MODULAR INVARIANTS AND THE DOUBLE OF THE HAAGERUP SUBFACTOR

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ABSTRACT. We analyse the modular invariants of the quantum double of the Haagerup subfactor.

CONTENTS

1. INTRODUCTION

In the framework of braided factors we consider a finite system of endomorphisms ${}_N\mathscr{X}_N$ of a type III factor N with a (non-degenerate) braiding on it, which leads to a Verlinde fusion rule algebra and produce the modular data S and T [9, 35]. Sources of modular data from braided factors arise from the WZW models and Ocneanu's asymptotic subfactor, which is regarded as the subfactor analogue of Drinfeld's quantum double construction. This *quantum double subfactor* is basically the same as the Longo-Rehren inclusion [28] and gives a way of yielding braided systems from not necessarily commutative systems. The S- and T- matrices for the WZW subfactor models have been constructed by A. Wassermann [40], proving that they indeed coincide with the Kac-Peterson S and T matrices. The modular data from a quantum double subfactor was first established by Ocneanu [15, Section 12.6] using topological insight, and later, with an algebraic flavour, by Izumi [22] (see also [37, 25]). Given S and T modular data, a coupling matrix Z that commutes with S and T subject to the constraints

$$Z_{\lambda,\mu} = 0, 1, 2, 3, \ldots \text{ and } Z_{00} = 1 \tag{1}$$

is called a *modular invariant*. These constraints reflect the physical background of the problem. The condition $Z_{00} = 1$ reflects the uniqueness of the vacuum.

Fixing a braided system of endomorphisms ${}_N\mathscr{X}_N$ on a type III factor N, we look for inclusions $\iota : N \hookrightarrow M$ whose dual canonical endomorphism $\theta = \bar{\iota}\iota$ decomposes as a sum of endomorphisms from ${}_N\mathscr{X}_N$. To produce a modular invariant from such an inclusion,

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we first employ the Longo-Rehren $\alpha^{\pm}$-induction method [28] of extending endomorphisms $\lambda \in {}_N\mathscr{X}_N$ to those $\alpha^{\pm}_{\lambda}$ in M, and then compute the dimensions of the intertwiner spaces where

$$Z_{\lambda\mu} = \langle \alpha^{+}_{\lambda}, \alpha^{-}_{\mu} \rangle = \dim \mathrm{Hom}(\alpha^{+}_{\lambda}, \alpha^{-}_{\mu}), \qquad \lambda, \mu \in {}_N\mathscr{X}_N. \tag{2}$$

The matrix $Z_{N\subset M} = [Z_{\lambda,\mu}]$ thus constructed from a braided subfactor $N \subset M$ is a modular invariant [9, 13] – which we call *sufferable*. A modular invariant $Z_{N\subset M}$ encodes the rich structure of the inclusion $N \subset M$. The induced systems, which may well be non-commutative and hence not braided, encode the symmetries of the physical situation we started with. Given the list of all modular invariants of a S- and T-model, it is an interesting problem to determine which ones can be produced by braided subfactors through α-induction. This is in general a difficult task and is in turn related to the problem of classification of subfactors. The present work features the several known constraints a modular invariant has to fulfill in order to be produced by an inclusion, as well as the new ones arising from the fusion rule structure of those realized by subfactors. There do exist modular invariants that are insufferable, that is, are not of the form $Z_{N\subset M}$, see [16]. In fact, in Subsection 4.3.1 we study the modular data arising from the quantum double of the Haagerup subfactor with index $(5+\sqrt{13})/2$, and find for the first time a model where the number of (normalized) sufferable modular invariants are sparse compared to the full list of (normalized) modular invariants.

In the course of previous work [8, 12, 17], it was observed that modular invariants satisfy remarkable fusion rules and an analysis was begun of this intriguing structure. This programme was completed in [16]. We showed in [16, Theorem 3.6] that the matricial product $Z_a Z_b$ between two sufferable modular invariants Z_a and Z_b can also be realized by a braided inclusion $N \subset M_{ab}$ such that $[Z_{N\subset M_{ab}}]_{00} = \dim(M'_{ab} \cap M_{ab})$. Moreover, by taking a partition of unity $\{p\}$ by minimal projections of the finite dimensional von Neumann algebra $M'_{ab} \cap M_{ab}$, we proved in [16, Theorem 3.6] that $Z_a Z_b$ decomposes into a sum of normalized sufferable modular invariants

$$Z_a Z_b = \sum Z_c. \tag{3}$$

This decomposition in particular shows that the matrix $Z_{N\subset M}$ defined as in Eq. (2) is a modular invariant even when M is not a factor. In Section 3 we illustrate how [16] works. Then in Section 4 we study the modular invariants arising from the quantum double of the Haagerup $(5+\sqrt{13})/2$ subfactor.

2. Preliminaries

In this section we recall the general framework of [9, 10, 16]. We shall consider type III von Neumann algebras with finite dimensional centres. A morphism between such algebras A and B shall be a faithful unital *-homomorphism $\rho : A \to B$, called a B-A morphism, and we write $\rho \in \mathrm{Mor}(A, B)$. We will only consider those of finite statistical dimension or inclusions of finite index. Then $d_\rho = [B : \rho(A)]^{1/2}$ is called the *statistical* (or *quantum*) *dimension* of ρ; here $[B : \rho(A)]$ is the *Jones index* of the inclusion $\rho(A) \subset B$. If ρ and σ are B-A morphisms with finite statistical dimensions, then the vector space of intertwiners $\mathrm{Hom}(\rho,\sigma)=\{t \in B : t\rho(a) = \sigma(a)t,\ a \in A\}$ is finite-dimensional, and we denote its dimension by $\langle \rho, \sigma \rangle$. A morphism conjugate to ρ will be denoted $\bar\rho : B \hookrightarrow A$.

Consider a type III inclusion $N \subset M$ with N a factor, and denote by ι the inclusion map. Then $\gamma = \iota\bar\iota$ and $\theta = \bar\iota\iota$ are the *canonical* and *dual canonical endomorphism* of $N \subset M$ respectively.

Let ${}_N\mathscr{X}_N$ denote a *finite system of irreducible endomorphisms* of a factor N, in the sense that different elements of ${}_N\mathscr{X}_N$ are inequivalent, for any $\lambda \in {}_N\mathscr{X}_N$ there is a representative $\bar{\lambda} \in {}_N\mathscr{X}_N$ of the conjugate sector $[\bar{\lambda}]$, and ${}_N\mathscr{X}_N$ is closed under composition and subsequent irreducible decomposition. We denote by $\Sigma({}_N\mathscr{X}_N)$ the set of representative endomorphisms of integral sums of sectors from ${}_N\mathscr{X}_N$ [9]. The quantity $\omega = \sum d_\lambda^2$ is the *global index* of the system ${}_N\mathscr{X}_N$. If ${}_N\mathscr{X}_N = \{\rho_\xi\}$ is a finite system and $j : N \to N^{\mathrm{opp}}$ is the natural anti-linear isomorphism, $\rho_\xi^{\mathrm{opp}} = j \cdot \rho_\xi \cdot j$, we set $B = N \otimes N^{\mathrm{opp}}$. Longo and Rehren [28] have shown that there exists a subfactor $A \subset B$ such that $\gamma = \bigoplus_\xi \ \rho_\xi \otimes \rho_\xi^{\mathrm{opp}}$ is its canonical endomorphism (the Longo-Rehren inclusion). This notion was further translated into the type II_1 setting by Masuda [29] who proved that the Longo-Rehren inclusion and Ocneanu's asymptotic inclusion are essentially the same object. Ocneanu [15, Chapter 12] has constructed a non-degenerate braiding on the A-A system from the above quantum double inclusion $A \subset B$, and later this was done in a more algebraic way by Izumi [22, 21]. For a subsystem Π of ${}_N\mathscr{X}_N$, Izumi's Galois correspondence [22, Proposition 2.4] asserts then that there exists an intermediate subfactor $A \subset B_\Pi \subset B$ such that $\gamma_\Pi = \bigoplus_{\xi\in\Pi} \rho_\xi \otimes \rho_\xi^{\mathrm{opp}}$ is a canonical endomorphism of the inclusion $B_\Pi \subset B$.

Whenever we have a non-degenerate braiding $\{\epsilon_{\lambda\mu}\}$ on a system ${}_N\mathscr{X}_N$, from the Hopf link and twist we can define S- and T-matrices of type $\#{}_N\mathscr{X}_N \times \#{}_N\mathscr{X}_N$ [35, 38], which satisfy the Verlinde formula [39]

$$\sum_{\rho\in{}_N\mathscr{X}_N} \frac{S_{\lambda,\rho} S_{\mu,\rho} S^*_{\nu,\rho}}{S_{\rho,0}} = \langle \lambda\mu, \nu\rangle. \tag{4}$$

The fusion matrices $N_\lambda = [N^\nu_{\lambda,\mu}]_{\mu,\nu}$, where $N^\nu_{\lambda,\mu} = \langle \lambda\mu, \nu\rangle$, recover the original fusion rules

$$N_\lambda \cdot N_\mu = \sum_{\nu\in{}_N\mathscr{X}_N} N^\nu_{\lambda,\mu} N_\nu. \tag{5}$$

A *nimrep* is a collection of nonnegative-integer matrices G_λ which give a representation of the fusion ring:

$$G_\lambda \cdot G_\mu = \sum_\eta N^\eta_{\lambda,\mu} G_\eta, \qquad G_0 = \mathrm{id}, \quad \text{and} \quad G_{\bar\lambda} = G_\lambda^{\mathrm{t}}, \quad \text{for all } \lambda,\mu, \tag{6}$$

where $N^\eta_{\lambda,\mu}$ are the *Verlinde fusion numbers*, see e.g. [16, Definition 2.1]. A modular invariant Z is said to be *nimble* if there is a nimrep whose spectrum is dictated by $\sigma_{G_\lambda} = \{S_{\lambda\mu}/S_{0\mu} : \text{ with multiplicity } Z_{\mu\mu}\}$. We note that a sufferable modular invariant is automatically nimble [10, p. 768] or [7, p. 10].

We can fix in the opposite system ${}_N\mathscr{X}_N^{\mathrm{opp}} = \{\lambda^{\mathrm{opp}}\}$ the braiding defined as $\epsilon_{\lambda^{\mathrm{opp}}\mu^{\mathrm{opp}}} := \epsilon^*_{\mu\lambda}$, whose modular data is S^* and T^*. In particular the fusion rules of both ${}_N\mathscr{X}_N^{\mathrm{opp}}$ and ${}_N\mathscr{X}_N$ coincide.

Fixing a braided subfactor $\iota : N \hookrightarrow M$, let ${}_N\mathscr{X}_M$ be the system of endomorphisms from irreducible representatives of $[\iota\lambda]$ with λ running in ${}_N\mathscr{X}_N$. Next, let ${}_M\mathscr{X}_M \subset \mathrm{End}(M)$ denote a system of endomorphisms of irreducible representatives from $[\iota\lambda\bar\iota]$, $\lambda \in {}_N\mathscr{X}_N$ (${}_M\mathscr{X}_M$ is called the *full M-M system*). Then we define the $\pm$-chiral systems ${}_M\mathscr{X}_M^\pm$ to be the subsystems of ${}_M\mathscr{X}_M$ consisting of irreducible representatives from $[\alpha_\lambda^\pm]$ with $\lambda \in {}_N\mathscr{X}_N$. The neutral system ${}_M\mathscr{X}_M^0$ is defined as the intersection ${}_M\mathscr{X}_M^0 = {}_M\mathscr{X}_M^+ \cap {}_M\mathscr{X}_M^-$, so that ${}_M\mathscr{X}_M^0 \subset {}_M\mathscr{X}_M^\pm \subset {}_M\mathscr{X}_M$ (see e.g. [8]). Their global indices (the sum of the squares of the quantum dimensions) are denoted by $\omega_0, \omega_\pm$ and ω, and are completely encoded in the modular invariant $Z = Z_{N\subset M}$ as defined in Eq. (2), namely [10, 6]:

$$\omega_\pm = \frac{\omega}{\sum_\lambda d_\lambda Z_{\lambda,0}} = \frac{\omega}{\sum_\lambda d_\lambda Z_{0,\lambda}}\,, \qquad \omega_0 = \omega_\pm^2/\omega. \tag{7}$$

The neutral system ${}_M\mathscr{X}_M^0$ inherits a non-degenerate braiding (therefore their fusion rules are commutative) by [6] whose modular matrices are denoted by S^{ext} and T^{ext}. The matrices $b^\pm$ of the branching coefficients $b^\pm_{\tau,\lambda} = \langle \tau, \alpha^\pm_\lambda\rangle$ intertwine the original and extended modular data [6, Theorem 6.5]: $S^{\mathrm{ext}} b^\pm = b^\pm S$ and $T^{\mathrm{ext}} b^\pm = b^\pm T$. Moreover we can compute the cardinality of the various systems as follows [9, 6]:

$$\#_N\mathscr{X}_M = \mathrm{Tr}(Z), \qquad \#_M\mathscr{X}_M = \mathrm{Tr}(ZZ^{\mathrm{t}}), \qquad \#_M\mathscr{X}_M^\pm = \mathrm{Tr}({}^{\mathrm{t}}b^\pm b^\pm). \tag{8}$$

The modular invariant Z is a permutation matrix if and only if ${}_M\mathscr{X}_M^0 = {}_M\mathscr{X}_M$ [9, 10]. The modular invariant Z produced by a subfactor $N\subset M$ is said to be of type I if $Z_{0,\lambda} = \langle\theta,\lambda\rangle$ for all $\lambda\in {}_N\mathscr{X}_N$ [10, 6] (in particular Z is symmetric). In this situation we say that the chiral locality holds for the subfactor $N\subset M$. In other words, the chiral locality holds if the dual canonical endomorphism is visible in the vacuum row (and hence column), so $[\theta] = \bigoplus Z_{0,\lambda}[\lambda]$. In the presence of chiral locality, we can usually recover the canonical endomorphism from the full M-M system [5] by computing the dimensions $\langle \alpha^+_\lambda\alpha^-_{\bar\mu},\gamma\rangle = \langle\alpha^+_\lambda,\alpha^+_\mu\rangle$.

