On pro-C*-algebras of profinite groups

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Noncommutative Geometry and applications
Poiana Braşov, Romania, 1 – 8 September 2013

Projective system or inverse system (of groups)

1. **directed set** $(I, \leq)$, groups G_α, $\alpha \in I$ and compatible cont. group hom's $\pi_{\alpha\beta} : G_\beta \to G_\alpha$ whenever $\beta \leq \alpha$ so that $\pi_{\alpha\alpha} = I$ and

$$\pi_{\alpha\beta} \circ \pi_{\beta\gamma} = \pi_{\alpha\gamma}, \quad \text{if } \alpha \leq \beta \leq \gamma$$

2. **projective limit or inverse limit** of the inverse system $((G_\lambda), (\pi_{\alpha\beta}), I)$ is the following group

$$G = \varprojlim_{\alpha \in I} G_\alpha = \left\{ g = (g_\alpha) \in \prod G_\alpha : \pi_{\alpha\beta}(g_\beta) = g_\alpha, \quad \alpha \leq \beta \right\}$$

Then G is a subgroup of $\prod G_\alpha$

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Then G is a subgroup of $\prod G_\alpha$
3. A **profinite group** is a topological group which is obtained as the inverse limit of a collection of finite groups, each given the discrete topology

Remark

Also projective limits of sets $\varprojlim_\alpha X_\alpha$, topological *-algs, etc....

Example

G top. group, and $\mathcal{N}_G := \{N : \ N$ closed normal finite index subgroup of $G\}$.

$$N_1,\ N_2 \in \mathcal{N}_G \implies N_1 \cap N_2 \in \mathcal{N}_G.$$

Then G/N is a finite group and a standard finite group result leads to a surjective hom

$$G/N_1 \to G/N_2 \text{ if } N_2 \leq N_1 \text{ (if } N_1 \subseteq N_2 \text{ by def.)}$$

Profinite completion of G is the projective limit $\widehat{G} = \varprojlim_{N \triangleleft_f G} G/N$.

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Profinite completion of G is the projective limit $\widehat{G} = \varprojlim_{N \lhd_f G} G/N$.

Remark

1. $\widehat{G}$ is a Hausdorff, **compact**, totally disconnected
2. Natural diagonal group hom $\phi : G \to \widehat{G}$ with

$$\phi(g) = (gN)_{N \in \mathcal{N}_G}.$$

ϕ is injective iff G is a residually finite group (RF),i.e. $\bigcap_{N \in \mathcal{N}_G} N = \{e\}$

G, $\widehat{G}$, $\phi : G \to \widehat{G}$, ϕ injective iff G is RF

Replace G by a C*-alg A

1. what is the C*-analog of the above map ϕ?
2. if such a map exists, when is it injective?
3. notion of a *profinite completion* $\widehat{A}$ of a C*-alg. A?

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Remark (G a locally compact group)

1. Group C^*-algebra $C^*(G) = \overline{L^1(G,\mu)}^{||\cdot||}$ with $||f|| = \sup_{\pi \ * - \text{rep. of } L^1(G)} ||\pi(f)||$
2. In general, a hom $G_1 \to G_2$ **does not extend** to a C*-hom

$$C^*(G_1) \to C^*(G_2) \text{ and in general } G_i \not\subseteq C^*(G_i)\ i = 1, 2.$$

If the groups are discrete, then yes, but the topology of $\widehat{G}$ is the product topology...

