1. Homework - 3

Exercise 1. *Show that a perfect t-error-correcting linear code of length n over $\mathbb{F}_q$ has exactly $\binom{n}{i}(q-1)^i$ cosets with coset leader of weight i, for $0 \leq i \leq t$ and no other cosets.*

Exercise 2. *Let C be a $[n,k,d]$-linear code over $\mathbb{F}_q$ such that for every $1 \leq i \leq n$ there exists a codeword $c = (c_1, \cdots, c_n)$ with $c_i \neq 0$.*

a) *Show that $\sum_{c \in C} w(c) = n(q-1)q^{k-1}$;*

b) *Show that $d \leq \frac{n(q-1)q^{k-1}}{q^k - 1}$;*

c) *Show that there cannot be a binary $[15,7,d]$-linear code for $d > 7$.*

Hint: *Consider the table whose rows are the codewords and compute the sum of the weights in two ways.*

Exercise 3 (**Plotkin bound**: general case)**.** *Suppose C is a (n,M,d) code over $\mathbb{F}_q$. Consider the table whose rows are the codewords and, for a given fixed column, denote by m_j the number of ocurrences of $j \in \mathbb{F}_q$ in that column.*
Denote by S the sum of distances between distinct codewords: $S = \sum_{c \neq c'} \operatorname{dist}(c, c')$.

a) *Verify that the contribution of the fixed column to S is*

$$\sum_{j \in \mathbb{F}_q} m_j(M - m_j) = M^2 - \sum_{j \in \mathbb{F}_q} m_j^2 \leq \theta M^2$$

where $\theta = \frac{q-1}{q}$.
Hint: *Apply Cauchy-Schwarz inequality to $\sum_j m_j$.*

b) *By summing over all ordered pairs of codewords, conclude that*

$$M(M-1)d \leq n\theta M^2 \Leftrightarrow d \leq n\theta \frac{M}{M-1}.$$

Exercise 4. *Consider the code $[6,3]$ C over $\mathbb{F}_3$ with generator matrix*

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

a) *Verify directly that C and $C^{\perp}$ have both minimal distance 3 and that*

$$A_0 = A_0^{\perp} = 1, \qquad A_3 = A_3^{\perp} = 6.$$

b) *Use the MacWilliams equalities to deduce the remaining weight coefficients.*

Exercise 5. *Show that, if $W_C(z)$ is the weight enumerator of a binary code C then*

a) $W_{C_e}(z) = \frac{1}{2}\left(W_C(z) + W_C(-z)\right)$ *where $C_e = \{c \in C : w(c) \equiv 0 \mod 2\}$;*

b) $W_{\hat{C}} = \frac{1}{2}\left((1+z)W_C(z) + (1-z)W_C(-z)\right)$ *where $\hat{C}$ is the parity extension of C.*

Exercise 6. *Find all possible weight enumerators of binary self-dual $[8,4]$ codes.*