is clear. Suppose that $\mathrm{OS} \neq \mathbb{Z}$. Note that

$$\mathrm{OS}_r = \{x \in \mathrm{OS} : x \cdot y = 0, \ \forall y \in \mathrm{OS} \backslash \mathbb{Z}\}.$$

If we know the modules $\mathrm{OS}_r, \dots, \mathrm{OS}_{r-i}$ and $\mathrm{OS}^{\langle i+1 \rangle} := \mathrm{OS}_r \oplus \cdots \oplus \mathrm{OS}_{r-i} \neq \mathrm{OS}$, (i.e., $r - i > 1$) the module OS_{r-i-1}, $i = 0, \dots, r-2$, can be defined recursively as follows

$$\mathrm{OS}_{r-i-1} = \{x \in \mathrm{OS} : x \cdot y \in \mathrm{OS}^{\langle i+1 \rangle}, \ \forall y \in \mathrm{OS} \backslash \mathbb{Z}\} / \mathrm{OS}^{\langle i+1 \rangle}. \qquad \square$$

The following basic theorem was independently discovered by Orlik and Solomon in 1980, Björner in 1982 and Jambu and Leborgne in 1986. See in [2] for a historical note.

THEOREM 2.4 ([1, 6, 7]). *The set* $\boldsymbol{nbc}^i := \{e_I : I \in \mathrm{Inter}^i(\mathcal{M})\}$ *is a linear basis of the free* $\mathbb{Z}$*-module* $\mathrm{OS}_i(\mathcal{M})$, $\forall i \in \mathbb{N}$. $\square$

Set $\boldsymbol{nbc} := \sum_{i=0}^{r} \boldsymbol{nbc}^i = \{e_I : I \in \mathrm{Inter}(\mathcal{M})\}$. $\boldsymbol{nbc}$ is termed the *no broken circuit basis* of the free $\mathbb{Z}$-module $\mathrm{OS}(\mathcal{M})$.

THEOREM 2.5. *Let e_X be a strongly decomposable element of* $\mathrm{OS}(\mathcal{M})$. *If e_X is express in the $\boldsymbol{nbc}$ basis of the $\mathbb{Z}$-module* $\mathrm{OS}(\mathcal{M})$, *then all the coefficients are* 0 *or,* ± 1.

Theorem 2.5 is a consequence of the following technical lemma. Let **G** be the direct graph such that:

- Its vertex set $\mathrm{V}(\mathbf{G})$ is the set of all the independent sets of the matroid $\mathcal{M}$.
- $\overrightarrow{XX'} \in \mathrm{E}(\mathbf{G})$ is a directed edge of **G** iff there is a pivotable pair (α, x) such that $X' = X\backslash x \cup \alpha$, where $\alpha = \alpha(X)$ and $x \in C(X, \alpha)\backslash\alpha$.

LEMMA 2.6. *For every pair of vertices X, X' of the graph* **G**, *there is at most one directed path from X to X'.*

PROOF OF LEMMA 2.6. Suppose that there is in **G** a directed path containing exactly the $k + 1$ vertices, $X_1, \ldots, X_{k+1}$. For every $i = 1, \ldots, k$ set $\alpha_i = \alpha(X_i)$, $C_i = C(X_i, \alpha_i)$, and set $x_i = X_i\backslash X_{i+1}$. We show first that:

$$\alpha_{i+1} \neq x_i, \tag{2.6.1}$$
$$\alpha_i \notin \mathrm{bc}(C_{i+1}), \tag{2.6.2}$$
$$\alpha_1 < \cdots < \alpha_k, \tag{2.6.3}$$
$$C_i = C(X_1, \alpha_i), \tag{2.6.4}$$
$$X_{i+1} = X_1\backslash\{x_1, \ldots, x_i\} \cup \{\alpha_1, \ldots, \alpha_i\},\ \ |X_1 \Delta X_{i+1}| = 2i, \tag{2.6.5}$$
$$C_i = C(X_1\backslash\{x_1, \ldots, x_{i-1}\}, \alpha_i). \tag{2.6.6}$$