1. $(A, ||\cdot||)$ C*-alg $||aa^*|| = ||a||^2$
2. $p : A \to \mathbb{R}_0^+$ C*-**seminorm** if p is an seminorm and $p(aa^*) = p(a)^2$
3. Then $p(a) \leq ||a||$ and $ker(p) = \{a \in A : p(a) = 0\}$ is a *-ideal
4. $A_p := A/ker(p)$ is a C*-alg.
5. C*-seminorms of $A \longleftrightarrow$ *-homs of A

$$A \xrightarrow[\pi_p]{\text{quotient map}} A_p \xrightarrow[\rho]{\text{GNS}} B(H_p)$$

6. p, q C*-seminorms $\Longrightarrow \max\{p, q\}$ C*-seminorm

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6. p, q C*-seminorms $\Longrightarrow \max\{p, q\}$ C*-seminorm
7. I a directed set, $(p_\alpha)_{\alpha \in I}$ C*-seminorms such that $\alpha \leq \beta$ implies $p_\alpha \leq p_\beta$, where $A_\alpha := A/ker(p_\alpha)$ we also get surjective maps $\pi_{\alpha\beta} : A_\beta \to A_\alpha$ if $\alpha \leq \beta$ (and $p_\alpha \leq p_\beta$ so $ker(p_\beta) \subseteq ker(p_\alpha)$).

   Get a projective system $(I, A_\alpha, (\pi_{\alpha\beta}))$

   Projective limit

$$\varprojlim A_\alpha := \left\{ a = (a_\alpha) \in \prod A_\alpha : \pi_{\alpha\beta}(a_\beta) = a_\alpha, \quad \alpha \leq \beta \right\}$$

   is a ***-top. alg**, the top τ is the weakest making the restriction to $\varprojlim A_\alpha$ of the projs $P_\alpha : \prod_\alpha A_\alpha \to A_\alpha$ conts.

- $\pi : A \to \varprojlim A_\alpha$ such that $a \mapsto (\pi_\alpha(a))$ with $\pi_\alpha : A \to A/ker(p_\alpha)$
- if $\varprojlim A_\alpha$ is a top *-alg, set

$$||a||_\infty := \sup_\alpha \; p_\alpha(a), \; p_\alpha \in \textit{all cont. } C^*\text{-seminorms of } \varprojlim A_\alpha$$

bounded part: $(\varprojlim A_\alpha)_b = \{a \in \varprojlim A_\alpha : ||\cdot||_\infty \leq \infty\}$.

Then $(\varprojlim A_\alpha)_b$ is a C*-alg. and $A = \pi(A) \subseteq (\varprojlim A_\alpha)_b$ if π is injective.

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Example ($A = C(X)$, X Hausdorff loc. cpt set)

compact $K \subseteq X \implies p_K(f) = \sup_{t\in K} |f(t)|$ a C*-seminorm. Then

$$(\varprojlim A_K)_b := C_b(X) \simeq C(\beta X)$$

where βX is the Stone-Čheck compactification of X.

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Definition

Given such projective system $(A, (p_\alpha))$, we naturally get a topology on A: a net $a_j \to a$ if $p_\alpha(a_j - a) \to 0$ for all α. Let $\overline{A}$ its completion.

$(A, (p_\alpha))$ equivalent to $(A, (q_i))$ if (p_α) and (q_i) define the same topology.

Definition (Voiculescu, J. Op. Theory 1987)

A pro-C*-alg is a C^*-alg A equipped with a directed family of C*-seminorms (p_α), such that

1. $\alpha \leq \beta$ implies $p_\alpha \leq p_\beta$
2. $\sup_\alpha(a) = ||a||$ **(FAITHFULL)**
3. $A_1 = \varprojlim_\alpha (A_\alpha)_1$ **(FULL)**

where 1 indicates the unit ball.

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1. If $(A, (p_\alpha))$ is a pro-C*-algebra then $(\varprojlim A_\alpha)_b = A$.
2. $(A, (p_\alpha))$ is a pro-C*-algebra $\iff$ $(\varprojlim A_\alpha)_b \simeq A$ isometrically
3. $(A, (p_\alpha))$ faithfull iff $\pi : A \to \varprojlim A_\alpha$ injective iff $\bigcap_\alpha ker(p_\alpha) = 0$
4. $(A, (p_\alpha))$ full iff $\pi(A) = (\varprojlim A_\alpha)_b$.