If a braided subfactor $N\subset M$ produces a modular invariant Z, then there are intermediate subfactors $N\subset M_\pm\subset M$, such that $N\subset M_\pm$ produce symmetric modular invariants $Z_\pm$ (type I parents) [6]. Moreover we have $Z_{\lambda,0} = Z^+_{\lambda,0}$ and $Z_{0,\lambda} = Z^-_{0,\lambda}$, and the dual canonical endomorphisms $\theta_\pm$ of $N\subset M_\pm$ are visible from the vacuum row and column of Z, i.e. $[\theta_+] = \bigoplus_\lambda Z_{\lambda,0}[\lambda]$ and $[\theta_-] = \bigoplus_\lambda Z_{0,\lambda}[\lambda]$.

3. Braided Q-systems

Let N be a factor and $\iota: N\to M$ be an inclusion in a von Neumann algebra M, with $\bar\iota: M\to N$ a conjugate endomorphism of ι. Then since $\bar\iota\iota$, $\iota\bar\iota$ contain the identity id_M, id_N respectively, there are intertwining isometries, v and w_1, in $\mathrm{Hom}(\mathrm{id}_N,\bar\iota\iota)$, $\mathrm{Hom}(\mathrm{id}_M,\iota\bar\iota)$ respectively [26, 21]. Then $w = \bar\iota(w_1)$ is an isometry in $\mathrm{Hom}(\theta,\theta^2)$, where $\theta = \bar\iota\iota$ is the dual canonical endomorphism which satisfy [27]

$$w^*\theta(w) = ww^*, \qquad w^2 = \theta(w)w, \qquad v^*w = w^*\theta(v) = 1/d \tag{9}$$

with $d = d(\iota)$, thus $d^2 = [M:N]$. The system $\Theta = (\theta, v, w)$ is called a Q-system by Longo [27], and conditions in Eq. (9) characterize precisely which endomorphisms can arise as dual canonical endomorphisms for $N\subset M$. It is convenient to represent intertwiners graphically [9, 16], reading the pictures downwards. With this convention, we write the isometries w and v as in Fig. 1 and Fig. 2, respectively, where a wire labelled by θ gives rise to two wires labelled by $\bar\iota$ and ι. Then the relations of Eq. (9) can be displayed graphically as in Figs. 3, 4 and 5. A dual endomorphism θ does not determine the subfactor uniquely up to conjugacy. This is an H^2-cohomological obstruction that has been studied in [20] and in Proposition 3.2 and Remark 3.3 of [16].

Let $\mathscr{C}$ be a modular tensor category, see e.g. [38, 30, 18], i.e. $\mathscr{C}$ is a semisimple ribbon category with ground field $\mathbb{C}$ that has only a finite number of isomorphism classes of simple objects, a non-degenerate S-matrix, and spherical in the sense that the left and right traces of endomorphisms coincide, see e.g. [30, Definition 2.6]. Then the above relations Eq. (9) mean that a Q-system is a Frobenius algebra $A = (\theta, m, e, \Delta, \epsilon)$, see [30], where θ is an object of $\mathscr{C}$ (an integral sum of simple objects in $\mathscr{C}$), $e\in\mathrm{Hom}(\mathbf{1},\theta)$,

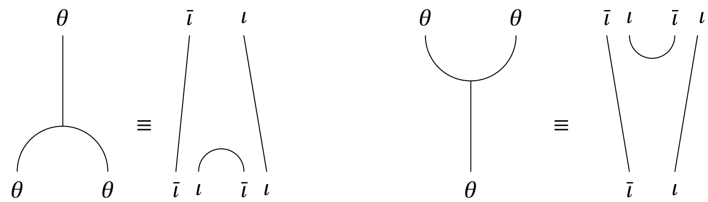

FIGURE 1. Diagrammatic representation of $\sqrt{d(\iota)}\,w$ and $\sqrt{d(\iota)}\,w^*$, respectively.

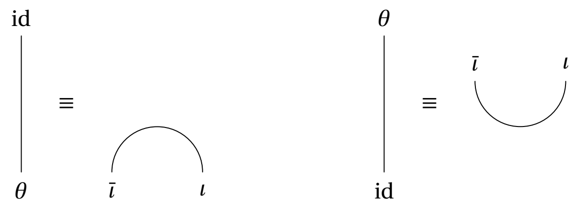

FIGURE 2. Diagrammatic representation of $\sqrt{d(\iota)}\,v$ and $\sqrt{d(\iota)}\,v^*$, respectively.

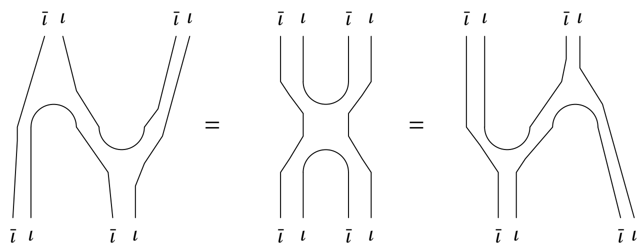

FIGURE 3. Q-system relation $d(\iota)\theta(w^*)w = d(\iota)ww^* = d(\iota)w^*\theta(w)$.

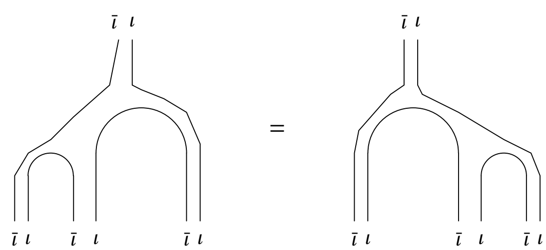

FIGURE 4. Q-system relation $d(\iota)w^2 = d(\iota)\theta(w)w$.

$m \in \mathrm{Hom}(\theta\otimes\theta,\theta)$, $\epsilon \in \mathrm{Hom}(\theta,\mathbf{1})$, $\Delta \in \mathrm{Hom}(\theta,\theta\otimes\theta)$, such that (θ,m,e) is an algebra and (θ,Δ,ϵ) is a co-algebra, with the algebraic and co-algebraic structure related by

$$(m\otimes \mathrm{id}_A)\circ(\mathrm{id}_A\otimes\Delta) = \Delta\circ m = (\mathrm{id}_A\otimes m)\circ(\Delta\otimes\mathrm{id}_A). \tag{10}$$

The intertwiners w and w^* of Fig. 1 are translated into Δ and m in Fig. 6, respectively, and the isometries v and v^* as in Fig. 2 are replaced by e and ϵ as in Fig. 6, respectively. The Q-system relation $d(\iota)\theta(w^*)w = d(\iota)ww^* = d(\iota)w^*\theta(w)$ of Fig. 3 becomes Eq. (10), see Fig. 7. The Q-system relations of Figs. 4 and 5 are translated to the relations in Frobenius algebras as in Figs. 8 and 9, respectively.

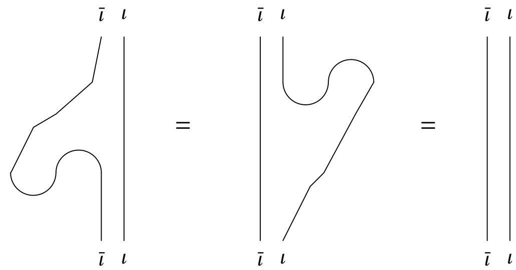

FIGURE 5. Q-system relation $d(\iota)v^* w = d(\iota)w^*\theta(v) = \mathrm{id}$.

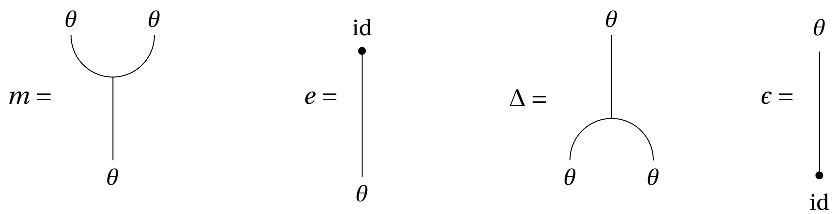

FIGURE 6. Diagrammatic representation of m, e, Δ and ϵ, respectively.

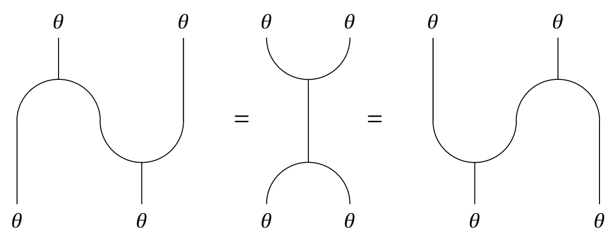

FIGURE 7. Frobenius algebra diagrammatical relations for Eq. (10).

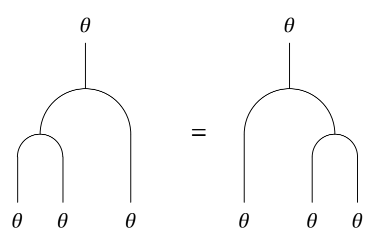

FIGURE 8. Co-associativity $(\Delta \otimes \mathrm{id}_A) \circ \Delta = (\mathrm{id}_A \otimes \Delta) \circ \Delta$ in Frobenius algebras.

A Frobenius algebra A is said to be *special* if the equations in Fig. 10 hold, and *symmetric* if the equality in the right-hand side of Fig. 12 holds, see [18, Definition 3.4]. As pointed out in [16, p. 320], a Q-system is automatically special with $\beta_1 = \beta_\Theta = d(\iota)$, and symmetric since $\mathrm{id} = d^2 v^* w^* \theta(w)\theta(v) = d^2\theta(v^*)\theta(w^*)wv$.

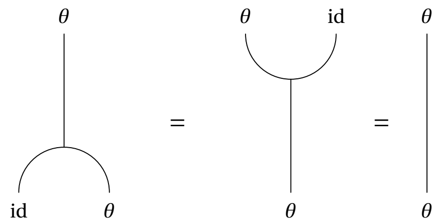

FIGURE 9. Frobenius algebra relation $(\epsilon \otimes \mathrm{id}_A) \circ \Delta = m \circ (\mathrm{id}_A \otimes e) = \mathrm{id}_A$.

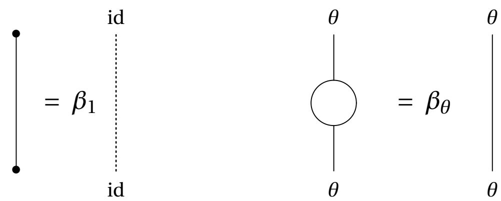

FIGURE 10. Special Frobenius algebra.

The notions of $H(\Theta)$ and $\mathcal{Z}(\Theta)$ in the subfactor setting [16, p. 327], see Fig. 11, are naturally translated into the tensor category language by $H(A) = \mathrm{Hom}(\mathbf{1}, \theta)$ and

$$\mathcal{Z}(A) := \{a \in H(A) : m \circ (a \otimes \mathrm{id}_A) = m \circ (\mathrm{id}_A \otimes a)\} \tag{11}$$

as in the left-hand side of Fig. 12.

Summarizing, a Q-system $\Theta = (\theta, v, w)$ is automatically a symmetric special Frobenius algebra as pointed out in [30] or [16, p. 320], with $m = w^*$, $e = v$, $\Delta = w$, and $\epsilon = v^*$. Moreover, a braided system ${}_N\mathcal{X}_N$ is a (unitary) modular tensor category, where the simple objects are the irreducible objects in ${}_N\mathcal{X}_N$, intertwiners in $\mathrm{Hom}(\rho, \sigma)$ are the morphisms between the objects $\rho, \sigma \in \Sigma({}_N\mathcal{X}_N)$, and the tensor product corresponds to composition of morphisms in $\Sigma({}_N\mathcal{X}_N)$ (see [30, 41]).

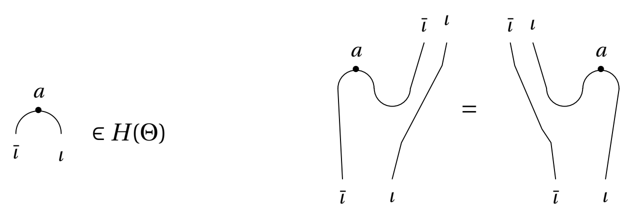

FIGURE 11. Centre $\mathcal{Z}(\Theta)$ of $H(\Theta)$ in Q-systems.

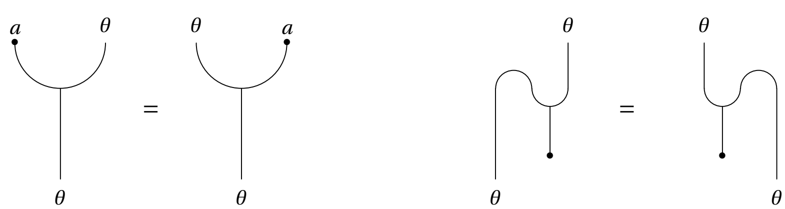

FIGURE 12. Centre $\mathcal{Z}(A)$ of $H(A)$ and symmetric Frobenius algebras.

3.1. **Decompositions for Q-systems and Frobenius algebras.** Given a braided Q-system $\Theta = (\theta, v, w)$ with associated inclusion $\iota : N \subset M$, the left Reciprocity Frobenius map $\mathscr{L}_{\bar\iota}$ provides an isomorphism of algebras $\mathscr{L}_{\bar\iota} : \mathcal{Z}(\Theta) \to M' \cap M$, see [16, Lemma 3.4]. Let $p \in \mathcal{Z}(\Theta)$ be a projection and $P = \mathscr{L}_{\bar\iota}(p)$, see the right-hand side of Fig. 13. As in [16], for every minimal projection p, we get a cut-down Q-system

$$\Theta_p = (\theta_p, v_p, w_p) := \left(\bar\iota_p \iota_p, s^* v, s^* \theta(s^*) w\right) \tag{12}$$

with $\iota_p(n) = p\iota(n)$, $n \in N$, and fixing an isometry $s \in \mathrm{Hom}(\theta_p, \theta)$ implementing P, i.e. $s^* s = 1$ and $ss^* = P$, see Fig. 14. For example, that $w_p^* w_p = 1$ is shown in Fig. 15, where the last equality holds since $s^* P s = 1$. Since the modular invariant of the direct sum of Q-systems is the sum of the modular invariants, we proved in [16, Theorem 3.6], by taking a partition of unity by minimal projections, that any (braided) Q-system is a sum of Q-systems Θ_a whose modular invariants Z_a have normalized vacuum $[Z_a]_{00} = 1$.