(2.6.1). Suppose for a contradiction that $\alpha_{i+1} = x_i (\neq \alpha_i)$. Then

$$\alpha_i, \alpha_{i+1} \in C_i = C_{i+1} \subset X_i \cup \alpha_i = X_{i+1} \cup \alpha_{i+1},$$

and we find the contradiction $\alpha_i < \alpha_{i+1}$ and $\alpha_{i+1} < \alpha_i$.
(2.6.2). Suppose for a contradiction that $\alpha_i \in \mathrm{bc}(C_{i+1})$. So $\alpha_{i+1} = \alpha(X_{i+1}) < \alpha_i$. From the circuit elimination axiom we know that there is a circuit C'_{i+1} such that

$$\alpha_{i+1} \in C'_{i+1} \subset \{C_i \cup C_{i+1}\}\backslash\alpha_i \subset X_i \cup \{X_i \cup \alpha_{i+1}\} \subset X_i \cup \alpha_{i+1}.$$

So $C'_{i+1}\backslash\alpha_{i+1}$ is a broken circuit contained in X_i, and $\alpha_i = \alpha_i(X_i) < \alpha_{i+1}$, a contradiction.
(2.6.3). From (2.6.3), we see that $\mathrm{bc}(C_{i+1}) \subset X_i$. We conclude that $\alpha_i < \alpha_{i+1}$.
(2.6.4). From the definitions we know that

$$\alpha_i \in C_i \subset X_i \cup \alpha_i \subset X_1 \cup \{\alpha_1, \ldots, \alpha_i\}.$$

By our hypothesis we know that $C_i\backslash\alpha_i$ is a broken circuit. We have proved in (2.6.3) that $\alpha_1 < \cdots < \alpha_{i-1} < \alpha_i$ so $C_i \cap \{\alpha_1, \ldots, \alpha_{i-1}\} = \emptyset$, and (2.6.4) follows.
(2.6.5). It is clear that $|X_1 \Delta X_2| = 2$. Suppose inductively that

$$X_i = X_1\backslash\{x_1, \ldots, x_{i-1}\} \cup \{\alpha_1, \ldots, \alpha_{i-1}\}, \qquad \text{and} \qquad |X_1 \Delta X_i| = 2(i-1).$$

From (2.6.4) we know that $x_i \notin \{\alpha_1, \ldots, \alpha_{i-1}\}$. So (2.6.5) follows.
(2.6.6). From (2.6.5) we know that $X_i \cap \{x_1, \ldots, x_{i-1}\} = \emptyset$, so C_i is disjoint of $\{x_1, \ldots, x_{i-1}\}$. Making use of (2.6.4) we conclude that

$$C_i = C(X_1\backslash\{x_1, \ldots, x_{i-1}\}, \alpha_i).$$

We are now able to complete the proof of Lemma 2.6. We prove by induction on the length of the paths. Suppose that the Lemma 2.6 is true for all paths of length $\ell \leq k$. Consider a new directed path $X_1 = X'_1 \to \cdots \to X'_i \to \cdots \to X'_{k'+1} = X_{k+1}$. From (2.6.5) we know that $k' = k$. For every $i = 1, \ldots, k$, set $a'_i := \alpha(X'_i)$, $x'_i := X'_{i+1} \backslash X'_i$ and $C'_i = C(X'_i, \alpha'_i)$. As $X'_{k'+1} = X_{k+1}$, we know

$$\{\alpha_1, \ldots, \alpha_k\} = \{\alpha'_1, \ldots, \alpha'_k\} \qquad \text{and} \qquad \{x_1, \ldots, x_k\} = \{x'_1, \ldots, x'_k\}.$$

From (2.6.3) we get $\alpha_1 < \cdots < \alpha_k$ and $\alpha'_1 < \cdots < \alpha'_k$, so $\alpha_i = \alpha'_i$, for every every $i = 1, \ldots, k$. From (2.6.4) we conclude that $C_k = C(X_1, \alpha_k) = C(X_1, \alpha_{k'}) = C_{k'}$. (2.6.6) entails that

$$\begin{aligned} x_k, x_{k'} \in C_k &= C(X_1 \backslash \{x_1, \ldots, x_{k-1}\}, \alpha_k) \\ &= C_{k'} = C(X_1 \backslash \{x'_1, \ldots, x'_{k-1}\}, \alpha_k), \end{aligned}$$

so $x_k = x'_k$ and $X_k = X'_k$. By the induction hypothesis we conclude that $X_i = X'_i$, $\forall\, i = 2, \ldots, k-1$. $\square$