Definition

1. A projective system $(A, (p_\alpha))$ is profinite if $\dim(A_\alpha) < \infty$, for all α.
2. The projective system $(A, (p_\alpha))$ of all C*-seminorms p_α of A so that $\dim(A_\alpha) < \infty$ is the **profinite completion** of A (denote it by $\widehat{A}$).

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This profinite completion projective syst is faithfull iff A is **residually finite dimensional** (RFD) = if *-hom $A \hookrightarrow \prod_{i \in \mathbb{N}} M_{n(i)}(\mathbb{C})$ for some $n(1), n(2), ... \in \mathbb{N}$.

Example

X loc. cpt and $A = C_0(X)$ is RFD. The topology determined by the profinite completion of A is that of pointwise convergence and

$$\widehat{A} = F(X)$$

all functions on X (not necessarily cont. even bounded).
In general, this profinite struture is not FUll, even if X is cpt.

Example

if $A = \bigoplus_{i \in \mathbb{N}} M_{n(i)}(\mathbb{C})$ for $n(1), n(2), \ldots \in \mathbb{N}$. Then

1. $M(A) = \{a \in \prod_{i \in \mathbb{N}} M_{n(i)}(\mathbb{C}) : \sup ||a_{n(i)}|| < \infty\}$
2. $\widehat{A} = \widehat{M(A)} = \prod_{i \in \mathbb{N}} M_{n(i)}(\mathbb{C})$
3. $(\varprojlim A_\alpha)_b = (\varprojlim M(A)_\alpha)_b = M(A)$ (bounded parts of profinite systems)
4. profinite proj structure of A is faithfull
5. profinite completion structure of $M(A)$ is faithfull + full (Voiculescu pro-C*-alg)

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• The C*-seminorms for A are given by $p_F(a) = \sup_{s \in F} ||a_s||$ with finite set $F \subseteq \{n(1), n(2), ...\}$;

• A surjective hom $\rho : A \to B$ between C*-algs leads to $\tilde{\rho} : M(A) \to M(B)$

• If p_α C*-seminorm on A and $\pi_\alpha : A \to A/ker(p_\alpha)$, then we get $\widetilde{\pi_\alpha} : M(A) \to M(A/ker(p_\alpha))$ and so $a \mapsto ||\widetilde{\pi_\alpha}(a)||$ defines a C*-seminorm on $M(A)$.

• A RFD $\Longrightarrow M(A)$ RFD.

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Example

The profinite completion $M(\widehat{C^*(G)})$ is full if G is a **compact group**!!

• G loc. compact group, then

unitary reps of $G \longleftrightarrow$ nondegenerate *-reps of $C(G)$

Given $v : G \to B(H)$ then $\pi(f) : C^*(G) \to B(H)$ is given by

$$\langle \pi(f)\xi, \eta \rangle = \int_G f(g) \langle v_g \xi, \eta \rangle \, d\mu(g)$$

Given $\pi : C^*(G) \to U(H)$, extend it $\widetilde{\pi} : M(C^*(G)) \to B(H)$. Since $G \subset M(C^*(G))$ using a map $g \to u_g$, then $v_g := \widetilde{\pi}(u_g)$ does the job! [Pedersen textbook]

• $\mathcal{N}_G$ the directed set as before

$\kappa_N : G \to G/N$ the quotient map extends to $\kappa_N : C^*(G) \to C^*(G/N)$ and $\tilde{\kappa}_N : M(C^*(G)) \to C^*(G/N)$

$p_N(a) = ||\kappa_N(a)||$ leads to a proj. system $(p_N)_{N \in \mathcal{N}_G}$ on $C^*(G)$

$q_N(a) = ||\tilde{\kappa}_N(a)||$ leads to a proj. system $(q_N)_{N \in \mathcal{N}_G}$ on $M(C^*(G))$

Lemma

If G is a profinite group, then any unitary rep $v : G \to B(H)$ on a f.dim. Hilbert space H factors through some rep of $w : G/N \to B(H)$ for some $N \in \mathcal{N}_G$.

