

On semisimplicity of reflexive amenable operator algebras

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Problem: Galé, Ransford and White (1992)

A amenable + reflexive Banach algebra \implies A finite dimensional?



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Known answers:

- 1 A finite dimensional $\implies A$ semisimple and direct sum of matrix algs \checkmark
- 2 A is a abelian Banach algebra [Galé, Ransford and White (1992)] \checkmark
if A is a abelian Banach algebra of operators, see [R El Harti (2010)] \checkmark
- 3 If A is C^* -alg Galé, Ransford and White (1992) \checkmark
- 4 If A is non-abelian, little is known...

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Rachid+PP (2013)

- ① A is reflexive + amenable operator alg. \implies A semisimple, $A = \bigoplus_{\text{finite}} A_i$
- ② If A is simple op. alg. and $A \cap K(H) \neq \{0\} \implies A = M_n(\mathbb{C})$.

Let A be a Banach algebra

- 1 A is **reflexive** if the underlying Banach space of A is reflexive, i.e. the canonical map $\varphi : A \rightarrow A^{**}$ such that $\varphi_a(f) = f(a)$ is **surjective**
- 2 A is **amenable** Banach algebra if every derivation $D \rightarrow E^*$ is inner (there is b such that $D(a) = a \cdot b - b \cdot a$), for every A -bimodule E .
- 3 A is **semi-simple** algebra if the Jacobson radical is trivial

$$\text{rad}(A) = \bigcap_{\text{maximal ideals}} M = \{0\}$$

Note that if E is A -bimodule so is E^* with $(a \cdot f)(\xi) = f(\xi \cdot a)$ and $(f \cdot a)(\xi) = f(a \cdot \xi)$, $a \in A$, $f \in E^*$, $\xi \in E$

Remark

- 1 If A is finite dim, then A amenable $\iff A$ semisimple.
- 2 If A is a radical algebra $\text{rad}(A) = A$ then A is non-semisimple.
- 3 $K(H)$ is amenable, non-reflexive and semisimple.
- 4 $B(H)$ non-amenable, non-reflexive and semi-simple
- 5 $l^1(\mathbb{Z})$ is amenable, non-reflexive and non-semisimple?

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- 2 Let G be a locally compact group . Then the Banach algebra $A = L^1(G)$ is Arens regular iff G is **finite** [N.J. Young, 1973]
- 3 Any C^* -algebra A is Arens regular [Á. Rodríguez-Palacios, 1987]

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A operator algebra $\not\Rightarrow$ A self adjoint!

Remark

- 1 $B(H)$ is Arens regular
- 2 If A is an operator algebra, then A is Arens regular
- 3 A C^* -algebra $\implies A \hookrightarrow B(H)$ for a Hilbert sp [Gelfand, Naimark 1943]
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Definition

A bounded approximate identity for a normed algebra A is a bounded net $\{e_\alpha\}_{\alpha \in I}$ in A with the property that

$$\lim_{\alpha} e_\alpha a = \lim_{\alpha} a e_\alpha = a, \quad a \in A.$$

Examples: C^* -algebras and amenable Banach algebras

Lemma (Rachid+PP)

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Proof: Let P be a primitive ideal. Then $B := A/P$ is a primitive operator algebra. Is B simple? For a maximal two-sided ideal M_B in B . Then M_B has a bounded approximate identity (since $M_B = (M_A + P)/P$ for some maximal two-sided ideal M_A in A).

Then $\overline{M_B}^{w^*}$ is a two-sided ideal in B^{**} and $\overline{M_B}^{w^*} = B^{**}p$ with $p \in B^{**}$ some central idempotent [Effros]. Besides this,

$$B^{**} = B^{**}p \oplus B^{**}(1_{B^{**}} - p), \quad (3.1)$$

Since the reflexivity property passes to quotients we have that B is also reflexive. Thus from (3.1) we conclude that $B = Bp \oplus B(1 - p)$ with Bp and $B(1 - p)$ being two-sided ideals in B . However every non-trivial two-sided ideal in the primitive algebra B is essential (an ideal I is said to be essential if $I \cap J$ is non-trivial for all non-trivial ideal J). It follows that $Bp = \{0\}$ or $B(1 - p) = \{0\}$. Since $M_B = Bp \neq B$ and $p \neq 1$ we conclude that $Bp = \{0\}$. Hence $M_B = \{0\}$ and so B is simple.

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Let A be a reflexive operator algebra such that each maximal two sided ideal has a bounded approximate identity. Then A is semisimple. Moreover, A is a finite direct sum of simple operator algebras.