PROOF OF THEOREM 2.5. If $C \in \mathfrak{C}(\mathcal{M})$, $|C| > 1$, then we have $e_C = 0$. Indeed pick an element $e \in C$. Then $e_C = \pm e \cdot \partial(C) = 0$. So $e_D = 0$, for every dependent set D of $\mathcal{M}$. It is clear that $e_{X'} \in \boldsymbol{nbc}$ iff X' is a sink of $\mathbf{G}$. We see $\mathbf{G}$ as an edge-labelled graph:

- Let $\overrightarrow{YY'_{i_1}}$ be an arbitrary edge where $\alpha = \alpha(Y) = Y'_{i_1} \backslash Y$, and set $C = C(Y, \alpha)$. Suppose that $\mathrm{bc}(C) = \{y_1, \ldots, y_{i_1}, \ldots, y_m\}$ and $Y'_{i_1} = Y \backslash y_{i_1} \cup \alpha$. Consider the expansion of the element $\partial(e_\alpha \cdot e_Y) \in \Lambda E$. The elements $\partial(e_\alpha \cdot e_Y)$ and $e_C \cdot \partial(e_{Y \backslash C})$ are members of the ideal $\mathcal{I}$, so

$$e_Y = \sum_{i=1}^{m} \zeta_i e_{Y'_i}, \qquad \text{with} \qquad \zeta_i = \pm 1 \qquad \text{and} \qquad Y'_i = Y \backslash y_i \cup \alpha. \tag{2.1}$$

  We label the edge $\overrightarrow{YY'_{i_1}}$ with the scalar ζ_{i_1}.

Let $\mathfrak{P}_1, \ldots, \mathfrak{P}_\ell$ be the list of the maximal length directed paths of $\mathbf{G}$, beginning with the vertex X. We denote by T_i the last vertex of the path $\mathfrak{P}_i$. T_i is a sink of $\mathbf{G}$, so $e_{T_i} \in \boldsymbol{nbc}$. From Eqn. (2.1) and Lemma 2.6 we conclude that

$$e_{X_1} = \sum_{i=1}^{\ell} \xi_i e_{T_i}, \qquad e_{T_i} \in \boldsymbol{nbc}, \qquad \xi_i \pm 1, \tag{2.2}$$

where ξ_i is the product of the labels of all the edges of the path $\mathfrak{P}_i$. $\square$

The following corollary provides an useful algorithm to compute the support set of e_X, $\mathrm{supp}(e_X) := \{e_{T_1}, \ldots, e_{T_\ell}\}$.

COROLLARY 2.7. *On the conditions of Theorem* 2.5, $e_{T_i} \in \mathrm{supp}(e_X)$ *iff there is a maximal sequence of pairs* $(\alpha_1, x_1), (\alpha_2, x_2), \ldots, (\alpha_k, x_k)$ *in* $\mathrm{EA}(X) \times X$*, satisfying the following three conditions:*

- $x_i \in C(X, \alpha_i)$, $\forall i \in \{1, \ldots, k\}$.
- $\alpha_1 = \alpha(X_1)$ *and* $\forall i \in \{2, \ldots, k\}$, α_i *is the smallest element of* $\mathrm{EA}(X)$ *such that* $C(X, \alpha_i) \cap \{x_1, \ldots, x_{i-1}\} = \emptyset$.
- $T_i = X \backslash \{x_1, \ldots, x_k\} \cup \{\alpha_1, \ldots, \alpha_k\}$.