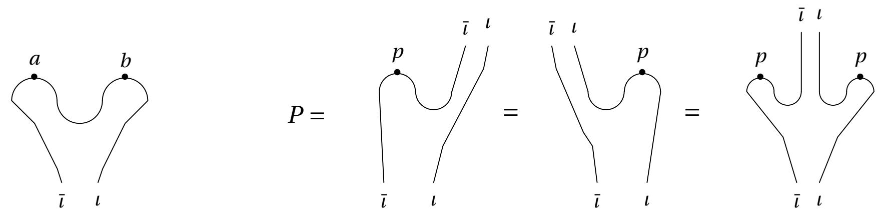

FIGURE 13. Projections in $\mathrm{Hom}(\theta, \theta)$.

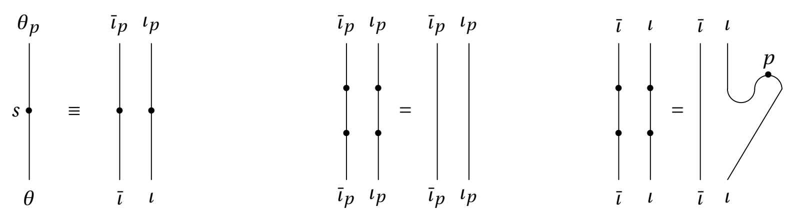

FIGURE 14. Isometry $s \in \mathrm{Hom}(\theta_p, \theta)$.

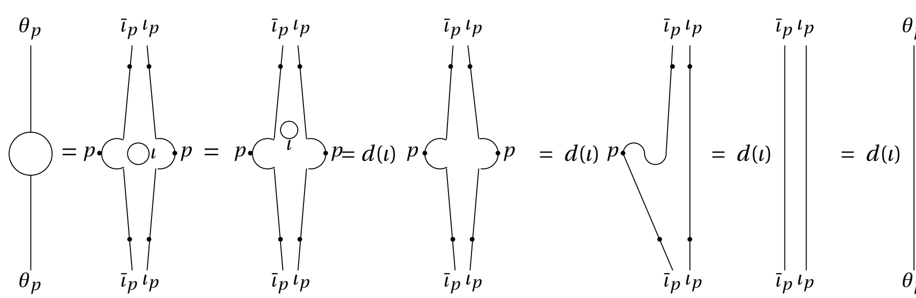

FIGURE 15. The proof of the equality $w_p^* w_p = 1$.

Let A be a symmetric special Frobenius algebra and take a projection $p \in \mathcal{Z}(A)$. Since p is central, the equalities in Fig. 16 hold as for the subfactor setting in Fig. 13 (cf. the definition of $\mathcal{Z}(A)$ in Eq. (11) or the left-hand side of Fig. 12 for Q-systems). The morphism P in Fig. 16 is indeed a projection by simple manipulations using the co-associativity, see Fig. 7.

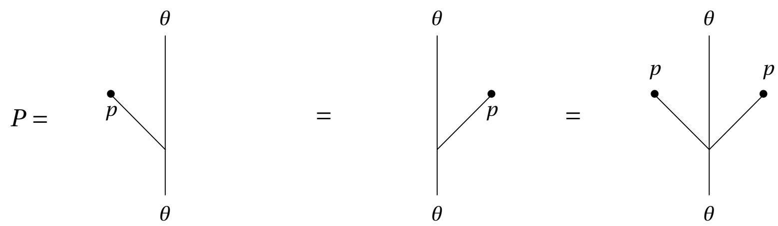

FIGURE 16. Projection $P \in \mathrm{Hom}(\theta,\theta)$.

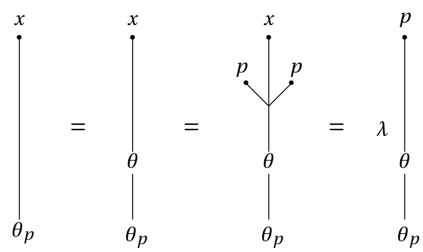

FIGURE 17. Proving that $\mathcal{Z}(A_p) \simeq \mathbb{C}$.

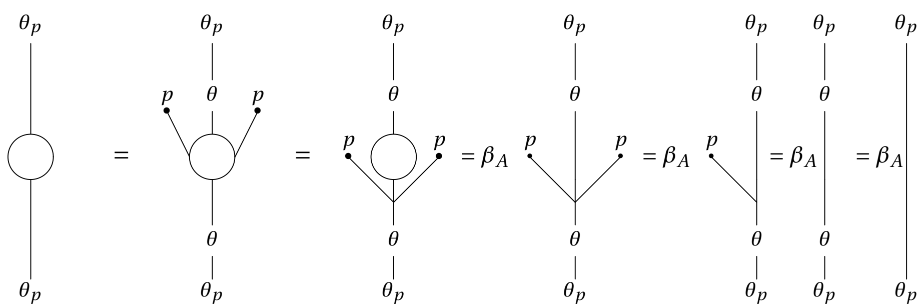

FIGURE 18. Specialness of A_p.

Since $\theta_p := P(\theta)$ is a sub-object of θ and the underlying category $\mathscr{C}$ is semi-simple, there are morphisms t and r, $t \in \mathrm{Hom}(\theta_p,\theta)$ and $r \in \mathrm{Hom}(\theta,\theta_p)$, satisfying $t \circ r = P$, $r \circ t = \mathrm{id}_{\theta_p}$ as in, e.g., [30, p. 5]. Remark that the morphisms t and r stand for s and s^*, respectively, seen above in the subfactor language. As for Θ_p in Eq. (12), we can define the cut-down $A_p = (\theta_p, m_p, e_p, \Delta_p, \epsilon_p)$ by

$$m_p = r \circ m_A \circ (t \otimes t), \quad e_p = r \circ e_A, \quad \Delta_p = (r \otimes r) \circ \Delta_A \circ t, \quad \epsilon_p = \epsilon_A \circ t. \tag{13}$$

Then A_p is an algebra. The translation of Fig. 15 in the subfactor setting to the tensor category setting is as in Fig. 18, and moreover A_p is a special algebra as proven in Fig. 18: the second equality holds because $t \circ r = P$ and that p is central, the third is because of the associativity of the product. Then we use the fact that A is special, the penultimate equality holds since $P \circ t = t$, and the final equality follows because $r \circ t = \mathrm{id}_{A_p}$. Following again [16] and that $\mathcal{Z}(A)$ is a semi-simple algebra, take a partition of unity by minimal projections $\{p\}$ of $\mathcal{Z}(A)$. Then $A \simeq \bigoplus_p A_p$ and since p is minimal, $pxp = \lambda p$ for any $x \in \mathcal{Z}(A)$ for some $\lambda \in \mathbb{C}$, thus $\mathcal{Z}(A_p) \simeq \mathbb{C}$ as in Fig. 17 where $r \circ P = r$ is used in the second equality, and the last one holds by minimality of p. Therefore the associated modular invariant Z_{A_p} has $[Z_{A_p}]_{00} = 1$.

Consequently, every symmetric special Frobenius algebra $A = (\theta, m, e, \Delta, \epsilon)$ in a modular tensor category $\mathscr{C}$ decomposes into a direct sum of symmetric special Frobenius algebras A_p with $[Z_{A_p}]_{00} = 1$, see [16, Theorem 3.6].

4. Examples of Modular Invariants and their classification

Here we discuss, in some detail, a few examples of the classification of modular invariants for the quantum double modular data. When taking the quantum double of a subfactor, it is not necessary that the original system of the underlying subfactor is braided, not even commutative. The E_6 system is commutative but not braided – we emphasize this in Subsection 4.1. The analysis of the corresponding quantum double has already been carried out in [11, 17]. In Subsection 4.2 we study the quantum double of the Ising system. This system ${}_N\mathscr{X}_N$ is non-degenerately braided, but nevertheless, the modular data of the double is of independent interest on ${}_N\mathscr{X}_N \otimes {}_N\mathscr{X}_N^{\mathrm{opp}}$. We compare this in the modular data and classification of modular invariants on ${}_N\mathscr{X}_N \otimes {}_N\mathscr{X}_N$, as well as with the quantum double of a quasi Ising system. The quasi Ising system has the same fusion rules as the Ising system, but it is not braided although it is commutative. Finally in Subsection 4.3, we analyse the modular invariants for the modular data of the quantum double of the Haagerup subfactor.

4.1. **On degeneracy of E_6.** The N-N system of a subfactor $N \subset M$ with E_6 the Dynkin diagram as principal graphs cannot have a non-degenerate braiding on it.

One can use topological quantum field theory TQFT. For a nondegenerate braiding on the even system ${}_N\mathscr{X}_N$ of E_6 we could conclude that the Turaev-Viro invariant $\mathrm{TV}_{\mathscr{P}}$ associated to a triangulated manifold $\mathscr{P}$ splits $\mathrm{TV}_{\mathscr{P}} = \mathrm{RT}_{\mathscr{P}}\overline{\mathrm{RT}}_{\mathscr{P}}$, where RT denotes the Reshetikhin-Turaev invariant associated to a closed 3-manifold constructed by surgery (see e.g. [15]). In particular TV does not detect the change of orientation since it is a real number. But this contradicts a computation of Niţică and Török for the lens spaces $L(3,1)$ which is not real (see [32, p. 56]).

Alternatively, if there is a nondegenerate braiding on the even system ${}_N\mathscr{X}_N$ (with three elements) of E_6, then by [11] the S and T matrices arising from the quantum double have $3 \times 3 = 9$ irreducible endomorphisms, but this contradicts Izumi's computation [23] of the quantum double system with 10 irreducible endomorphisms. See also [17] for further study of this model.

Actually there is no braiding at all. If we have a braiding $\mathscr{E}$, then it gives rise to half braidings. Then [23, Example 6.1] gives all half braidings for E_6, and [22, Definition 4.2(ii)] implies that, if we have a braiding, then the adjoint of course is also a braiding. So that, when we take the braiding $\mathscr{E}$, then it means that one of the half braidings on the list is part of a braiding, but also that its adjoint is also a half braiding - hence must be on the list [23, Example 6.1]. In proof of (3) in [23, p. 642], it says that $d\phi_\rho(\mathscr{E}_\rho(\rho)) = \omega$. When we take complex conjugate of this equation, then the ω corresponding to $\mathscr{E}^*$ is the complex conjugate of that of $\mathscr{E}$ (here $\mathscr{E} = \mathscr{E}_\rho(\rho)$ in the notation of Izumi [23]). But, looking at table on [23, Page 648] in Example 6.1, whichever half braiding we take, it is never the case that the conjugate of ω is on the list. Therefore the even part of E_6 cannot have a braiding.

4.2. **Genuine and quasi Ising systems.** Here we consider the genuine (braided) Ising system ${}_N\mathscr{X}_N$ corresponding to the Kac-Peterson matrices of $SU(2)_2$. We determine the dual canonical endomorphisms θ that arise from irreducible braided subfactors $N \subset M$ in the Ising model. The modular data is

$$S_0 = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}, \qquad T_0 = \mathrm{diag}(1, \exp(3\pi \mathrm{i}/8), -1). \tag{14}$$

We take the type III_1 factor $N = \pi^0(L_I SU(2)_2)''$ where π^0 denotes the level 1 vacuum representation of $LSU(2)$ [40], and let ${}_N\mathscr{X}_N = \{\lambda_0, \lambda_1, \lambda_2\}$ be the underlying non-degenerately N-N system [5, 40]. The (commutative) Ising fusion rules are, cf. [2],

$$[\lambda_2]^2 = [\lambda_0], \quad [\lambda_1]^2 = [\lambda_0] \oplus [\lambda_2], \quad [\lambda_1][\lambda_2] = [\lambda_1]. \tag{15}$$

Take $\theta = \lambda_0 \oplus n_1\lambda_1 \oplus n_2\lambda_2 \in \Sigma({}_N\mathscr{X}_N)$ and assume that θ is a dual endomorphism (denote by $\iota : N \subset M$ the underlying inclusion map). We now use the fusion rules to meet the constraints on integers n_1 and n_2 in order to get the system ${}_N\mathscr{X}_M$. By the fusion rules $\langle\theta\lambda_0, \lambda_0\rangle = \langle\theta\lambda_2, \lambda_2\rangle = 1$ and $\langle\theta\lambda_1, \lambda_1\rangle = 1 + n_2$; therefore $\iota\lambda_0$ and $\iota\lambda_2$ are irreducibles, which implies that $n_2 = 0$ or $n_2 = 1$, as $\langle\theta\lambda_0, \lambda_2\rangle = n_2$.

If $n_2 = 1$, then $\langle\theta\lambda_1, \lambda_1\rangle = 2$, but since $\langle\theta\lambda_0, \lambda_1\rangle = n_1$, we have $n_1 \leq 1$. Hence we have three candidates: only $\lambda_0, \lambda_0 \oplus \lambda_2$ and $\lambda_0 \oplus \lambda_1 \oplus \lambda_2$. Suppose $\theta = \lambda_0 \oplus \lambda_1 \oplus \lambda_2$ is a dual endomorphism; then $[M : N] = d_\theta = 2 + \sqrt{2} = 4\cos^2(\pi/8)$, which implies by [24] that the principal graph of $N \subset M$ is one of the Dynkin diagrams A_7 or D_5. However, by [31], D_5 is not realized as principal graph. Therefore it must be A_7; but this is a contradiction since the dual endomorphism of the A_7-subfactor has two components. Finally we can use either [36, Lemma 4.4] or [16, Lemma 3.8] to prove that $\lambda_0 \oplus \lambda_2$ is a dual canonical endomorphism, because $T^2_{\lambda_2,\lambda_2} = 1$.

