Schatten Ideals and Respective Sequence Spaces

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Abstract

This text is a brief description of the analogy between l^p, as Banach algebras, and the corresponding Schatten ideals, which will be defined below, and are their noncommutative analog. Only the cases of l^1, l^2, c_0 and l^∞ will be discussed in some detail. The case of l^p, for other p, will only be hinted. Here H will be a separable Hilbert space of infinite dimension.

1 The Sequence Spaces as Banach Algebras

First let's note that l^p , $p \in [1, \infty]$, is a Banach space. We can define a componentwise product in l^p, which is $(x_n)_{n\in\mathbb{N}}.(y_n)_{n\in\mathbb{N}} = (x_n y_n)_{n\in\mathbb{N}}$. It is obviously associative and distributive. To show that it is a Banach algebra we must show that $||xy||_p \leq ||x||_p||y||_p$. $||xy||_p^p = \sum_{n=1}^{\infty} |x_n y_n|^p \leq \sum_{n=1}^{\infty} |x_n|^p (\sum_{n=1}^{\infty} |y_n|^p) = (\sum_{n=1}^{\infty} |x_n|^p)(\sum_{n=1}^{\infty} |y_n|^p) = ||x||_p^p||y||_p^p$. In case $p = \infty$, the inequality holds: $||xy||_\infty = sup\{|x_n y_n|\} \leq sup\{|x_n|||y||_\infty\} = ||x||_\infty||y||_\infty$. c_0 is also a Banach algebra which is proven using similar arguments to the case of l^∞.

2 l^∞ and $L(H)$

Let's define the linear function $\Phi : l^\infty \longrightarrow L(H)$ by $\Phi(\lambda) = T(x) = \sum_{n=1}^{\infty} \lambda_n < x, e_n > e_n$ (we will use the notation λ to denote $(\lambda_n)_{n\in\mathbb{N}}$). We have to see that the $T(x)$ is bounded, in order for Φ to be well defined: $||T(x)||_H^2 = \sum_{n=1}^{\infty} |\lambda_n|^2| < x, e_n > |^2 \leq ||\lambda||_\infty^2||x||^2$. We can also prove that Φ is an isometry: $||\Phi(\lambda)|| = ||T|| =$
$sup\{||T(x)||_H : ||x||_H = 1\} \geq sup\{||T(e_n)||_H\} = sup\{|\lambda_n|\} = ||\lambda||_\infty$. Therefore, $||\Phi(\lambda)|| = ||\lambda||_\infty$ and Φ is an isometry. Φ is also an algebra homomorphism: $\Phi(\lambda) \circ \Phi(\mu) = \Phi(\lambda)(\sum_{n=1}^{\infty} \mu_n < x, e_n > e_n) = \sum_{n=1}^{\infty} \lambda_n < \sum_{i=1}^{\infty} \mu_i < x, e_i > e_i, e_n > e_n =$
$\sum_{n=1}^{\infty} \lambda_n \mu_n < x, e_n > e_n = \Phi(\lambda\mu)$. Φ is not surjective though, since the right shift operator is not diagonal, for example. So the image of l^∞ is the set of diagonal operators with eigenvalues in l^∞

3 c_0 and $L_0(H)$

If $\lambda \in c_0$ then the diagonal operator T is compact: let
$T_n = \sum_{i=1}^{n} \lambda_i < x, e_i > e_i$, where (λ_i) is the preimage of T, be a finite rank operator. Then $||T(x) - T_n(x)||_H^2 =$
$|| \sum_{i=n+1}^{\infty} \lambda_i < x, e_i > e_i||_H^2 = \sum_{i=n+1}^{\infty} |\lambda_i|^2| < x, e_i > |^2 \leq$
$(\sum_{i=n+1}^{\infty} |\lambda_i|^2)||x||_H^2$. So $\frac{||T(x)-T_n(x)||_H^2}{||x||_H^2} \leq \sum_{i=n+1}^{\infty} |\lambda_i|^2$. Therefore, if $\lambda \in c_0$ then the diagonal operator T is compact. If an operator is compact it has as the spectrum a sequence in c_0. So a diagonal T is in $L_0(H)$ iff $\lambda \in c_0$. Therefore $\Phi^{-1}(L_0(H)) = c_0$. Since

the norm in c_0 is the same as the norm of l^∞, Φ is an isometric homomorphism from c_0 to $L_0(H)$. However, $\Phi(c_0) \neq L_0(H)$, since $T(x) = \sum_{n=1}^{\infty} \lambda_n < x, e_n > e_{n+1}$ is compact, if $\lambda_n \to 0$, but T is not a diagonal operator. So the image of c_0 is the set of diagonal operators with eigenvalues in c_0.

