On semisimplicity of reflexive amenable operator algebras

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Workshop Analyse Harmonique et Inégalitiés Fonctionnelles,
UNIVERSITE MOULAY ISMAIL FACULTE DES SCIENCES,
Département de Mathématiques et Informatique
Meknès, Morocco, 19 – 20 June 2013

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A amenable + reflexive Banach algebra $\Longrightarrow$ A finite dimensional?

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Known answers:

1. A finite dimensional $\Longrightarrow$ A semisimple and direct sum of matrix algs $\surd$
2. A is a abelian Banach algebra [Galé, Ransford and White (1992)] $\surd$
   if A is a abelian Banach algebra of operators, see [R El Harti (2010)] $\surd$
3. If A is C*-alg Galé, Ransford and White (1992) $\surd$
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Rachid+PP (2013)

1. A is reflexive + amenable operator alg. $\Longrightarrow$ A semisimple, $A = \bigoplus_{\text{finite}} A_i$
2. If A is simple op. alg. and $A \cap K(H) \neq \{0\}$ $\Longrightarrow$ $A = M_n(\mathbb{C})$.

Let A be a Banach algebra

1. A is **reflexive** if the underlying Banach space of A is reflexive, i.e. the canonical map $\varphi : A \to A^{**}$ such that $\varphi_a(f) = f(a)$ is **surjective**.
2. A is **amenable** Banach algebra if every derivation $D \to X^*$ is inner (there is b such that $D(a) = a \cdot b - b \cdot a$), for every A-bimodule X.
3. A is **semi-simple** algebra if the Jacobson radical is trivial

$$\mathrm{rad(A)} = \bigcap_{\text{irreducible rep A}} \ker(\pi) = \{0\}.$$

Note that if X is A-bimodule so is X^* with $(a \cdot f)(\xi) = f(\xi \cdot a)$ and $(f \cdot a)(\xi) = f(a \cdot \xi)$, $a \in A$, $f \in X^*$, $\xi \in X$

Remark

1. If A if finite dim, then A amenable $\iff$ A semisimple.
2. If A is a radical algebra $\mathrm{rad}(A) = A$ then A is non-semisimple.
3. $K(H)$ is amenable, non-reflexive and semisimple.
4. $B(H)$ non-amenable, non-reflexive and semi-simple
5. $l^1(\mathbb{Z})$ is amenable, non-reflexive and non-semisimple?

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2. Let G be a locally compact group. Then the Banach algebra $A = L^1(G)$ is Arens regular iff G is **finite** [N.J. Young, 1973]
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Definition

A is an **operator algebra** if $A \subseteq B(H)$ is an alg. closed under the norm topology, for some Hilbert space H.

A operator algebra $\not\Rightarrow$ A self adjoint!

Remark

1. $B(H)$ is Arens regular (it is a C*-algebra)
2. If A is an operator algebra, then A is Arens regular
3. A C*-algebra $\implies A \hookrightarrow B(H)$ for a Hilbert sp [Gelfand, Naimark 1943]
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Definition

A bounded approximate identity for a normed algebra A is a bounded net $\{e_\alpha\}_{\alpha \in I}$ in A with the property that

$$\lim_\alpha e_\alpha a = \lim_\alpha a e_\alpha = a, \quad a \in A.$$

Examples: C*-algebras and amenable Banach algebras

Lemma (Rachid+PP)

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Proof: Let P be a primitive ideal. Then $B := A/P$ is a primitive operator algebra. Is B is simple? For a maximal two-sided ideal M_B in B. Then M_B has a bounded approximate identity (since $M_B = (M_A + P)/P$ for some maximal two-sided ideal M_A in A).
Then $\overline{M_B}^{w^*}$ is a two-sided ideal in B^{**} and $\overline{M_B}^{w^*} = B^{**}p$ with $p \in B^{**}$ some central idempotent [Effros]. Besides this,

$$B^{**} = B^{**}p \oplus B^{**}(1_{B^{**}} - p), \tag{3.1}$$

Since the reflexivity property passes to quotients we have that B is also reflexive. Thus from (3.1) we conclude that $B = Bp \oplus B(1-p)$ with Bp and $B(1-p)$ being two-sided ideals in B. However every non-trivial two-sided ideal in the primitive algebra B is essential (an ideal I is said to be essential if $I \cap J$ is non-trivial for all non-trivial ideal J. It follows that $Bp = \{0\}$ or $B(1-p) = \{0\}$. Since $M_B = Bp \neq B$ and $p \neq 1$ we conclude that $Bp = \{0\}$. Hence $M_B = \{0\}$ and so B is simple.