The endomorphisms λ_0 and $\lambda_0 \oplus \lambda_2$ are the only dual canonical endomorphisms of (irreducible) braided subfactors $N \subset M$ in the Ising model.

Also note that $\lambda_0 \oplus \lambda_2$ has an unique Q-system structure by [16]. We also remark that both λ_0 and $\lambda_0 \oplus \lambda_2$ in the Ising model produce the (normalized) modular invariant $Z = 1$, which in turn is the only normalized modular invariant for the Ising S- and T-model. Since the Φ-product of Q-systems realizes the matricial product of modular invariants, we can also conclude that $([\lambda_0] \oplus [\lambda_2])^2 = 2[\lambda_0] \oplus 2[\lambda_2]$ is also a dual canonical sector producing $Z^2 = 1$, whose underlying inclusion $N \subset M$ is indeed a subfactor (not irreducible though as $\dim(N' \cap M) = 2$). Considering also the direct sum of Q-systems [16, p. 321], we thus conclude that $(p + q)(\lambda_0) \oplus p(\lambda_2)$ is a canonical endomorphism for $p, q \in \mathbb{N}$.

4.2.1. *Quantum doubles of the Ising fusion rules.* We can produce three sets of modular data from the fusion rules, Eq. (15). One is by considering the non-braided Ising fusion rules as in [14] – we call this system the *quasi Ising system*. The quantum double of the quasi Ising system has been obtained in [23, p. 616], which after relabelling is as follows:

(1) $S_0 \otimes S_0$, $T = \mathrm{Ad}(\mathrm{V} \otimes \mathrm{V})(\mathrm{T}_0 \otimes \mathrm{T}_0)$, where S_0, T_0 is the modular data of the braided Ising case written above in Eq. (14), and

$$V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The other two arise from the braided Ising fusion rules $SU(2)_2$:

(2) $S_0 \otimes S_0$, $T_0 \otimes T_0^*$,

(3) $S_0 \otimes S_0$, $T_0 \otimes T_0$.

Here (2) arises from the non-degenerately braided system ${}_N\mathscr{X}_N \otimes {}_N\mathscr{X}_N^{\mathrm{opp}}$ (which is the system of the Longo-Rehren inclusion [28] of the Ising systems on $N \otimes N^{\mathrm{opp}}$) whereas (3) arises from the non-degenerately braided system ${}_N\mathscr{X}_N \otimes {}_N\mathscr{X}_N$ on $N \otimes N$. If no confusion arises, in the sequel we denote by ${}_A\mathscr{X}_A = \{\lambda_{ij} : i, j = 0,1,2\}$ any of the three braided systems. The S matrix is the same in all three cases, with the same fusion rules:

$$[\lambda_{ia}][\lambda_{jb}] = \sum_{k,c} N_{ij}^k N_{ab}^c [\lambda_{kc}], \tag{16}$$

where N_{ij}^k and N_{ab}^c are the Ising fusion rules, see Eq. (15). The modular invariants in the models (1) and (2) coincide (and are $\mathrm{Ad}(V \otimes V)$ invariant):

$$\begin{aligned} Z_1 &= |\chi_{00}|^2 + |\chi_{01}|^2 + |\chi_{02}|^2 + |\chi_{10}|^2 + |\chi_{11}|^2 + |\chi_{12}|^2 + |\chi_{20}|^2 + |\chi_{21}|^2 + |\chi_{22}|^2, \\ Z_2 &= |\chi_{00} + \chi_{22}|^2 + |\chi_{02} + \chi_{20}|^2 + 2|\chi_{11}|^2, \\ Z_3 &= |\chi_{00} + \chi_{11} + \chi_{22}|^2, \end{aligned}$$

with fusion $Z_2^2 = 2Z_2$, $Z_3^2 = 3Z_3$, $Z_2 Z_3 = 2Z_2 = Z_3 Z_2$. The final model also has three modular invariants, namely the previous Z_1 and Z_2, and a permutation invariant

$$Z = |\chi_{00}|^2 + \chi_{01}\chi_{10}^* + \chi_{10}\chi_{01}^* + \chi_{02}\chi_{20}^* + \chi_{20}\chi_{02}^* + \chi_{12}\chi_{21}^* + \chi_{21}\chi_{12}^* + |\chi_{11}|^2 + |\chi_{22}|^2$$

with fusion $Z_2^2 = 2Z_2$, $Z_2 Z = Z Z_2 = Z_2$, $Z^2 = Z_1$.

Since $T^2_{(22),(22)} = 1$, by [16, Lemma 3.8] $\lambda_{00} \oplus \lambda_{22}$ is a dual endomorphism in (1) as $\lambda_{22} = \lambda_2 \otimes \lambda_2^{\mathrm{opp}}$ or $\lambda_2 \otimes \lambda_2$, respectively. Its braided subfactor $\iota : A \subset A \rtimes \mathbb{Z}_2$ produces a trace 6 modular invariant (since ${}_A\mathscr{X}_B = \{\iota\lambda_{00}, \iota\lambda_{01}, \iota\lambda_{02}, a, b\}$ and $\iota\lambda_{00} = \iota\lambda_{22}, \iota\lambda_{01} = \iota\lambda_{21}, \iota\lambda_{02} = \iota\lambda_{20}, \iota\lambda_{10} = \iota\lambda_{12}$ and $\iota\lambda_{11} = a \oplus b$). Thus Z_2 is sufferable.

As in [23, p. 616], the dual canonical endomorphism of Longo-Rehren $\iota : A \subset B$ from the double of the quasi Ising rules is $\theta = \lambda_{00} \oplus \lambda_{11} \oplus \lambda_{22}$. This θ is also a dual canonical endomorphism for the model (2) by [11]. We obtain in both models ${}_A\mathscr{X}_B = \{\iota\lambda_{00}, \iota\lambda_{01}, \iota\lambda_{02}\}$ and $\iota\lambda_{00} = \iota\lambda_{22}$, $\iota\lambda_{20} = \iota\lambda_{02}$, $\iota\lambda_{21} = \iota\lambda_{12} = \iota\lambda_{10} = \iota\lambda_{01}$, $\iota\lambda_{11} = \iota\lambda_{00} \oplus \iota\lambda_{02}$. Then $\#{}_A\mathscr{X}_B = 3$, thence $Z_{A \subset B} = Z_3$. Moreover, since the systems $\{\lambda_0, \lambda_1, \lambda_2\}$ of the (braided) Ising model are all self-conjugate, the $6j$-symbols are real and therefore the $6j$-symbols for both models (2) and (3) essentially coincide. Then thanks to the co-ordinate form of Q-systems [16, Eq. (14)], the dual endomorphisms of the models (2) and (3) coincide. Therefore $\theta = \lambda_{00} \oplus \lambda_{11} \oplus \lambda_{22}$ is also a dual endomorphism for the model (3), producing a trace 3 modular invariant, thence Z is sufferable. The full system ${}_B\mathscr{X}_B$ has global index 16 in any of the three models.

Full system for Z_2: the global indices of the $\pm$-chiral system ${}_B\mathscr{X}_B^{\pm}$ and the neutral system ${}_B\mathscr{X}_B^0$ are $\omega_\pm = 8$, $\omega_0 = 4$, respectively, and $\#_B\mathscr{X}_B = \mathrm{Tr}(Z_2 Z_2^t) = 12$. Then ${}_B\mathscr{X}_B^0 = \{\alpha_{00}, \alpha_{02}, \alpha_{11}^{(1)}, \alpha_{11}^{(2)}\}$ where $\alpha_{11}^+ = \alpha_{11}^- = \alpha_{11}^{(1)} \oplus \alpha_{11}^{(2)}$ and ${}_B\mathscr{X}_B^{\pm} = {}_B\mathscr{X}_B^0 \cup \{\alpha_{01}^{\pm}, \alpha_{10}^{\pm}\}$. We still need to find four further irreducible B-B sectors for the full system ${}_B\mathscr{X}_B$. They appear in the irreducible decompositions of $\alpha_{10}^+\alpha_{01}^- = (\alpha_{10}^+\alpha_{01}^-)^{(1)} \oplus (\alpha_{10}^+\alpha_{01}^-)^{(2)}$, $\alpha_{10}^+\alpha_{10}^- = (\alpha_{10}^+\alpha_{10}^-)^{(1)} \oplus (\alpha_{10}^+\alpha_{10}^-)^{(2)}$, and where the following holds: $\alpha_{01}^+\alpha_{10}^- = \alpha_{10}^+\alpha_{01}^-$ and $\alpha_{01}^+\alpha_{01}^- = \alpha_{10}^+\alpha_{10}^-$.

Since $\omega_0 = 4$ and $\#_B\mathscr{X}_B^0 = 4$, the sectors of ${}_B\mathscr{X}_B^0$ are isomorphic to either $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$. Our next objective is to prove that modular data S^{ext} and T^{ext} of ${}_B\mathscr{X}_B^0$ coincides with that of the quantum $\mathbb{Z}_2$ double [16] in the modular data (1) and (2), and thus ${}_M\mathscr{X}_M^0$ is indeed isomorphic to $\mathbb{Z}_2\times\mathbb{Z}_2$ as sectors. The branching coefficient matrix $b=[b_{\tau,\lambda}]$ is

$$b=\begin{pmatrix}1&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&0&0&1\\0&0&0&0&0&0&0&0&1\\0&1&0&1&0&0&0&0&0\end{pmatrix}.$$

The modular matrices S_1, T_1 of the quantum $\mathbb{Z}_2$ double satisfy $S_1b=bS$ and $T_1b=bS$, as well as for $T^{\rm ext}$ and $S^{\rm ext}$, see [6], thus $(T^{\rm ext}-T_1)bb^{\rm t}=0$ and $(S^{\rm ext}-S_1)bb^{\rm t}=0$. Then the matrix $T^{\rm ext}$ is automatically determined to be $T_1=\mathrm{diag}(1,1,1,-1)$. Then the other equation together with the two constraints $(S^{\rm ext})^2$ a permutation matrix and $T^{\rm ext}S^{\rm ext}T^{\rm ext}S^{\rm ext}T^{\rm ext}=S^{\rm ext}$, see e.g. [7], determines $S^{\rm ext}$ to be precisely S_1 – this computation holds in the models (1) and (2). In the modular data (3), Z_2 is realised by the conformal inclusion $SU(2)_2\times SU(2)_2\subset SU(4)_1$ with dual endomorphism $\lambda_{00}\oplus\lambda_{22}$, whose neutral system is given by that of $SU(4)_1$, therefore with fusion isomorphic to $\mathbb{Z}_4$.

Full system for Z_3: the global indices are $\omega_\pm=4$, $\omega_0=1$, and $\#_B\mathscr{X}_B=\mathrm{Tr}(Z_3Z_3^{\rm t})=9$. We can see that ${}_B\mathscr{X}_B^\pm=\{\alpha_{00},\alpha_{10}^\pm,\alpha_{20}^\pm\}$ and ${}_B\mathscr{X}_B={}_B\mathscr{X}_B^+\times{}_B\mathscr{X}_B^-$. Remark that the fusion rules of the ${}_B\mathscr{X}_B^\pm$ sectors are given by the Ising ones. The full system for the permutation invariant Z is obtained by permuting λ_{01} and λ_{10}, λ_{02} and λ_{20}, λ_{12} and λ_{21} in those for Z_1.

4.3. **The exotic model from quantum double of Haagerup $(5+\sqrt{13})/2$ subfactor.** The first irreducible finite depth subfactor $A\subset B$ with index greater than 4 is that of Haagerup [1] with index $(5+\sqrt{13})/2$. Let $\Delta=\{\mathrm{id},\alpha,\alpha^2,\rho,\alpha\rho,\alpha^2\rho\}$ be the A-A system of the Haagerup $(5+\sqrt{13})/2$ subfactor, whose noncommutative fusion rules are in turn

$$[\alpha]^3=[\mathrm{id}],\qquad[\alpha\rho]=[\rho][\alpha^2],\qquad[\rho^2]=[\mathrm{id}]\oplus[\rho]\oplus[\alpha\rho]\oplus[\alpha^2\rho],$$

and $\hat\Delta=\{\mathrm{id},a,b,c\}$ the commutative B-B system whose fusion rules are as follows:

$$[a]^2=[\mathrm{id}]\oplus[a]\oplus[b]\oplus[c],\qquad[b]^2=[\mathrm{id}]\oplus[c],\qquad[c]^2=[\mathrm{id}]\oplus2[a]\oplus[b]\oplus2[c],$$
$$[a][b]=[a]\oplus[c],\qquad[a][c]=[a]\oplus[b]\oplus[c],\qquad[b][c]=[a]\oplus[b]\oplus[c].$$

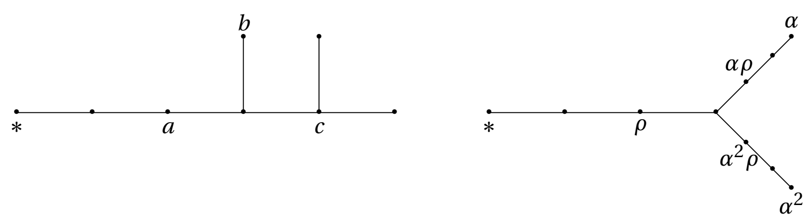

FIGURE 19. Principal graphs of the Haagerup $(5+\sqrt{13})/2$ subfactor.