Lemma

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Pf.: Choose an open set W of the unitary group $U(H)$ such that W contains no subgroups of $U(H)$ other than $\{1\}$. Let

$$V = \{g \in G\colon v_g \in W\}.$$

Then V is an open subset of G. Since G is profinite, there is $N \in \mathcal{N}_G$ such that $N \subset V$.

Since W contains no nontrivial subgroups, it follows that $v_g = 1$ for all $g \in N$.

Therefore v induces a representation w of G/N on H.

Lemma

If G is profinite, then $(C^*(G), p_N)$ is a faithfull and $(M(C^*(G)), q_N)$ faithfull and full.

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Pf.: We first prove that this profinite structure on $C^*(G)$ is equivalent to the one defining the profinite completion of $C^*(G)$. Since the C*-algebras of finite groups are finite dimensional, it suffices to prove that if p is a C* seminorm on $C^*(G)$ such that $C^*(G)/\ker(p)$ is finite dimensional, then there is closed normal subgroup N of finite index in G such that $\|\cdot\|_N \geq p$.
Represent $C^*(G)/\ker(p)$ unitally and faithfully on a finite dimensional Hilbert space H. Thus, we have a homomorphism $\pi\colon C^*(G) \to B(H)$ whose range contains 1 and such that $p(a) = \|\pi(a)\|$ for all $a \in C^*(G)$. Then π comes from a unitary representation $v \mapsto v_g$ of G on H. Let N and $w\colon G/N \to B(H)$ be as in Lemma. Let $\psi\colon C^*(G/N) \to B(H)$ be the corresponding representation of $C^*(G/N)$. Then $\psi \circ \kappa_N$ and π are both nondegenerate homomorphisms from $C^*(G)$ to $B(H)$ whose extensions to homomorphisms $M(C^*(G)) \to B(H)$ send u_g to v_g for all $g \in G$. Therefore $\psi \circ \kappa_N = \pi$. For all $a \in A$, we thus have $\|\kappa_N(a)\| \geq \|\pi(a)\| = p(a)$. This proves the equivalence of pro-C*-algebra structures.

The rest follows from the example above (as G is cpt).

Theorem

1. *There exists a unique hom* $\varphi_G : C^*(G) \to \varprojlim C^*(G/N)$ *such that* $\pi_M = P_M \circ \varphi$ *where* $\pi_M : C^*(G) \to C^*(G/M)$ *and* $P_M : \varprojlim C^*(G/N) \to C^*(G/M)$ *is the natural projection.*
2. $(\varprojlim C^*(G/N))_b \simeq M(C^*(\widehat{G}))$.
3. $ker(\varphi) = \bigcap_\pi ker(\pi)$ *where* π *runs over *-reps of* $C^*(G)$ *associated with finite range rep of* G

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Pf.: (1) Universal property for inverse limits...
(2) Known that since $C^*(G)$ is a C*-alg $\varphi(C^*(G)) \subseteq (\varprojlim C^*(G/N))_b$ and by above Example, $M(C^*(\widehat{(G)})) = (\varprojlim C^*(G/N))_b$, as $\widehat{G}$ is a compact group...
(3) Let $a \in \ker(\varphi_G)$. Let $v : G \to B(H)$ be an arbitrary representation of G with finite range, with associated representation $\pi : C^*(G) \to B(H)$. Then v factors through $\overline{G}$, so Lemma implies that π factors through $M\left(C^*\left(\overline{G}\right)\right)$. Therefore $\pi(a) = 0$. This shows that $a \in I$.

Conversely, let $a \in \bigcap \ker(\pi)$. Then $\kappa_N(a) = 0$ for all $N \in \mathcal{N}_G$. Therefore

$\widetilde{\kappa_N}(\varphi_G(a)) = 0$ for all $N \in \mathcal{N}_G$. Now since $M(C^*(\widehat{G}))$ is faithfull and full, $\varphi_G(a) = 0$.