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Proof: Let Π_A be the space of all primitive ideals in A equipped with the hull kernel topology. If $P \in \Pi_A$, then P is maximal by Lemma. Therefore P has a bounded approximate identity and $P^{**} = A^{**}p$ for some central idempotent p by [Effros]. Since A is reflexive, $P = Ap$. Using the same argument in [Galé Ransford, White], we conclude that Π_A is discrete and compact. Hence Π_A is a finite set, say $\Pi_A = \{P_1, ..., P_n\}$ with central idempotents $p_1, ..., p_n$, respectively. It is easy to check that

$$A = Ap_1p_2...p_n \oplus \bigoplus_{i=1}^{n} A(1-p_i), \qquad \mathrm{Rad}(A) = Ap_1p_2...p_n = \bigcap_{i=1}^{n} Ap_i.$$

Therefore $\mathrm{Rad}(A) = \{0\}$ and $A(1-p_i)$ is a minimal two sided ideal (for every $i = 1, ..., n$). Thus A is semisimple and moreover A is a finite direct sum of simple algebras.

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Theorem (Rachid+PP)

Every reflexive and amenable operator algebra A is semisimple and it is a finite direct sum of simple Banach algebras of operators.

Proof: Let $\pi : A \to B(H)$ be a bounded representation of A on some Hilbert space H. Let M be a closed invariant subspace of $\pi(A)$ and take the following admissible short sequence

$$0 \longrightarrow M \longrightarrow H \to H/M \longrightarrow 0.$$

By [Curtis], this sequence splits, therefore A has the total reduction property. It follows from [REIHarti] that every closed two-sided ideal of A has a bounded approximate identity. Therefore the result is now an easy consequence of our Theorem.

Definition

1. For $T \in B(H)$, let $A_T^0 = \overline{\{p(T) : p \text{ polynomial}, \ p(0) = 0\}}^{||\cdot||}$ be the nonunital Banach algebra generated by $T \in B(H)$.
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1. The amenability of operator algebras generated by single operators have been investigated.
2. It is not known whether such algebras are reflexive.
3. Recall that if T is a non trivial quasi-nilpotent operator, then A_T^0 is a radical algebra $\text{Rad}(A_T^0) = A_T^0$ (hence A_T^0 is a non semisimple algebra)
4. If we further assume that A_T^0 is amenable then A_T^0 cannot be reflexive by Theorem above.
5. The following result shows that A_T^0 cannot contain non trivial compact operators!!

Application

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If A_T^0 amenable and contains a non trivial compact operator K, then T is non quasinilpotent.

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Proof: • We show that $TK \neq 0$. Indeed, if that is not the case, then since $K \in A_T^0$, K is a limit of polynomial $P_n(T)$ with $P_n(0) = 0$. So K^2 is the limit of $P_n(T)K$. Note now that $P_n(T)K = 0$ for all n, so $K^2 = 0$ thus K is nilpotent. This implies that A_K^0 is finite dimensional because it is generated by a nilpotent operator as a linear space. Since A_K^0 is an amenable Banach, A_K^0 is a semisimple algebra. Then $K = 0$ which is a contradiction (remark that commutative semisimple Banach algebras do not contain non trivial quasi nilpotent or nilpotent elements).
• So since $TK \neq 0$, there exists a trace-class operator $N \in C(H)$ such that $\mathrm{tr}(TKN) \neq 0$. Let D_N be the derivation from A_T^0 to $(A_{TK}^0)^\top$ defined by $D_N(A) := NA - AN$ for all $A \in A_T^0 \subseteq B(H) = C(H)^*$, where $(A_{TK}^0)^\top$ is the annihilator of A_{TK}^0 taken in $C(H)$ (note that $A_{TK}^0 \subseteq A_T^0$, so $D : A_T^0 \to (A_T^0)^\top \subseteq (A_{TK}^0)^\top$). Besides this, $(A_{TK}^0)^\top$ is a Banach A_T^0-bimodule in $K(H)$. Since A_T^0 is amenable D_N is inner, so there exists an $M \in (A_{TK}^0)^\top$ such that $D_N(A) = MA - AM$ for all $A \in A_T^0$. This means that $KT(N - M) = (N - M)KT$ and $\mathrm{tr}(KT(N - M)) = \mathrm{tr}((N - M)KT) \neq 0$. Hence $\sigma(KT(N - M)) \neq \{0\}$ where $\sigma(KT(N - M))$ denotes the spectrum of $KT(N - M)$. Since KT and $N - M$ commute we have $\sigma(KT(N - M)) \subseteq \sigma(KT)\sigma(N - M)$ and therefore $\sigma(KT) \neq \{0\}$. Similarly, $\sigma(KT) \subseteq \sigma(K)\sigma(T)$, whence $\sigma(T) \neq \{0\}$. Therefore T is non quasinilpotent.

Example (Volterra operator)

Let $H = L^2[0,1]$, $\kappa(s,t) := \chi_{\{s \leq t\}} = 1$ iff $s \leq t$, and define $V \in L(H)$ by

$$Vf(t) = \int_0^t f(s)\,ds = \int_0^1 \kappa(s,t)f(s)\,ds, \quad f \in L^2,\ t \in [0,1].$$

Then V is a compact, $||V|| = \frac{2}{\pi}$, and quasinilpotent operator because

$$|V^n f(t)| \leq \frac{||\kappa||_\infty^n t^n}{n!}\,||f||, \text{ thus } ||V^n|| \leq \frac{||\kappa||_\infty^n}{n!}$$

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Our Thm $\Longrightarrow$ A_V^0 is not amenable!!!!

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