Here we will consider the Longo-Rehren inclusion $N\subset M$ of the A-A system of the Haagerup $(5+\sqrt{13})/2$ subfactor $A\subset B$. The corresponding modular data was computed in [23]:

$$A=\frac{1}{\lambda}\begin{pmatrix}
1&d^2&1+d^2&1+d^2&1+d^2&1+d^2&3d&3d&3d&3d&3d&3d\\
d^2&1&1+d^2&1+d^2&1+d^2&1+d^2&-3d&-3d&-3d&-3d&-3d&-3d\\
1+d^2&1+d^2&2+2d^2&-1-d^2&-1-d^2&-1-d^2&0&0&0&0&0&0\\
1+d^2&1+d^2&-1-d^2&2+2d^2&-1-d^2&-1-d^2&0&0&0&0&0&0\\
1+d^2&1+d^2&-1-d^2&-1-d^2&-1-d^2&2+2d^2&0&0&0&0&0&0\\
1+d^2&1+d^2&-1-d^2&-1-d^2&2+2d^2&-1-d^2&0&0&0&0&0&0\\
3d&-3d&0&0&0&0&A_{(1,1)}&A_{(1,2)}&A_{(1,3)}&A_{(1,4)}&A_{(1,5)}&A_{(1,6)}\\
3d&-3d&0&0&0&0&A_{(1,2)}&A_{(2,2)}&A_{(2,3)}&A_{(2,4)}&A_{(2,5)}&A_{(2,6)}\\
3d&-3d&0&0&0&0&A_{(1,3)}&A_{(2,3)}&A_{(3,3)}&A_{(3,4)}&A_{(3,5)}&A_{(3,6)}\\
3d&-3d&0&0&0&0&A_{(1,4)}&A_{(2,4)}&A_{(3,4)}&A_{(4,4)}&A_{(4,5)}&A_{(4,6)}\\
3d&-3d&0&0&0&0&A_{(1,5)}&A_{(2,5)}&A_{(3,5)}&A_{(4,5)}&A_{(5,5)}&A_{(5,6)}\\
3d&-3d&0&0&0&0&A_{(1,6)}&A_{(2,6)}&A_{(3,6)}&A_{(4,6)}&A_{(5,6)}&A_{(6,6)}
\end{pmatrix}$$

and $T = \mathrm{diag}(w)$ where $w = (1,1,1,1,\exp(2\pi i/3),\exp(-2\pi i/3),\rho^4,\rho^{-4},\rho^{10},\rho^{-10},\rho^{12},\rho^{-12})$ with $d = (3+\sqrt{13})/2$, $r = (3+i\sqrt{4d-1})/(2d-2)$, $\lambda = 3(1+d^2)$, $\rho = \exp(\pi i/13)$, where $d^2 = 3d+1$ and $C_p = \{C_p(x,y) : x,y = 0,1\}$ is computed as follows $(p = 1,2,3,4,5,6)$:
$C_p(0,0) = -(1+\overline{w}(p))/(d-1)$,
$C_p(1,2) = (w(p)+\overline{w}(p)+d-2)(r-1+(d-1)(w(p)^2+\overline{w}(p)^2+w(p)+\overline{w}(p)))/((d-1)(w(p)+1)(r+\overline{r}))$,
$C_p(2,1) = (w(p)+\overline{w}(p)+d-2)(-\overline{r}+1-(d-1)(w(p)^2+\overline{w}(p)^2+w(p)+\overline{w}(p)))/((d-1)(w(p)+1)(r-\overline{r}))$,
$C_p(0,2) = ((d-1+w(p)-\overline{r})C_p(1,2)-\overline{w}(p)\overline{r}(d-1+w(p)+C_p(2,1)))/(w(p)+\overline{r})$,
$C_p(2,0) = (-w(p)^2(d-1+\overline{w}(p))C_p(1,2)+w(p)\overline{r}(d-1+\overline{w}(p)-r))C_p(2,1))/(w(p)+\overline{r})$,
$C_p(0,1) = (-\overline{w}(p)r(d-1+w(p))C_p(1,2)+(d-1+w(p)-r)C_p(2,1))/(w(p)+r)$,
$C_p(1,0) = (w(p)r(d-1+\overline{w}(p))C_p(1,2)-w(p)^2(d-1+\overline{w}(p)-\overline{r})C_p(2,1))/(w(p)+r)$,
$C_p(1,1) = (w(p)C_p(0,1)+rC_p(1,0))/(w(p)+r)$,
$C_p(2,2) = (w(p)C_p(0,2)+\overline{r}C_p(2,0))/(w(p)+\overline{r})$.

Finally we set

$$\begin{aligned}A_{(p,q)} = 3[&w(p)w(q)+d(\overline{C}_p(0,0)\overline{C}_q(0,0)+\overline{C}_p(0,1)\overline{C}_q(0,1)+\overline{C}_p(0,2)\overline{C}_q(0,2)+\overline{C}_p(1,1)\overline{C}_q(2,0)\\&+\overline{C}_p(1,2)\overline{C}_q(2,1)+\overline{C}_p(1,0)\overline{C}_q(2,2)+\overline{C}_p(2,2)\overline{C}_q(1,0)+\overline{C}_p(2,0)\overline{C}_q(1,1)+\overline{C}_p(2,1)\overline{C}_q(1,2))].\end{aligned}$$

4.3.1. *Verlinde matrices of the quantum double of the Haagerup $(5+\sqrt{13})/2$ subfactor.* The fusion rules of the N-N system ${}_N\mathcal{X}_N$ arising from the Longo-Rehren inclusion $N\subset M$ of the Haagerup $(5+\sqrt{13})/2$ subfactor are computed by means of the Verlinde matrices, see Eq. (4), which as quadratic functions are as follows:

$$\begin{aligned}
N_0 &= |\chi_0|^2+|\chi_1|^2+|\chi_2|^2+|\chi_3|^2+|\chi_4|^2+|\chi_5|^2+|\chi_6|^2+|\chi_7|^2+|\chi_8|^2+|\chi_9|^2\\
&\quad +|\chi_{10}|^2+|\chi_{11}|^2,\\
N_1 &= \chi_0\chi_1^*+\chi_1\chi_0^*+|\chi_2|^2+|\chi_3|^2+|\chi_4|^2+|\chi_5|-|\chi_6|^2-|\chi_7|^2-|\chi_8|^2-|\chi_9|^2\\
&\quad -|\chi_{10}|^2-|\chi_{11}|^2+|\chi_1+\chi_2+\chi_3+\chi_4+\chi_5+\chi_6+\chi_7+\chi_8+\chi_9+\chi_{10}+\chi_{11}|^2,\\
N_2 &= (\chi_0+\chi_1)\chi_2^*+\chi_2(\chi_0+\chi_1)^*+|\chi_2|^2+\chi_3(\chi_4+\chi_5)^*+(\chi_4+\chi_5)\chi_3^*+\chi_4\chi_5^*+\chi_5\chi_4^*\\
&\quad +|\chi_1+\chi_2+\chi_3+\chi_4+\chi_5+\chi_6+\chi_7+\chi_8+\chi_9+\chi_{10}+\chi_{11}|^2,\\
N_3 &= (\chi_0+\chi_1)\chi_3^*+\chi_3(\chi_0+\chi_1)^*+\chi_2(\chi_4+\chi_5)^*+(\chi_4+\chi_5)\chi_2^*+|\chi_3|^2+\chi_4\chi_5^*+\chi_5\chi_4^*\\
&\quad +|\chi_1+\chi_2+\chi_3+\chi_4+\chi_5+\chi_6+\chi_7+\chi_8+\chi_9+\chi_{10}+\chi_{11}|^2,\\
N_4 &= (\chi_0+\chi_1)\chi_4^*+\chi_4(\chi_0+\chi_1)^*+(\chi_2+\chi_3)\chi_5^*+\chi_5(\chi_2+\chi_3)^*+\chi_2\chi_3^*+\chi_3\chi_2^*+|\chi_4|^2\\
&\quad +|\chi_1+\chi_2+\chi_3+\chi_4+\chi_5+\chi_6+\chi_7+\chi_8+\chi_9+\chi_{10}+\chi_{11}|^2,\\
N_5 &= (\chi_0+\chi_1)\chi_5^*+\chi_5(\chi_0+\chi_1)^*+(\chi_2+\chi_3)\chi_4^*+\chi_4(\chi_2+\chi_3)^*++\chi_2\chi_3^*+\chi_3\chi_2^*\\
&\quad +|\chi_5|^2+|\chi_1+\chi_2+\chi_3+\chi_4+\chi_5+\chi_6+\chi_7+\chi_8+\chi_9+\chi_{10}+\chi_{11}|^2,\\
N_6 &= \chi_0\chi_6^*+\chi_6\chi_0^*-\chi_1\chi_6^*-\chi_6\chi_1^*-\chi_7(\chi_{10}+\chi_{11})^*-(\chi_{10}+\chi_{11})\chi_7^*-\chi_6\chi_9^*-\chi_9\chi_6^*\\
&\quad +|\chi_1+\chi_2+\chi_3+\chi_4+\chi_5+\chi_6+\chi_7+\chi_8+\chi_9+\chi_{10}+\chi_{11}|^2-|\chi_{11}|^2\\
&\quad -\chi_8(\chi_9+\chi_{10})^*-(\chi_9+\chi_{10})\chi_8^*,\\
N_7 &= \chi_0\chi_7^*+\chi_7\chi_0^*-\chi_1\chi_6^*+\chi_6\chi_1^*-\chi_6(\chi_{10}+\chi_{11})^*-(\chi_{10}+\chi_{11})\chi_6^*-\chi_7\chi_8^*-\chi_8\chi_7^*\\
&\quad +|\chi_1+\chi_2+\chi_3+\chi_4+\chi_5+\chi_6+\chi_7+\chi_8+\chi_9+\chi_{10}+\chi_{11}|^2-|\chi_{10}|^2\\
&\quad -(\chi_8+\chi_{11})\chi_9^*-\chi_9(\chi_8+\chi_{11})^*,\\
N_8 &= \chi_0\chi_8^*+\chi_8\chi_0^*-\chi_1\chi_8^*+\chi_8\chi_1^*-\chi_6(\chi_9+\chi_{10})^*-(\chi_9+\chi_{10})\chi_6^*-|\chi_7|^2-\chi_7\chi_9^*\\
&\quad -\chi_9\chi_7^*+|\chi_1+\chi_2+\chi_3+\chi_4+\chi_5+\chi_6+\chi_7+\chi_8+\chi_9+\chi_{10}+\chi_{11}|^2-\chi_8\chi_{11}^*\\
&\quad -\chi_{11}\chi_8^*-\chi_{10}\chi_{11}^*-\chi_{11}\chi_{10}^*,
\end{aligned}$$

$$
\begin{aligned}
N_9 &= \chi_0\chi_9^* + \chi_9\chi_0^* - \chi_1\chi_9^* + \chi_9\chi_1^* - |\chi_6|^2 - (\chi_6+\chi_7)\chi_8^* - \chi_8(\chi_6+\chi_8)^* - \chi_7\chi_{11}^* \\
&\quad -\chi_{11}\chi_7^* + |\chi_1+\chi_2+\chi_3+\chi_4+\chi_5+\chi_6+\chi_7+\chi_8+\chi_9+\chi_{10}+\chi_{11}|^2 - \chi_9\chi_{10}^* \\
&\quad -\chi_{10}\chi_9^* - \chi_{10}\chi_{11}^* - \chi_{11}\chi_{10}^*, \\
N_{10} &= \chi_0\chi_{10}^* + \chi_{10}\chi_0^* - \chi_1\chi_{10}^* + \chi_{10}\chi_1^* - \chi_6(\chi_7+\chi_8)^* - (\chi_7+\chi_8)\chi_6^* - |\chi_9|^2 \\
&\quad +|\chi_1+\chi_2+\chi_3+\chi_4+\chi_5+\chi_6+\chi_7+\chi_8+\chi_9+\chi_{10}+\chi_{11}|^2 - \chi_7\chi_9^* - \chi_9\chi_7^* \\
&\quad -(\chi_8+\chi_9)\chi_{11}^* - \chi_{11}(\chi_8+\chi_9)^*, \\
N_{11} &= \chi_0\chi_{11}^* + \chi_{11}\chi_0^* - \chi_1\chi_{11}^* + \chi_{11}\chi_1^* - \chi_6(\chi_7+\chi_{11})^* - (\chi_7+\chi_{11})\chi_6^* - \chi_7\chi_9^* - \chi_9\chi_7^* \\
&\quad +|\chi_1+\chi_2+\chi_3+\chi_4+\chi_5+\chi_6+\chi_7+\chi_8+\chi_9+\chi_{10}+\chi_{11}|^2 - |\chi_8|^2 \\
&\quad -(\chi_8+\chi_9)\chi_{10}^* - \chi_{10}(\chi_8+\chi_9)^*.
\end{aligned}
$$

All the irreducible N-N sectors λ are self-conjugate. The Frobenius-Schur indicator, see e.g. [16, p. 321], is $\mathrm{FS}_\lambda = \omega^{-1}\sum_{\mu,\nu} N_{\mu,\nu}^{\lambda} d_\mu d_\nu \omega_\mu^2/\omega_\nu^2 = 1$ for all $\lambda \in {}_N\mathscr{X}_N$, where ω is the global index of ${}_N\mathscr{X}_N$.

4.3.2. *Modular invariants of the exotic model.* Let Z be a matrix in $\{S,T\}'$. Then using the linear constraints $TZ = ZT$ and $TS = ST$, together with $Z_{00} = 1$, we find that $Z_{3,0}$, $Z_{3,2}$, $Z_{2,3}$, $Z_{3,3}$, $Z_{6,6}$ are free variables. Then the other possibly nonvanishing entries are:
$Z_{0,1} = -Z_{6,6}+1$, $\quad Z_{0,2} = -Z_{6,6}+2-Z_{2,3}/2+Z_{3,0}-Z_{3,2}/2-Z_{3,3}$,
$Z_{0,3} = Z_{2,3}/2+Z_{3,0}-Z_{3,2}/2$, $\quad Z_{1,0} = -Z_{6,6}+1$, $\quad Z_{1,1} = 1$,
$Z_{1,2} = -Z_{6,6}+2-Z_{2,3}/2+Z_{3,0}-Z_{3,2}/2-Z_{3,3}$, $\quad Z_{1,3} = Z_{2,3}/2+Z_{3,0}-Z_{3,2}/2$,
$Z_{2,0} = Z_{3,0}-Z_{6,6}+2-Z_{3,2}-Z_{3,3}$, $\quad Z_{2,1} = Z_{3,0}-Z_{6,6}+2-Z_{3,2}-Z_{3,3}$,
$Z_{2,2} = -2Z_{6,6}+4-Z_{2,3}-Z_{3,2}-Z_{3,3}$, $\quad Z_{3,3} = Z_{4,4} = -2Z_{3,0}+Z_{3,2}+Z_{3,3}$, $\quad Z_{3,1} = Z_{3,0}$,
$Z_{3,3} = Z_{6,6}$, $\quad Z_{4,4} = Z_{5,5} = -2Z_{3,0}+Z_{3,2}+Z_{3,3}$, $\quad Z_{6,6} = Z_{7,7} = Z_{8,8} = Z_{9,9} = Z_{10,10} = Z_{11,11}$,
and $\{S,T\}'$ is a six-dimensional space.