4 l^2 and $C_2(H)$

Let's now see what a Hilbert-Schmidt operator is. But first we have to define a special norm in $L_0(H)$. If $T \in L_0(H)$, then $||T||_2 = (\sum_{n=1}^{\infty} ||T(e_n)||_H^2)^{1/2}$, where $\{e_n\}$ is an orthonormal basis of H. Let $C_2(H) := \{T : ||T||_2 < \infty\}$. The members of $C_2(H)$ are called Hilbert-Schmidt operators and form a normed space. We will not prove the following proposition.

Proposition: 1- $||T||_2$ is independent of the chosen basis, for any $T \in C_2(H)$.
2- $||T||_2 = ||T^*||_2$
3- $||T|| \leq ||T||_2$

It's now easy to prove that $C_2(H)$ is a normed algebra. Let $S \in L(H)$. Then $||ST||_2^2 = \sum_{n=1}^{\infty} ||ST(e_n)||_H^2 \leq ||S||^2 \sum_{n=1}^{\infty} ||T(e_n)||_H^2 = ||S||^2 ||T||_2^2$. Therefore $C_2(H)$ is a left ideal of $L(H)$. Let's now compute $||TS||_2^2 = \sum_{i=1}^{\infty} ||TS(e_i)||_H^2 = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} | < TS(e_i), e_n > |^2 = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} | < S^*T^*(e_n), e_i > |^2 = \sum_{n=1}^{\infty} ||S^*T^*(e_n)||_H^2 \leq ||S^*||^2 ||T^*||_2^2 = ||S||^2 ||T||_2^2$. Therefore, $C_2(H)$ is also a right ideal of $L(H)$, and thus $C_2(H)$ is an ideal of $L(H)$, called a Schatten ideal. Since it is an ideal of $L(H)$, it's an algebra. If in these computations we choose $S \in C_2(H)$ we have that $||ST||_2 \leq ||S||_2 ||T||_2$ and that $||TS||_2 \leq ||S||_2 ||T||_2$, because of point 3 of the proposition. Therefore $C_2(H)$ is a also a normed algebra.

Theorem: $(C_2(H), ||.||_2)$ is a Hilbert space with inner product $< T, S >_2 = \sum_{i=1}^{\infty} < T(e_i), S(e_i) >$.

Proof: Apply the polarization identity. ■

Let $T \in C_2(H)$. Then, since it is compact, it can be written as $T(x) = \sum_{i=1}^{\infty} a_i < x, e_i > f_i$, where $\{f_i\}$ is an orthonormal basis for H. Let's compute $||T||_2^2 = \sum_{n=1}^{\infty} ||T(e_n)||_H^2 = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} | < T(e_n), e_j > |^2 =$

$\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}| < \sum_{i=1}^{\infty} a_i < e_n, e_i > f_i, e_j > |^2 =$
$\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}| < a_n f_n, e_j > |^2 = \sum_{j=1}^{\infty}| < \sum_{n=1}^{\infty} a_n f_n, e_j > |^2 = ||(a_n)||_H^2$. So $||T||_2 = ||(a_n)||_H$. Therefore $T \in C_2(H)$ iff the sequence of its coefficients is in l^2. In particular $\Phi^{-1}(C_2(H)) = l^2$. Therefore Φ is an isometric homomorphism. However, $\Phi(l^2) \neq C_2(H)$, because the $T(x) = \sum_{n=1}^{\infty} \lambda_n < x, e_n > e_{n+1}$ is in $C_2(H)$ if $\lambda \in l^2$, but it's not diagonal. So the image of l^2 is the set of diagonal operators with eigenvalues in l^2.

5 l^1 and $C_1(H)$

Let's now turn to the study of $C_1(H)$, which is formed by operators called the trace class operators. First let's define the respective norm. $||T||_1 := sup\{| < T, B >_2 | : B \in C_2(H) \wedge ||B|| \leq 1\}$. We define $C_1(H) = \{T \in C_2(H) : ||T||_1 < \infty\}$. $C_1(H)$ is a normed space. The following proposition will not be proven:

Proposition: 1- $||T||_1 = ||T^*||_1$
2- $||T||_2 \leq ||T||_1$

If $S, T \in C_1(H)$ then $||ST||_1 = sup\{| < ST, B >_2 | : ||B|| \leq 1\} = sup\{| < T, S^*B >_2 | : ||B|| \leq 1\} \leq$
$sup\{||T||_2||S^*B||_2 : ||B|| \leq 1\} \leq sup\{||T||_2||S^*||_2||B|| : ||B|| \leq 1\} \leq ||T||_2||S||_2 \leq ||T||_1||S||_1$. Therefore $C_1(H)$ is a normed subalgebra of $C_2(H)$. Then, from the above computations, if $S, T \in C_2(H)$, then $||ST||_1 \leq ||S||_2||T||_2 < \infty$. Therefore $C_1(H)$ is an ideal of $C_2(H)$ and $C_2(H)C_2(H) = C_1(H)$. Since $L(H)C_1(H)L(H) =$
$L(H)C_2(H)C_2(H)L(H) = C_2(H)C_2(H) = C_1(H)$, $C_1(H)$ is an ideal of $L(H)$. It is also called a Schatten ideal.

We define the trace of an operator in $C_1(H)$ as
$tr(T) := \sum_{n=1}^{\infty} < T(e_n), e_n >$, for an orthonormal basis $\{e_n\}$ of H. It's the analogous of the trace of a matrix in infinite dimension. The following proposition is left without proof:

Proposition: 1- $tr(T)$ is absolutely convergent.
2- $tr(T)$ does not depend on the orthonormal basis.
3- $tr \in C_1(H)^*$ and $||tr|| = 1$
4- $tr(TS) = tr(ST)$.
Remark: in point 4 T and S need not be in $C_1(H)$, just their compo-

sition.

Let $T \in C_1(H)$. Since it is compact, it can be written as
$T(x) = \sum_{n=1}^{\infty} \lambda_n < x, e_n > f_n$, where both $\{e_n\}$ and $\{f_n\}$ are orthonormal basis ($\{f_n\}$ is not the same for all $T(x)$).

Proposition: $||T||_1 = \sum_{n=1}^{\infty} |\lambda_n| = ||\lambda||_1$.

Proof: Let $B \in C_2(H)$ be such that $||B|| \leq 1$. Then $| < T, B >_2 | \leq \sum_{n=1}^{\infty} |\lambda_n| | << \cdot, e_n > f_n, B >_2 | = \sum_{n=1}^{\infty} |\lambda_n| | < f_n, B(e_n) > | \leq \sum_{n=1}^{\infty} |\lambda_n| ||f_n||_H ||B|| ||e_n||_H \leq \sum_{n=1}^{\infty} |\lambda_n| = ||\lambda||_1$.
To prove the other indequality, choose $b_n \in \mathbb{C}$ such that $\overline{b_n}\lambda_n = |\lambda_n|$ and $|b_n| = 1$. Define $B_N(x) := \sum_{n=1}^{N} b_n < x, e_n > f_n$. We have
$B_N \in C_2(H)$, because $(b_1, b_2, ..., b_N, 0, ...) \in l^2$. Plus, $||B_N|| = ||(b_n)||_\infty = 1$. Therefore $||T||_1 \geq | < T, B_N >_2 | =$
$| \sum_{n=1}^{\infty} < T(e_n), B_N(e_n) > | = | \sum_{n=1}^{N} < \lambda_n f_n, b_n f_n > | = | \sum_{n=1}^{N} \lambda_n \overline{b_n} | = \sum_{n=1}^{N} |\lambda_n| \longrightarrow ||\lambda||_1$.
QED ■

So we have $T \in C_1(H)$ iff $\lambda \in l^1$. In particular $\Phi^{-1}(C_1(H)) = l^1$. Thus Φ is an isometric homomorphism (for the respective norms $||.||_1$). Again, $\Phi : l^1 \longrightarrow C_1(H)$ is not surjective, since
$T(x) = \sum_{n=1}^{\infty} \lambda_n < x, e_n > e_{n+1}$, with $\lambda \in l^1$, is not diagonal.

6 The Analogous Relations

Now we are in conditions to make one part of the analogy: just as we have $l^1 \subset l^2 \subset c_0 \subset l^\infty$, we have $C_1(H) \subset C_2(H) \subset L_0(H) \subset L(H)$. Each of the former normed algebras can be put in the latter normed algebras, through an isometric homomorfism, as the diagonal operators in the respective operator algebras. Just as the former normed (Banach) algebras are ideals of l^∞, the latter normed algebras are ideals of $L(H)$. These operator algebras are the noncommutative analog of the respective commutative algebras of sequences.