PROOF. The proof is a straightforward consequence of Lemma 2.6 and left to the reader. □

REMARK 2.8. Theorem 2.5 does not characterize the ***nbc*** basis of OS. Indeed, consider the rank 2 uniform matroid $\mathcal{M}$ on the ground set $E = \{e_1 < e_2 < e_3 < e_4\}$. Let $\mathbf{B}$ be the basis of OS

$$\mathbf{B} := \{1, e_1, e_2, e_3, e_4, e_1 \cdot e_2, e_1 \cdot e_3, e_3 \cdot e_4\}.$$

Note that $\mathbf{B} \neq \boldsymbol{nbc} = \{1, e_1, e_2, e_3, e_4, e_1 \cdot e_2, e_1 \cdot e_3, e_1 \cdot e_4\}$. However, every product $e_i \cdot e_j$, $i, j \in \{1, 2, 3, 4\}$ can be written as a linear combination of the elements of $\mathbf{B}$, with coefficients 0 or ± 1. The following problem is open:

- Give a topological interpretation of Theorem 2.5.

3. APPENDIX

We remember that a flat F of $\mathcal{M}$ is a termed a *cyclic flat* if $F = \emptyset$ or F is the union of circuits.

PROPOSITION 3.1. *Let $\mathfrak{F}_c$ be the set of cyclic flats of $\mathcal{M}(E)$. Then*

$$\sum_{F \in \mathfrak{F}_c} \dim(\mathrm{OS}(\mathcal{M}/F)) \times \dim(\mathrm{OS}(\mathcal{M}^*/(E \backslash F))) = 2^n.$$

PROOF. We prove that for every subset $S \subset E$ there exists one and only one cyclic flat F such that:

$$S \backslash F \in \mathrm{Inter}(\mathcal{M}/F), \tag{3.1.1}$$

$$F \backslash S \in \mathrm{Inter}(\mathcal{M}^\star/(E \backslash F)). \tag{3.1.2}$$

Note that F is a cyclic flat of $\mathcal{M}$ iff $E \backslash F$ is a cyclic flat of $\mathcal{M}^*$. So (3.1.1) and (3.1.2) are equivalent. We make use of the following two results:

(a) *Given a subset S of E, there exists one and only one basis B of $\mathcal{M}$ such that $B \backslash \mathrm{IA}(B) \subset S \subset B \cup \mathrm{EA}(B)$, see* [1, *Proposition* 7.3.6]*:*

(b) *Given a basis B of $\mathcal{M}$, there is one and only one cyclic flat F of $\mathcal{M}$ such that $(B \backslash F, F \backslash B) \in \mathrm{Inter}(\mathcal{M}/F) \times \mathrm{Inter}(\mathcal{M}^\star/(E \backslash F))$, see* [4].

(3.1.1). Fix a subset $S \subset E$. Let B be the basis of $\mathcal{M}$ associated to S by (a). Let F be the cyclic flat associated to B by (b). By hypothesis $B \backslash F \in \mathrm{Inter}(\mathcal{M}/F)$, so $B \backslash F$ is a basis of $\mathcal{M}/F$. We claim that $S \backslash F \subset B \backslash F$. It is clear that this inclusion imply (3.1.1). From (a) we se that $S \backslash F \subset (B \backslash F) \cup (\mathrm{EA}(B) \backslash F)$. So it is enough to prove that $\mathrm{EA}(B) \subset F$. Suppose for a contradiction that $x \in \mathrm{EA}(B) \backslash F \subset B^\star \backslash F$. Note that $B^\star \backslash F = (E \backslash F) \backslash (B \backslash F)$. So

$$C_{\mathcal{M}/F}(B \backslash F, x) \subset C_{\mathcal{M}}(B, x) \backslash F.$$

As by hypothesis $x \in \mathrm{EA}(B)$, x is the smallest element of $C_{\mathcal{M}}(B, x)$ and hence it is also the smallest element of $C_{\mathcal{M}/F}(B \backslash F, x)$. So $B \backslash F \notin \mathrm{Inter}(\mathcal{M}/F)$ a contradiction. □

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RAUL CORDOVIL
Departamento de Matemática,
Instituto Superior Técnico,
Av. Rovisco Pais,
1049-001 Lisboa,
Portugal
E-mail: cordovil@math.ist.utl.pt
AND

GWIHEN ETIENNE
Université Pierre et Marie Curie,
Case 189, Combinatoire# 4, Place Jussieu,
75005 Paris,
France
E-mail: gwihen@easynet.fr