Then using $Z_{\lambda\mu} \le d_\lambda d_\mu$, see [8], we found that there are precisely 28 (normalised) modular invariants. The identity matrix Z_1 and the permutation matrix Z_2 corresponding to the interchange $2 \leftrightarrow 3$. Then there are three symmetric modular invariants:

$$|\chi_0+\chi_1|^2 + 2|\chi_2|^2 + 2|\chi_3|^2 + 2|\chi_4|^2 + 2|\chi_5|^2 + k(\chi_2\chi_3^* + \chi_3\chi_2^* - |\chi_2|^2 - |\chi_3|^2)$$

where $k = 0,1,2$, producing the matrices Z_5, Z_6 and Z_7 respectively. Next, we have two more symmetric modular invariants ($k = \pm 1$), whose matrices we denote by Z_3, Z_4, respectively:

$$|\chi_0+\chi_1+\chi_2+\chi_3|^2 + k(\chi_2\chi_3^* + \chi_3\chi_2^* - |\chi_2|^2 + |\chi_3|^2).$$

Set $s_1 = \chi_0+\chi_1+\chi_2+\chi_3$, $s_2 = \chi_0+\chi_1+2\chi_2$ and $s_3 = \chi_0+\chi_1+2\chi_3$. Then for every pair (i,j), $i,j = 1,2,3$, $Z_{ij} := s_i s_j^*$ is a modular invariant. Finally we have the following modular invariants:

$$
\begin{aligned}
Z_{(1)} &= |\chi_0+\chi_1+\chi_3|^2 + \chi_2\chi_3^* + \chi_3\chi_2^* + |\chi_3|^2 + |\chi_4|^2 + |\chi_5|^2, \\
Z_{(2)} &= |\chi_0+\chi_1+\chi_3|^2 + |\chi_2|^2 + 2|\chi_3|^2 + |\chi_4|^2 + |\chi_5|^2, \\
Z_{(3)} &= |\chi_0+\chi_1+\chi_2|^2 + |\chi_2|^2 + \chi_2\chi_3^* + \chi_3\chi_2^* + |\chi_4|^2 + |\chi_5|^2, \\
Z_{(4)} &= |\chi_0+\chi_1+\chi_2|^2 + 2|\chi_2|^2 + |\chi_3|^2 + |\chi_4|^2 + |\chi_5|^2, \\
Z_{(5)} &= |\chi_0+\chi_2|^2 + |\chi_1+\chi_2|^2 + |\chi_6|^2 + |\chi_7|^2 + |\chi_8|^2 + |\chi_9|^2 + |\chi_{10}|^2 + |\chi_{11}|^2, \\
Z_{(6)} &= |\chi_0+\chi_3|^2 + |\chi_1+\chi_3|^2 + |\chi_6|^2 + |\chi_7|^2 + |\chi_8|^2 + |\chi_9|^2 + |\chi_{10}|^2 + |\chi_{11}|^2,
\end{aligned}
$$

$$\begin{aligned}
Z_{(56)} &= (\chi_0+\chi_3)(\chi_0+\chi_2)^* + (\chi_1+\chi_3)(\chi_1+\chi_2)^* + |\chi_6|^2 + |\chi_7|^2 + |\chi_8|^2 + \\
&\quad |\chi_9|^2 + |\chi_{10}|^2 + |\chi_{11}|^2, \\
Z_{(65)} &= Z^{\mathrm{t}}_{(56)}, \\
Z_{(21)} &= (\chi_0+\chi_1+\chi_3)(\chi_0+\chi_1+\chi_2)^* + 2\chi_3\chi_2^* + \chi_2\chi_3^* + |\chi_4|^2 + |\chi_5|^2, \\
Z_{(12)} &= Z^{\mathrm{t}}_{(21)}, \\
Z_{(61)} &= (\chi_0+\chi_1+\chi_3)(\chi_0+\chi_1+\chi_2)^* + \chi_3\chi_2^* + \chi_2\chi_2^* + |\chi_3|^2 + |\chi_4|^2 + |\chi_5|^2, \\
Z_{(16)} &= Z^{\mathrm{t}}_{(61)},
\end{aligned}$$

where Z^{t} denotes the transpose modular invariant of Z.

4.3.3. *Canonical endomorphisms from Galois correspondence.* The only nontrivial subsystem Δ_0 of Δ is $\Delta_0 = \{\mathrm{id}, \alpha, \alpha^2\}$. (It is isomorphic to the cyclic group $\mathbb{Z}_3$ as sectors.) The dual sector $[\theta_{\mathrm{LR}}]$ of the LR inclusion $N \subset M$ of the Haagerup $(5+\sqrt{13})/2$ subfactor is $[\lambda_0] \oplus [\lambda_1] \oplus 2[\lambda_2]$ by Izumi [23]. Let $N \subset P \subset M$ be the intermediate subfactor obtained by applying Izumi's Galois correspondence to the subsystem Δ_0. Let $\theta_{N\subset P}$ be its dual endomorphism. Since $[M:P]=3$, $[M:N]=d(\theta_{\mathrm{LR}})$, and $\theta_{N\subset P} \prec \theta_{\mathrm{LR}}$, we easily conclude that $[\theta_{N\subset P}] = [\lambda_0] \oplus [\lambda_1]$.

4.3.4. *Some insufferable modular invariants.* First note that the global index ω of our system ${}_N\mathscr{X}_N$ is $\omega = 9(1+d^2)^2$.

Case Z_6. Suppose there is a subfactor $N \subset M$ producing the modular invariant Z_6. The global indices of the chiral systems ${}_M\mathscr{X}_M^{\pm}$ are $\omega_\pm = \omega/(1+d^2) = 9(1+d^2)$, whereas the global index of the neutral system ${}_M\mathscr{X}_M^0$ is $\omega_0 = \omega_\pm^2/\omega = \omega_\pm^2/\omega = 9$. Using Frobenius reciprocity and the definition of Z_6 through α-induction we obtain

$$\langle \alpha_\lambda^+, \alpha_\mu^+ \rangle = \langle \alpha_\lambda^+\alpha_\mu^+, \alpha_0^- \rangle = \sum N_{\lambda\mu}^{\xi} \langle \alpha_\xi^+, \alpha_0^- \rangle = \sum N_{\lambda\mu}^{\xi} [Z_6]_{\xi,0} = N_{\lambda\mu}^0 + N_{\lambda\mu}^1.$$

Hence by the fusion matrices N_0 and N_1 we get

$$\langle \alpha_\lambda^+, \alpha_\lambda^+ \rangle = \begin{cases} 1, & \lambda = 0,6,7,8,9,10,11 \\ 2, & \lambda = 1 \\ 3, & \lambda = 2,3,4,5. \end{cases}$$

Hence $\alpha_0, \alpha_6^+, \alpha_7^+, \alpha_8^+, \alpha_9^+, \alpha_{10}^+$ and α_{11}^+ are irreducible M-M sectors. Since $\langle \alpha_0, \alpha_\lambda^+ \rangle = 0$ for $\lambda \neq 0,1$, and $\langle \alpha_6^+, \alpha_\lambda^+ \rangle = 1$ for $\lambda = 7,8,9,10,11$, so $[\alpha_6^+] = [\alpha_7^+] = [\alpha_8^+] = [\alpha_9^+] = [\alpha_{10}^+] = [\alpha_{11}^+]$. Since we have $\langle \alpha_0, \alpha_1^+ \rangle = \langle \alpha_1^+, \alpha_6^+ \rangle = 1$, we have the decomposition $[\alpha_1^+] = [\alpha_0] \oplus [\alpha_6^+]$. It remains to decompose $\alpha_2^+, \alpha_3^+, \alpha_4^+$ and α_5^+. Observe that α_0 does not appear in these decompositions and $\langle \alpha_2^+, \alpha_3^+ \rangle = \langle \alpha_2^+, \alpha_4^+ \rangle = \langle \alpha_2^+, \alpha_5^+ \rangle = \langle \alpha_3^+, \alpha_4^+ \rangle = \langle \alpha_3^+, \alpha_5^+ \rangle = \langle \alpha_4^+, \alpha_5^+ \rangle = 1$.

Hence we have the following irreducible decomposition:

$$[\alpha_2^+] = [\alpha_6^+] \oplus [\alpha_2^{+(1)}] \oplus [\alpha_2^{+(2)}], \quad [\alpha_3^+] = [\alpha_6^+] \oplus [\alpha_3^{+(1)}] \oplus [\alpha_3^{+(2)}],$$
$$[\alpha_4^+] = [\alpha_6^+] \oplus [\alpha_4^{+(1)}] \oplus [\alpha_4^{+(2)}], \quad [\alpha_5^+] = [\alpha_6^+] \oplus [\alpha_5^{+(1)}] \oplus [\alpha_5^{+(2)}].$$

Therefore ${}_M\mathscr{X}_M^+ = \{\alpha_0, \alpha_6^+, \alpha_2^{+(i)}, \alpha_3^{+(i)}, \alpha_4^{+(i)}, \alpha_5^{+(i)} : i = 1,2\}$. Since Z_6 is symmetric we obtain a similar structure for the system ${}_M\mathscr{X}_M^-$, i.e. $[\alpha_0]$, $[\alpha_1^-] = [\alpha_0] \oplus [\alpha_6^-]$, $[\alpha_6^-] = [\alpha_7^-] = [\alpha_8^-] = [\alpha_9^-] = [\alpha_{10}^-] = [\alpha_{11}^-]$, $[\alpha_\lambda^-] = [\alpha_6^-] \oplus [\alpha_\lambda^{-(1)}] \oplus [\alpha_\lambda^{-(2)}]$, $\lambda = 2,3,4,5$.

Next we compute the neutral system ${}_M\mathscr{X}_M^0$. First note that $\alpha_6^\pm \notin {}_M\mathscr{X}_M^0$ as $\langle \alpha_6^+, \alpha_\lambda^- \rangle = [Z_6]_{6,\lambda} = 0 = [Z_6]_{\lambda,6} = \langle \alpha_\lambda^+, \alpha_6^- \rangle$ for all $\lambda \in {}_N\mathscr{X}_N$. Since $\langle \alpha_2^+, \alpha_2^- \rangle = [Z_6]_{2,2} = 1$, we have $\alpha_1^{+(i)} = \alpha_1^{-(j)}$ or $\alpha_2^{+(i)} = \alpha_2^{-(j)}$ for some $i,j = 1,2$. By making a choice, we may assume that $\alpha_2^{+(1)} = \alpha_2^{-(1)}$ (hereafter denoted by $\alpha_2^{(1)}$). Using again the entries of Z_6 we can also see that $[\alpha_2^{+(2)}] = [\alpha_3^{-(2)}]$, $[\alpha_2^{-(2)}] = [\alpha_3^{+(2)}]$, $[\alpha_3^{+(1)}] = [\alpha_3^{-(1)}]$, and $[\alpha_4^{\pm(i)}] = [\alpha_4^{\mp(i)}]$, $[\alpha_5^{\pm(i)}] = [\alpha_5^{\mp(i)}]$ with $i = 1,2$. Then the neutral system is ${}_M\mathcal{X}_M^0 = \{\alpha_0, \alpha_2^{(1)}, \alpha_2^{+(2)}, \alpha_2^{-(2)}, \alpha_3^{(1)}, \alpha_4^{(i)}, \alpha_5^{(i)} : i = 1,2\}$ and the chiral systems ${}_M\mathcal{X}_M^{\pm} = {}_M\mathcal{X}_M^0 \cup \{\alpha_6^{\pm}\}$. Since $\mathrm{Tr}(ZZ^{\mathrm{t}}) = 16$, we still need to find another five irreducible M-M sectors for the full system ${}_M\mathcal{X}_M$. They have to arise from the irreducible decomposition of $\alpha_6^+\alpha_6^-$. We have $\langle \alpha_6^+\alpha_6^-, \alpha_6^+\alpha_6^-\rangle = \sum N_{66}^{\xi} N_{66}^{\mu} Z_{\xi,\mu} = 9$, using the fusion matrices and the entries of the matrix Z_6. We can also check that the dimension of the intertwiner spaces between $\alpha_6^+\alpha_6^-$ and every sector in the ${}_M\mathcal{X}_M^{\pm}$ is null. Therefore, we find that $[\alpha_6^+\alpha_6^-] = 3[x_1]$, $[x_1] \oplus 2[x_2] \oplus 2[x_3]$, $[x_1] \oplus [x_2] \oplus [x_3] \oplus [x_4] \oplus [x_5] \oplus 2[x_6]$ or $[x_1] \oplus \cdots \oplus [x_9]$, where $[x_i]$ are distinct irreducible sectors. In any case we do not get $\mathrm{Tr}(ZZ^{\mathrm{t}}) = 16$ irreducible M-M sectors. Therefore Z_6 is insufferable.