The analogy extends to the l^p and $C_p(H)$, for any $p \in [1, \infty[$. $C_p(H)$ is defined as a class of operators which norm $||.||_p$ is finite. For $T \in C_p(H)$ a diagonal operator, we have $||T||_p = ||\lambda||_p$. So there is an isometric homomorphism $\Phi : l^p \longrightarrow C_p(H)$ (which is not surjective). The $C_p(H)$ are Banach algebras which are ideals of $L(H)$ and are called Schatten ideals.

But there are more relations that enrich this analogy. Just as we have $(c_0)^* \cong l^1$ and $(l^1)^* \cong l^\infty$, we have

Proposition: 1- $L_0(H)^* \cong C_1(H)$
2- $C_1(H)^* \cong L(H)$

Proof: 1- Let $f : C_1(H) \longrightarrow L_0(H)^*$ such that $f_S(T) = tr(TS)$. Obviously f is linear. Note that, because $tr(TS) = tr(ST)$, we have $f_T(S) = f_S(T)$. Therefore, $||f_T|| = sup\{|f_S(T)| : ||S|| \leq 1\} = sup\{|tr(TS)| : ||S|| \leq 1\} =$
$sup\{|\sum_{n=1}^{\infty} < TS(e_n), e_n > | : ||S|| \leq 1\} =$
$sup\{|\sum_{n=1}^{\infty} < e_n, TS(e_n) > | : ||S|| \leq 1\} =$
$sup\{|\sum_{n=1}^{\infty} < T^*(e_n), S(e_n) > | : ||S|| \leq 1\} =$
$sup\{| < T^*, S >_2 | : ||S|| \leq 1\} = ||T^*||_1 = ||T||_1$. Therefore f is an isometry. Let $g \in L_0(H)^*$. Then, for $S \in C_2(H)$, we have $g(S) \leq ||g||||S|| \leq ||g||||S||_2$. Therefore $g|_{C_2(H)} \in C_2(H)^*$. We will use the equality $< T, S >_2 =< S^*, T^* >_2$, which is valid for any $S, T \in C_2(H)$, without proving it. Let's use Riesz representation theorem to find the only S such that $g|_{C_2(H)}(T) =< T, S >_2 =< S^*, T^* >_2 = tr(TS^*)$. We know that $||g|_{C_2(H)}|| < \infty$ and so $||g|_{C_2(H)}|| =$
$sup\{| < T, S >_2 | : ||T|| \leq 1\} = sup\{| < S^*, T^* >_2 | : ||T|| \leq 1\} = sup\{| < S^*, T^* >_2 | : ||T^*|| \leq 1\} = ||S^*||_1 < \infty$. Therefore
$S^* \in C_1(H)$ (!) . Since $\overline{C_2(H)}^{||.||} = L_0(H)$, there exists a unique extension of $g|_{C_2(H)}$ to $L_0(H)$, which is g. Hence $g(T) = tr(TS^*)$ and therefore f is surjective.

2- Let $f : L(H) \longrightarrow C_1(H)^*$ be a function defined with the same expression as in point 1. It can be proven that it is an isometry. Let $g \in C_1(H)^*$ and $u, v \in H$. We have $|| <\cdot, u > v||_1 =$
$||u||_H||v||_H|| <\cdot, \frac{u}{||u||_H} > \frac{v}{||v||_H}||_1 = ||u||_H||v||_H$. The map
$h(v) = g(<\cdot, u > v)$ is in H^* with norm 1. Hence, by the Riesz representation theorem, there is a unique g_u such that $h(v) =< v, g_u >= g(<\cdot, u > v)$ with $||g_u||_H^2 \leq ||g||||u||_H||g_u||_H$. Thus $B : H \longrightarrow H$, $u \longmapsto g_u$ is a bounded operator. Furthermore, if $||u||_H = ||v||_H = 1$ we have $g(<\cdot, u > v) =< v, B(u) >=<<\cdot, u > v, B >_2 =$
$tr(<\cdot, u > vB^*)$. Denoting by $C_{fin}(H)$ the space of finite rank operators, then $\overline{C_{fin}(H)}^{||.||_1} = C_1(H)$. Thus there is a unique extension of $g|_{C_{fin}(H)}$ to $C_1(H)$, which is g. Therefore every $g \in C_1(H)^*$ has the form $g(T) = tr(TB^*)$. So every g is the image of one $B \in L(H)$.
QED ■

Corollary: $C_1(H)$ is a Banach space (and thus a Banach algebra).

In fact, the analogy goes even further: just as $(l^p)^* \cong l^q$, if $\frac{1}{p} + \frac{1}{q} = 1$, we have $(C_p(H))^* \cong C_q(H)$.

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