Proposition

φ_G *injective* $\Longrightarrow$ *G RF.*

Remark

1. Let $G = S^1$. Then $N \in \mathcal{N}_{S^1} \Longrightarrow N = S^1$. So $\widehat{S^1} = 1$.

   $\varphi : C^*(S^1) \to M(C^*(\widehat{S^1})) = \mathbb{C}$ NOT injective.
2. $G = SL(3, \mathbb{Z})$ is RF, but $C^*(G)$ is NOT RFD [M. Bekka 1999].
   Hence $\varphi : C^*(SL(3, \mathbb{Z}) \to M(C^*(\widehat{SL(3, \mathbb{Z})}))$ cannot be injective as the latter alg is RFD and RFD passes to subalgs.

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Problem

Example of a RF G such that $C^*(G)$ is RFD but φ is NOT injective????
Maybe the free group $\mathbb{F}_2$....

Theorem

G is a discrete, profinite and amenable group $\Longrightarrow \varphi_G$ is injective.

Pf.: Since G is amenable, $G \subseteq B(l^2(G))$ via the regular representation. Let $a \in C^*(G)$ be nonzero. We show that a is not in the ideal $\cap \ker(\pi)$ of above result.
Assume that $\|a\| = 1$. Choose $b \in C^*(G)$ such that $\|b - a\| < \frac{1}{4}$ and b is a finite linear combination of the standard unitaries u_g. So, there are a finite set $S \subset G$ and numbers $\beta_g \in \mathbb{C}$ for $g \in S$ such that $b = \sum_{g \in S} \beta_g u_g$. Then $\|b\| > \frac{3}{4}$, so there is $\xi \in l^2(G)$ with finite support such that $\|\xi\| = 1$ and $\|b\xi\| > \frac{1}{2}$. There are a finite set $T \subset G$ and numbers $\alpha_g \in \mathbb{C}$ for $g \in T$ such that $\xi = \sum_{g \in T} \alpha_g \delta_g$. Let

$$ST = \{gh\colon g \in S \text{ and } h \in T\} \quad \text{and} \quad T^{-1} = \{g^{-1}\colon g \in T\}.$$

Since G is residually finite, there is $N \in \mathcal{N}_G$ such that the restriction to ST of the quotient map $G \to G/N$ is injective.
Let $v\colon G \to B(l^2(G/N))$ be the composition of this quotient map with the regular representation of G/N. Let $\pi\colon C^*(G) \to B(l^2(G/N))$ be the corresponding homomorphism.

Set $\eta = \sum_{g \in T} \alpha_g \delta_{gN}$. Then $\|\eta\| = 1$ since the vectors δ_{gN} are orthonormal. For $g \in ST$, define

$$\lambda_g = \sum_{h \in S \cap gT^{-1}} \beta_h \alpha_{h^{-1}g}.$$

Then

$$b\xi = \sum_{g \in ST} \lambda_g \delta_g \quad \text{and} \quad \pi(b)\eta = \sum_{g \in ST} \lambda_g \delta_{gN}.$$

As g runs through ST, the elements δ_g and δ_{gN} form orthonormal systems in $l^2(G)$ and in $l^2(G/N)$. Therefore

$$\|\pi(b)\eta\|^2 = \sum_{g \in ST} |\lambda_g|^2 = \|b\xi\|^2.$$

So

$$\|\pi(a)\| > \|\pi(b)\| - \tfrac{1}{4} \geq \|\pi(b)\eta\| - \tfrac{1}{4} = \|b\xi\| - \tfrac{1}{4} > \tfrac{1}{2} - \tfrac{1}{4} = \tfrac{1}{4}.$$

Therefore $\pi(a) \neq 0$. Since π comes from a representation of G which factors through the finite group G/N, this shows that $a \notin \cap \ker(\pi)$. $\square$

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