Cases $Z_3, Z_4, Z_{11}, Z_{12}, Z_{21}, Z_{13}$ *and* Z_{31}. If $Z \in \{Z_3, Z_4, Z_{11}, Z_{12}, Z_{21}, Z_{13}, Z_{31}\}$ is sufferable, then by the type I parent theorem [6, Theorem 4.7], $\theta = \lambda_0 \oplus \lambda_1 \oplus \lambda_2 \oplus \lambda_3 \in \Sigma({}_N\mathcal{X}_N)$ has to be a dual canonical endomorphism of a subfactor $\iota : N \subset M$. Then $\langle \theta\lambda_0, \lambda_0\rangle = 1$, $\langle \theta\lambda_1, \lambda_1\rangle = 4$, $\langle \theta\lambda_2, \lambda_2\rangle = \langle \theta\lambda_3, \lambda_3\rangle = 6$, $\langle \theta\lambda_0, \lambda_1\rangle = \langle \theta\lambda_0, \lambda_2\rangle = \langle \theta\lambda_0, \lambda_3\rangle = 1$. Hence $\iota\lambda_0$ is irreducible and we have the irreducible decomposition $\iota\lambda_1 = \iota\lambda_0 \oplus a_1 \oplus a_2 \oplus a_2 \oplus a_3$. On the other hand, $\iota\lambda_2 = \iota\lambda_1 \oplus a_4 \oplus a_5$ with $[a_4]$ and $[a_5]$ two new sectors. But since $\langle \theta\lambda_2, \lambda_3\rangle = 3$ we cannot decompose $\iota\lambda_3$ in a consistent manner. Therefore Z is insufferable.

Cases $Z_{(5)}, Z_{(56)}$ *and* $Z_{(65)}$. Again, let us assume that $Z \in \{Z_{(5)}, Z_{(56)}, Z_{(65)}\}$ is sufferable. Then by the type I parent theorem $\lambda_0 \oplus \lambda_2$ is a dual canonical endomorphism. But this is a contradiction, by Subsection 4.3.3, as a consequence of Galois correspondence.

Cases $Z_{(3)}, Z_{(4)}, Z_{(12)}, Z_{(21)}, Z_{(16)}$ *and* $Z_{(61)}$. Let $Z \in \{Z_{(3)}, Z_{(4)}, Z_{(12)}, Z_{(21)}, Z_{(16)}, Z_{(61)}\}$. Then by the type I parent theorem $\lambda_0 \oplus \lambda_1 \oplus \lambda_2$ is a dual canonical endomorphism. But this is a contradiction, by Subsection 4.3.3, as a consequence of Galois correspondence.

4.3.5. *Subfactor realization for exotic model modular invariants.*

Proposition 4.1. *The dual endomorphism $\theta = \lambda_0 \oplus \lambda_1$ from the intermediate subfactor $N \subset P$ in Subsection* 4.3.3 *produces the modular invariant Z_5.*

Proof. We use $\theta := \theta_{N\subset P}$ and the fusion matrices to compute: $\langle \theta\lambda_0, \lambda_0\rangle = \langle \theta\lambda_6, \lambda_6\rangle = \langle \theta\lambda_7, \lambda_7\rangle = \langle \theta\lambda_8, \lambda_8\rangle = \langle \theta\lambda_9, \lambda_9\rangle = \langle \theta\lambda_{10}, \lambda_{10}\rangle = \langle \theta\lambda_{11}, \lambda_{11}\rangle = 1$, $\langle \theta\lambda_1, \lambda_1\rangle = 2$, $\langle \theta\lambda_2, \lambda_2\rangle = 3$, $\langle \theta\lambda_3, \lambda_3\rangle = \langle \theta\lambda_4, \lambda_4\rangle = \langle \theta\lambda_5, \lambda_5\rangle = 3$, $\langle \theta\lambda_0, \lambda_i\rangle = \delta_{0,i} + \delta_{1,i}$, $\langle \theta\lambda_i, \lambda_j\rangle = 1$ with $i \neq j$; $i, j \neq 0, 1$. Hence, we infer that $[\iota\lambda_0]$, $[\iota\lambda_1] = [\iota\lambda_0] \oplus [a_1]$, $[\iota\lambda_2] = [a_1] \oplus [a_2] \oplus [a_3]$, $[\iota\lambda_6] = [\iota\lambda_7] = [\iota\lambda_8] = [\iota\lambda_9] = [\iota\lambda_{10}] = [\iota\lambda_{11}] = [a_1]$, $[\iota\lambda_3] = [a_1] \oplus [a_4] \oplus [a_5]$, $[\iota\lambda_4] = [a_1] \oplus [a_6] \oplus [a_7]$, $[\iota\lambda_5] = [a_1] \oplus [a_8] \oplus [a_9]$. So we have ten N-P irreducible sectors implying that $\mathrm{Tr}(Z_{N\subset P}) = 10$, hence by Subsection 4.3.4 $Z_{N\subset P} = Z_2$ or Z_5. We will use the curious identity [12, Proposition 3.3]

$$\bigoplus_{a \in {}_M\mathcal{X}_N} [a\bar{a}] = \bigoplus_{\lambda,\mu \in {}_N\mathcal{X}_N} Z_{\lambda,\mu}[\lambda\bar{\mu}]. \tag{17}$$

to eliminate Z_2. Observe that since $\iota\lambda_0$ and $\iota\lambda_6$ are irreducible M-M sectors, $[\theta] \oplus [\theta][\lambda_6]^2$ is a subsector of the left-hand side of Eq. (17) in which $[\lambda_1]$ appears 11 times. But in the right-hand side, $[\lambda_1]$ appears only 6 times for $Z = Z_2$, so by the curious identity, see Eq. (17) above, we finally conclude that $Z_{N\subset P} = Z_5$. □

Lemma 4.2. *The modular S^{ext} and T^{ext} matrices of the neutral system of Z_5 are the S_0, T_0 modular data of quantum $\mathbb{Z}_3$ double as in* [16, Section 5].

Proof. Let us first compute the chiral systems for Z_5. The global indices of the chiral systems ${}_M\mathscr{X}_M^\pm$ are $\omega_\pm = \omega/(1+d^2)$ and the global index of the neutral system ${}_M\mathscr{X}_M^0$ is $\omega_0 = \omega_\pm^2/\omega = 9$. We easily obtain the following irreducible decompositions: $[\alpha_0]$, $[\alpha_1^\pm] = [\alpha_0]\oplus[\alpha_6^\pm]$, $[\alpha_2^\pm] = [\alpha_6^\pm]\oplus[\alpha_2^{\pm(1)}]\oplus[\alpha_2^{\pm(2)}]$, $[\alpha_{11}^\pm] = [\alpha_{10}^\pm] = [\alpha_9^\pm] = [\alpha_8^\pm] = [\alpha_7^\pm] = [\alpha_6^\pm]$, $[\alpha_3^\pm] = [\alpha_6^\pm]\oplus[\alpha_3^{\pm(1)}]\oplus[\alpha_3^{\pm(2)}]$, $[\alpha_4^\pm] = [\alpha_6^\pm]\oplus[\alpha_4^{\pm(1)}]\oplus[\alpha_4^{\pm(2)}]$, $[\alpha_5^\pm] = [\alpha_6^\pm]\oplus[\alpha_5^{\pm(1)}]\oplus[\alpha_5^{\pm(2)}]$. We now use the entries of the modular invariant Z_5 to compute the neutral system ${}_M\mathscr{X}_M^0$ and conclude that $[\alpha_2^{\pm(i)}] = [\alpha_2^{\mp(i)}]$, $i = 1,2$, which then leads to two sectors $[\alpha_2^{(1)}]$ and $[\alpha_2^{(2)}]$ in ${}_M\mathscr{X}_M^0$. Similarly we have $[\alpha_3^{\pm(i)}] = [\alpha_3^{\mp(i)}]$, $[\alpha_4^{\pm(i)}] = [\alpha_4^{\mp(i)}]$, and $[\alpha_5^{\pm(i)}] = [\alpha_5^{\mp(i)}]$ for $i = 1,2$. In this way, we get the neutral system ${}_M\mathscr{X}_M^0 = \{\alpha_0, \alpha_2^{(i)}, \alpha_3^{(i)}, \alpha_4^{(i)}, \alpha_5^{(i)} : i = 1,2\}$ and ${}_M\mathscr{X}_M^\pm = {}_M\mathscr{X}_M^0 \cup \{\alpha_6^\pm\}$.

Let us now find the S^{ext} and T^{ext} modular matrices of the neutral system ${}_M\mathscr{X}_M^0$. Since $\langle\alpha_2^+\alpha_2^-, \alpha_0\rangle = \langle\alpha_2^+\alpha_2^-, \alpha_6^+\rangle = 2$, α_6^+ and α_0 appear twice in the irreducible decomposition of $\alpha_2^+\alpha_2^-$. Also $\langle\alpha_2^+\alpha_2^-, \alpha_2^+\rangle = 4$ and $\langle\alpha_2^+\alpha_2^-, \alpha_3^+\rangle = \langle\alpha_2^+\alpha_2^-, \alpha_4^+\rangle = \langle\alpha_2^+\alpha_2^-, \alpha_5^+\rangle = 2$, thus $\alpha_j^{(i)}$, $j = 2,3,4,5$, $i = 1,2$, appear in the irreducible decomposition of $\alpha_2^+\alpha_2^-$ only for $i = 2$. This implies that $\{\alpha_0, \alpha_2^{(1)}, \alpha_2^{(2)}\}$ is a copy of $\mathbb{Z}_3$. We also obtain a similar result for $\alpha_4^{(i)}$ and $\alpha_5^{(i)}$. Therefore ${}_M\mathscr{X}_M^0$ cannot be $\mathbb{Z}_9$ so has to be $\mathbb{Z}_3\times\mathbb{Z}_3$. Next the branching coefficient matrix ($b_{\tau,\lambda} = \langle\tau, \alpha_\lambda^\pm\rangle$ with $\tau\in{}_M\mathscr{X}_M^0, \lambda\in{}_N\mathscr{X}_N$) is

$$b = \begin{pmatrix} 1&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0 \end{pmatrix}.$$

Then as $T^{\text{ext}}b = bT$ and $T_0 b = bT$, we conclude that $(T^{\text{ext}} - T_0)bb^{\text{t}} = 0$. This equation alone determines T^{ext} to be T_0. By [35] or [9, Eq. (12)] we have

$$S^{\text{ext}}_{\lambda,\mu} = S^{\text{ext}}_{00}\sum_{\rho\in{}_M\mathscr{X}_M^0}\frac{\omega_\lambda\omega_\mu}{\omega_\rho}\,N_{\lambda,\mu}^\rho = S^{\text{ext}}_{00}\,\frac{\omega_\lambda\omega_\mu}{\omega_{\lambda\mu}}\,, \tag{18}$$

and since we have now both the fusion rules of ${}_M\mathscr{X}_M^0$ and its T^{ext}, we conclude that $S^{\text{ext}} = S_0$. Therefore the S^{ext} and T^{ext} modular data coincide with the quantum $\mathbb{Z}_3$ double modular data. □

Theorem 4.3. *The modular invariants* $\{Z_1, Z_5, Z_7, Z_{22}, Z_{33}, Z_{23}, Z_{32}\}$ *of the quantum double of Haagerup are sufferable.*

Proof. The modular invariant Z_5 is sufferable by Proposition 4.1. Moreover, as in Lemma 4.2, the extended modular data is given by the quantum $\mathbb{Z}_3$ modular data, therefore their modular invariants $\mathcal{Z}_i$, $i = 1,\cdots,8$, coincide with those of the quantum $\mathbb{Z}_3$ double, see [16, Section 5.1]. Moreover by the argument in [16, Lemma 3.9], every subgroup H of $\mathbb{Z}_3\times\mathbb{Z}_3$ gives rise to a dual endomorphism $\theta_H \in \Sigma({}_M\mathscr{X}_M^0)$. Therefore the same arguments of [16, Section 5.1] apply so that the 8 (normalized) modular invariants of the neutral system ${}_M\mathscr{X}_M^0$ are sufferable. Let b denote the branching coefficient matrix as in Subsection 4.3.6. By [8], $b^{\text{t}}\mathcal{Z}_i b$ is a sufferable modular invariant of the quantum

Haagerup model. Since

$$b^{\mathrm{t}}\mathcal{Z}_1 b = Z_5, \quad b^{\mathrm{t}}\mathcal{Z}_2 b = Z_7, \quad b^{\mathrm{t}}\mathcal{Z}_3 b = Z_7, \quad b^{\mathrm{t}}\mathcal{Z}_4 b = Z_{33},$$
$$b^{\mathrm{t}}\mathcal{Z}_5 b = Z_{22}, \quad b^{\mathrm{t}}\mathcal{Z}_6 b = Z_5, \quad b^{\mathrm{t}}\mathcal{Z}_7 b = Z_{32}, \quad b^{\mathrm{t}}\mathcal{Z}_8 b = Z_{23},$$

we conclude the proof of the theorem.

□

We therefore have also obtained the following result (the permutation invariant Z_2 remains to be decided).

Corollary 4.4. *Of the* 28 *modular invariants from the quantum double of the Haagerup* $(5+\sqrt{13})/2$ *subfactor,* 7 *are sufferable and* 20 *are insufferable.*

Also, since every sufferable modular invariant is nimble [10, 7], we automatically have the following.

Corollary 4.5. *The modular invariants* $\{Z_1, Z_5, Z_7, Z_{22}, Z_{33}, Z_{23}, Z_{32}\}$ *are nimble.*

	Z_1	Z_2	Z_{22}	Z_{33}	Z_{23}	Z_{32}	Z_5	Z_7
Z_1	Z_1	Z_2	Z_{22}	Z_{33}	Z_{32}	Z_{23}	Z_5	Z_7
Z_2	Z_2	Z_1	Z_{32}	Z_{23}	Z_{22}	Z_{33}	Z_7	Z_5
Z_{22}	Z_{22}	Z_{23}	$6Z_{22}$	$2Z_{23}$	$2Z_{22}$	$6Z_{23}$	$2Z_{22}$	$2Z_{23}$
Z_{23}	Z_{33}	Z_{32}	$2Z_{32}$	$6Z_{33}$	$6Z_{32}$	$2Z_{33}$	$2Z_{33}$	$2Z_{32}$
Z_{23}	Z_{23}	Z_{22}	$2Z_{22}$	Z_{23}	$6Z_{22}$	$2Z_{23}$	$2Z_{23}$	$2Z_{23}$
Z_{32}	Z_{32}	Z_{33}	Z_{32}	$2Z_{33}$	$2Z_{32}$	$6Z_{33}$	$2Z_{32}$	$2Z_{33}$
Z_5	Z_5	Z_7	$2Z_{22}$	$2Z_{33}$	$2Z_{32}$	$2Z_{23}$	$2Z_5$	$2Z_7$
Z_7	Z_7	Z_5	$Z2_{23}$	$2Z_{23}$	$2Z_{22}$	$2Z_{33}$	$2Z_7$	$2Z_5$

TABLE 1. Fusion $Z_a Z_b^{\mathrm{t}}$ of modular invariants.