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Proof: Let Π_A be the space of all primitive ideals in A equipped with the hull kernel topology. If $P \in \Pi_A$, then P is maximal by Lemma. Therefore P has a bounded approximate identity and $P^{**} = A^{**}p$ for some central idempotent p by [Effros]. Since A is reflexive, $P = Ap$. Using the same argument in [Galé Ransford, White], we conclude that Π_A is discrete and compact. Hence Π_A is a finite set, say $\Pi_A = \{P_1, \dots, P_n\}$ with central idempotents p_1, \dots, p_n , respectively. It is easy to check that

$$A = Ap_1p_2\dots p_n \oplus \bigoplus_{i=1}^n A(1 - p_i), \quad \text{Rad}(A) = Ap_1p_2\dots p_n = \bigcap_{i=1}^n Ap_i.$$

Therefore $\text{Rad}(A) = \{0\}$ and $A(1 - p_i)$ is a minimal two sided ideal (for every $i = 1, \dots, n$). Thus A is semisimple and moreover A is a finite direct sum of simple algebras.

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Every abelian, reflexive, amenable and unital operator algebra is isomorphic to \mathbb{C}^n for some n .

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Theorem (Rachid+PP)

Every reflexive and amenable operator algebra A is semisimple and it is a finite direct sum of simple Banach algebras of operators.

Proof: Let $\pi : A \rightarrow B(H)$ be a bounded representation of A on some Hilbert space H . Let M be a closed invariant subspace of $\pi(A)$ and take the following admissible short sequence

$$0 \longrightarrow M \longrightarrow H \longrightarrow H/M \longrightarrow 0.$$

By [Curtis], this sequence splits, therefore A has the total reduction property. It follows from [REIHarti] that every closed two-sided ideal of A has a bounded approximate identity. Therefore the result is now an easy consequence of our Theorem.

Definition

- 1 For $T \in B(H)$, let $A_T^0 = \overline{\{p(T) : p \text{ polynomial, } p(0) = 0\}}^{\|\cdot\|}$ be the nonunital Banach algebra generated by $T \in B(H)$.
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 - 2 T quasinilpotent if $\sigma(T) = \{0\}$
- 1 The amenability of operator algebras generated by single operators have been investigated.
 - 2 It is not known whether such algebras are reflexive.
 - 3 Recall that if T is a non trivial quasi-nilpotent operator, then A_T^0 is a radical algebra $\text{Rad}(A_T^0) = A_T^0$ (hence A_T^0 is a non semisimple algebra)
 - 4 If we further assume that A_T^0 is amenable then A_T^0 cannot be reflexive by Theorem above.
 - 5 The following result shows that A_T^0 cannot contain non trivial compact operators!!

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Proof: We show that $TK \neq 0$. Indeed, if that is not the case, then since $K \in A_T^0$, K is a limit of polynomial $P_n(T)$ with $P_n(0) = 0$. So K^2 is the limit of $P_n(T)K$. Note now that $P_n(T)K = 0$ for all n , so $K^2 = 0$ thus K is nilpotent. This implies that A_K^0 is finite dimensional because it is generated by a nilpotent operator as a linear space. Since A_K^0 is an amenable Banach, A_K^0 is a semisimple algebra. Then $K = 0$ which is a contradiction (remark that commutative semisimple Banach algebras do not contain non trivial quasi nilpotent or nilpotent elements).

So since $TK \neq 0$, there exists a trace-class operator $N \in C(H)$ such that $\text{tr}(TKN) \neq 0$. Let D_N be the derivation from A_T^0 to $(A_{TK}^0)^\top$ defined by $D_N(A) := NA - AN$ for all $A \in A_T^0 \subseteq B(H) = C(H)^*$, where $(A_{TK}^0)^\top$ is the annihilator of A_{TK}^0 taken in $C(H)$ (note that $A_{TK}^0 \subseteq A_T^0$, so $D : A_T^0 \rightarrow (A_T^0)^\top \subseteq (A_{TK}^0)^\top$). Besides this, $(A_{TK}^0)^\top$ is a Banach A_T^0 -bimodule in $K(H)$. Since A_T^0 is amenable D_N is inner (see [?]), so there exists an $M \in (A_{TK}^0)^\top$ such that $D_N(A) = MA - AM$ for all $A \in A_T^0$. This means that $KT(N - M) = (N - M)KT$ and $\text{tr}(KT(N - M)) = \text{tr}((N - M)KT) \neq 0$. Hence $\sigma(KT(N - M)) \neq \{0\}$ where $\sigma(KT(N - M))$ denotes the spectrum of $KT(N - M)$. Since KT and $N - M$ commute we have $\sigma(KT(N - M)) \subseteq \sigma(KT)\sigma(N - M)$ and therefore $\sigma(KT) \neq \{0\}$. Similarly, $\sigma(KT) \subseteq \sigma(K)\sigma(T)$, whence $\sigma(T) \neq \{0\}$. Therefore T is non quasinilpotent.

THANK YOU