We have chosen the indices of the sufferable modular invariants of the exotic model so that the map that sends every sufferable modular invariant in the exotic modular data $Z_i \mapsto Z_i$ in the quantum S_3 double model [16] is an injective fusion algebra homomorphism.

4.3.6. *Full systems of sufferable exotic modular invariants.*

Cases Z_{22}, Z_{33}, Z_{23} *and* Z_{32}. The full systems for Z_{33}, Z_{23} and Z_{32} can be obtained from the one arising from Z_{22} by permuting sectors. For $Z = Z_{22}$, the global indices of ${}_M\mathcal{X}_M^{\pm}$ and ${}_M\mathcal{X}_M^{0}$ are as follows:

$$\omega_{\pm} = \frac{\omega}{\sum d_\lambda Z_{\lambda 0}} = \frac{\omega}{(3+3d^2)} = 3(1+d^2), \qquad \omega_0 = \omega_{\pm}^2/\omega = 1.$$

Using

$$\langle \alpha_\lambda^+, \alpha_\mu^+ \rangle = \sum_\nu N_{\lambda,\mu}^{\nu} \langle \alpha_\nu^+, \alpha_0 \rangle = \sum_\nu N_{\lambda,\mu}^{\nu} [Z_{22}]_{\nu,0},$$

and similarly for α_λ^- noting that Z_{22} is symmetric, we get the following irreducible decomposition for $\alpha_\lambda^{\pm}$ with the obvious notations: $[\alpha_1^{\pm}] = [\alpha_0] \oplus [\alpha_1^{(1)\pm}] \oplus [\alpha_1^{(2)\pm}] \oplus [\alpha_1^{(3)\pm}]$, $[\alpha_2^{\pm}] = 2[\alpha_0] \oplus [\alpha_1^{(1)\pm}] \oplus [\alpha_1^{(2)\pm}] \oplus [\alpha_1^{(3)\pm}]$, $[\alpha_3^{\pm}] = [\alpha_1^{(1)\pm}] \oplus [\alpha_1^{(2)\pm}] \oplus [\alpha_1^{(3)\pm}] \oplus [\alpha_3^{(1)\pm}] \oplus [\alpha_3^{(2)\pm}]$, and $[\alpha_5^{\pm}] = [\alpha_4^{\pm}] = [\alpha_3^{\pm}]$, $[\alpha_1^{(1)\pm}] \oplus [\alpha_1^{(2)\pm}] \oplus [\alpha_1^{(3)\pm}] = [\alpha_6^{\pm}] = [\alpha_7^{\pm}] = [\alpha_8^{\pm}] = [\alpha_9^{\pm}] = [\alpha_{10}^{\pm}] =$

$[\alpha_{11}^{\pm}]$. Thence ${}_M\mathcal{X}_M^{\pm} = \{\alpha_0, \alpha_1^{(1)\pm}, \alpha_1^{(2)\pm}, \alpha_1^{(3)\pm}, \alpha_3^{(1)\pm}, \alpha_3^{(2)\pm}\}$ and ${}_M\mathcal{X}_M = {}_M\mathcal{X}_M^{+} \times {}_M\mathcal{X}_M^{-}$. Note that the full system of Z_{22} decomposes into 6 copies of ${}_M\mathcal{X}_M^{+}$ according to $Z_{22}^2 = 6Z_{22}$, cf. [16].

Cases Z_5 and Z_7. We have already computed in the proof of Lemma 4.2 the neutral ${}_M\mathcal{X}_M^0$ and the chiral systems ${}_M\mathcal{X}_M^{\pm}$ for the modular invariant Z_5. Since $\langle \alpha_6^+\alpha_6^-, \alpha_6^+\alpha_6^-\rangle = 9$ and the dimension of $\alpha_6^+\alpha_6^-$ with any other irreducible sector in ${}_M\mathcal{X}_M^{\pm}$ is null, we conclude that $[\alpha_6^+\alpha_6^-] = 3[x]$, $[x_1]\oplus 2[x_2]\oplus 2[x_3]$, $[x_1]\oplus[x_2]\oplus[x_3]\oplus[x_4]\oplus[x_5]\oplus 2[x_6]$ or $= [x_1]\oplus\cdots\oplus[x_9]$. However, since Z_5 is sufferable and we must have $\#_M\mathcal{X}_M = \mathrm{Tr}(ZZ^{\mathrm{t}}) = 20$ irreducible M-M sectors, we conclude that $\alpha_6^+\alpha_6^-$ decomposes into 9 further irreducible sectors henceforth denoted by $(\alpha_6^+\alpha_6^-)^{(i)}$ with $i = 1, \cdots, 9$, because we must have 20 irreducible M-M sectors. Note that when computing the fusion graph of α_6^+ we have

$$\langle \alpha_6^+\alpha_6^+\alpha_6^-, \alpha_6^+\alpha_6^-\rangle = \sum_{\xi,\eta,\zeta} N_{66}^{\xi} N_{66}^{\zeta} N_{\xi 6}^{\eta} [Z_5]_{\eta,\zeta} = 81.$$

Consequently, the fusion algebra $\Sigma({}_M\mathcal{X}_M)$ is in this case not easily computable as we do not know how to decompose $[\alpha_6^+][(\alpha_6^+\alpha_6^-)^{(i)}]$ into irreducible M-M sectors. However, the full system decomposes into two sheets, according to $Z_5^2 = 2Z_5$, with 10 elements each.

For the local realisation $N \subset M$ of Z_5 with $\theta = \lambda_0 \oplus \lambda_1$, we obtain the canonical sector $[\gamma] = [\alpha_0] \oplus [(\alpha_6^+\alpha_6^-)^{(i)}]$ by using [11, Corollary 3.19] and choosing one irreducible M-M sector from the decomposition of $\alpha_6^+\alpha_6^-$.

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References

[1] M. ASAEDA, U. HAAGERUP, Exotic subfactors of finite depth with Jones indices $(5+\sqrt{13})/2$ and $(5+\sqrt{17})/2$, *Comm. Math. Phys.* **202** (1999),1–63.
[2] J. BÖCKENHAUER, Localized endomorphisms of the chiral ising model, *Comm. Math. Phys.* **177** (1996), 265–304.
[3] J. BÖCKENHAUER, D.E. EVANS, Modular invariants, graphs and α-induction for nets of subfactors. I, *Comm. Math. Phys.* **197** (1998), 361–386.
[4] J. BÖCKENHAUER, D.E. EVANS, Modular invariants, graphs and α-induction for nets of subfactors. II, *Comm. Math. Phys.* **200** (1999), 57–103.
[5] J. BÖCKENHAUER, D.E. EVANS, Modular invariants, graphs and α-induction for nets of subfactors. III, *Comm. Math. Phys.* **205** (1999), 183–228.
[6] J. BÖCKENHAUER, D.E. EVANS, Modular invariants from subfactors: Type I coupling matrices and intermediate subfactors, *Comm. Math. Phys.* **213** (2000), 267–289.
[7] J. BÖCKENHAUER, D.E. EVANS, Modular invariants and subfactors, in *Mathematical Physics in Mathematics and Physics (Siena, 2000)*, R. Longo (Ed.), Fields Inst. Commun., vol. 30, Amer. Math. Soc., 2001, pp. 11–37.
[8] J. BÖCKENHAUER, D.E. EVANS, Modular invariants from subfactors, in *Quantum Symmetries in Theoretical Physics and Mathematics (Bariloche, 2000)*, R. Coquereaux et al. (Eds.), Contemp. Math., vol. 294, Amer. Math. Soc., Providence, RI, 2002, pp. 95–131.
[9] J. BÖCKENHAUER, D.E. EVANS, Y. KAWAHIGASHI, On α-induction, chiral generators and modular invariants for subfactors, *Comm. Math. Phys.* **208** (1999), 429–487.
[10] J. BÖCKENHAUER, D.E. EVANS, Y. KAWAHIGASHI, Chiral structure of modular invariants for subfactors, *Comm. Math. Phys.* **210** (2000), 733–784.

[11] J. BÖCKENHAUER, D.E. EVANS, Y. KAWAHIGASHI, Longo-Rehren subfactors arising from α-induction, *Publ. Res. Inst. Math. Sci. Kyoto Univ.* **37** (2001), 1–35.
[12] D.E. EVANS, Fusion rules of modular invariants, *Rev. Math. Phys.* **14** (2002), 709–732.
[13] D.E. EVANS, Critical phenomena, modular invariants and operator algebras, in *Operator Algebras and Mathematical Physics (Constanţa, 2001)*, J. Cuntz, G.A. Elliott, S. Stratila et al. (Eds.), The Theta Foundation, Bucharest, 2003, pp. 89–113.
[14] D.E. EVANS, Modular invariant partition functions in statistical mechanics, conformal field theory and their realisation by subfactors, in *Proceedings Congress of International Association of Mathematical Physics*, Lisbon 2003, to appear.
[15] D.E. EVANS, Y. KAWAHIGASHI, *Quantum Symmetries on Operator Algebras*, Oxford University Press, 1998.
[16] D.E. EVANS, P.R. PINTO, Subfactor realisation of modular invariants, *Comm. Math. Phys.* **237** (2003), 309–363.
[17] D.E. EVANS, P.R. PINTO, Modular invariants and their fusion rules, in *Advances in Quantum Dynamics (Mount Holyoke, 2002)*, G.L. Price et al. (Eds.), Contemp. Math. **335**, Amer. Math. Soc., Providence, RI, 2003, pp. 119-130.
[18] J. FUCHS, I. RUNKEL, C. SCHWEIGERT, TFT construction of RCFT correlators I: Partition functions, *Nucl. Phys. B* **646** (2002), 354–497.
[19] J. FUCHS, I. RUNKEL, C. SCHWEIGERT, TFT construction of RCFT correlators II: Unoriented world sheets, *Nucl. Phys. B* **678** (2004), 511–637.
[20] M. IZUMI, H. KOSAKI, On the subfactor analogue of the second cohomology, *Rev. Math. Phys.* **14** (2002), 733–757.
[21] M. IZUMI, Subalgebras of infinite C^*-algebras with finite Watatani indices. II. Cuntz-Krieger algebras, *Duke Math. J.* **91** (1998), 409–461.
[22] M. IZUMI, The structure of sectors associated with Longo-Rehren inclusions. I. General theory, *Comm. Math. Phys.* **213** (2000), 127–179.
[23] M. IZUMI, The structure of sectors associated with Longo-Rehren inclusions. II. Examples, *Rev. Math. Phys.* **13** (2001), 603–674.
[24] V.F.R. JONES, Index for subfactors, *Invent. Math.* **72** (1983), 1–25.
[25] Y. KAWAHIGASHI, N. SATO, M. WAKUI, $(2+1)$-dimensional topological quantum field theory from subfactors and Dehn surgery formula for 3-manifold invariants, *Adv. Math.* **195** (2005), 165–204; preprint math.OA/0208238.
[26] R. LONGO, Index of subfactors and statistics of quantum fields. II, *Comm. Math. Phys.* **130** (1990), 285–309.
[27] R. LONGO, Duality for Hopf algebras and for subfactors. I, *Comm. Math. Phys.* **159** (1994), 133–150.
[28] R. LONGO, K.-H. REHREN, Nets of subfactors, *Rev. Math. Phys.* **7** (1995), 567–597.
[29] T. MASUDA, An analogue of Longo's canonical endomorphism for bimodule theory and its application to asymptotic inclusions, *Internat. J. Math.* **8** (1997), 249–265.
[30] M. MÜGER, From subfactors to categories and topology. I. Frobenius algebras in and Morita equivalence of tensor categories, *J. Pure Appl. Algebra* **180** (2003), 81–157, or math.CT/0111204.
[31] A. OCNEANU, Quantized groups, string algebras, and Galois theory for von Neumann algebras, in *Operator Algebras and its Applications, vol. 2*, D.E. Evans, M. Takesaki (Eds.), London Math. Soc. Lecture Notes Ser., vol 136, Cambridge University Press, 1987, pp. 119–172.
[32] A. OCNEANU, Chirality for operator algebras, in *Subfactors (Kyuzeso, 1993)*, H. Araki, Y. Kawahigashi, H. Kosaki (Eds.), World Sci. Publishing, River Edge, NJ, 1994, pp. 39–63.
[33] V. OSTRIK, Module categories, weak Hopf algebras and modular invariants, *Transform. Groups* **8** (2003), 177–206; math.QA/0111139.
[34] V.B. PETKOVA, J.B. ZUBER, The many faces of Ocneanu cells, *Nucl. Phys. B* **603** (2001), 449–496.
[35] K.-H. REHREN, Braid group statistics and their superselection rules, in *The Algebraic Theory of Superselection Sectors (Palermo 1989)*, D. Kastler (Ed.), World Sci. Publishing, River Edge, NJ, 1990, pp. 333–355.
[36] K.-H. REHREN, Space-time fields and exchange fields, *Comm. Math. Phys.* **132** (1990), 461–483.
[37] N. SATO, M. WAKUI, Computations of Turaev-Viro-Ocneanu invariants of 3-manifolds from subfactors, *J. Knot Theory Ramifications* **12** (2003), 543–574.
[38] V.G. TURAEV, *Quantum Invariants of Knots and* 3*-manifolds*, de Gruyter Studies in Mathematics, vol. 18, Walter de Gruyter, Berlin, 1994.
[39] E. VERLINDE, Fusion rules and modular transformations in 2D conformal field theory, *Nucl. Phys. B* **300** (1988), 360–376.

[40] A. WASSERMANN, Operator algebras and conformal field theory. III: fusion of positive representations of LSU(n) using bounded operators, *Invent. Math.* **133** (1998), 467–538.

[41] A. WASSERMANN, Quantum subgroups and vertex algebras, MSRI talk, December 2000, http://www.msri.org/publications/ln/msri/2000/subfactors/wassermann/1.

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