Notes on Ordinary Differential Equations - VIII

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1 Stability of Orbits

Let $x \in \Omega$ be such that its orbit $\Psi(t,x)$ under the flow associated to the equation $\dot{x} = F(x)$, is defined for all $t > 0$. We introduce the following notions of stability:

Definition 1 *The solution $\Psi(t,x)$ is **stable** if, given $\varepsilon > 0$, there exists $\delta > 0$ such that $\|y - x\| < \delta$ implies that the solution $\Psi(t,y)$ is defined for every $t > 0$ and*

$$\|\Psi(t,y) - \Psi(t,x)\| < \varepsilon \ \forall t > 0.$$

*$\Psi(t,x)$ is **asymptotically stable** if it is stable and there exists δ_1 such that*

$$\|y - x\| < \delta_1 \implies \lim_{t \to +\infty} \|\Psi(t,y) - \Psi(t,x)\| = 0.$$

*If the solution is not stable, it is **unstable**.*

The interpretation of this concept is clear when considered in the extended phase space: a solution $\Psi(t,x)$ is stable if the graphs of all solutions with initial condition sufficiently close to x stay in a tubular neighborhood, with fixed given radius, of the graph of $\Psi(t,x)$; it is asymptotically stable if the same is true for a tubular neighborhood whose radius decreases to zero with t.
For a linear flow $e^{tA}x$ where the eigenvalues all have negative real part, the origin is asymptotically stable. If, for instance,

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

the origin is stable but not asymptotically stable. If A has some eigenvalue with positive real part, the origin is unstable for the linear flow $e^{tA}x$.

Example 2 *Consider the following equation in polar coordinates*

$$\begin{cases} \dot{r} = r(1-r^2) \\ \dot{\theta} = \sin^2(\frac{\theta}{2}) \end{cases}$$

The origin is clearly unstable. The other singularity with coordinates $(1,0)$ *is also unstable although for any point* (r,θ) *different from the origin,* $(1,0)$ *belongs to the its* ω*-limit.*

Remark 3 *It should be noticed that the second condition for asymptotic stability doesn't imply stability. A concrete examples is given by the equation*

$$\begin{cases} \dot{x} = x^2(y-x) + y^5 \\ \dot{y} = y^2(y-2x) \end{cases}$$

We will now focus on the stability of constant solutions, ie, equilibrium points. We start with an example:

Example 4 *Consider*

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\sin(x) \end{cases}$$

This equation models the movement of a pendulum, where x *represents the angular displacement and* y *the angular velocity, and all physical constants - the mass of the pendulum, lenght of the string to which it is attached, gravitation constant - have been rescaled to* 1*. As the vector field (which we will denote* $F(x,y)$*) is periodic in* x *with period* 2π*, it is natural to take the phase space to be a cylinder instead of the plane.*
As is well known,

$$E(x,y) = \frac{1}{2}y^2 + 1 - \cos(x),$$

which represents, up to a constant, the total energy of the system, is a ***first integral*** *of the equation, ie,* $\dot{E} = 0$*, and so all solutions are contained in level curves of* E*.*
There are, in the cylinder, two equilibrium points: $(0,0)$ *and* $(\pi,0)$*. The first one, corresponding to the pendulum being at rest in the downward vertical position, is stable, but not asymtotically stable: the neighborhood* $\{(x,y) : E(x,y) < 2\}$ *of* $(0,0)$ *is filled with closed orbits, corresponding to the usual periodic movements of the pendulum.*
The second equilibrium, which corresponds to the situation where the pendulum is at rest in the upward vertical position, is of course unstable. The analysis of the level set $E(x,y) = 2$ *containing* $(\pi,0)$ *shows that it contains two other orbits, each one with* α*-limit and* ω*-limit both equal to the equilibrium.*
To complete the picture, the trajectories contained in level sets $E(x,y) = c > 2$ *correspond to periodic motions of the pendulum making circles around its center of rotation.*

It is instructive to compare the phase portrait in the neighborhood of each one of the equilibrium points with the one for the flow of the linear equation defined by the derivative of the vector field at that point; in other words, this ***linearized flow*** *is obtained by replacing the vector field with its linear approximation at the equilibrium point. As*

$$DF(x,y) = \begin{pmatrix} 0 & 1 \\ -\cos(x) & 0 \end{pmatrix}$$

we have

$$DF(\pi,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which has eigenvalues 1 *with associated eigenvector* $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ *and* -1 *with associated eigenvector* $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. *The linearized equation equation at this equilibrium is of saddle type with the spaces generated by the two eigenvectors above being respectively the unstable and stable invariant spaces. The phase portrait of the nonlinear flow near* $(\pi,0)$ *is similar to this, even to the point that the two orbits that have the equilibrium as* ω*-limit approach that point with the same slope of the stable space of the linearized flow (and the same happens for the orbits with* α*-limit equal to* $(\pi,0)$ *and the unstable space of the linearized flow); this is easily verified if we recall that these orbits are the graphs of*

$$y = \pm\sqrt{2(1+\cos(x)}$$

and compute the limit of the slope of these functions as $x \to \pi$ *(we leave the details to the reader).*
As we'll see later, this is a general fact, determined by the property of $DF(\pi,0)$ *being hyperbolic.*

For the other equilibrium we have

$$DF(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which has purely imaginary eigenvalues; the orbits of the equation

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

are the equilibrium at the origin and circles centered at the origin traversed in the clockwise direction. so, again, the two phase portraits are similar. These two flows - the restriction of the nonlinear flow to a neighborhood of $(0,0)$ *and the linearized flow at that point - are in fact* ***equivalent*** *(see Remark 24 in Notes VI). However, in this case we don't have the stronger relation of conjugacy.*

Example 5 *We modify now the previous example by adding a friction term, proportional to velocity*

$$\left\{ \begin{array}{l} \dot{x} = y \\ \dot{y} = -\sin(x) - by \end{array} \right. ,$$

with $b > 0$.
We still have two equilibrium points at $(0,0)$ *and* $(\pi,0)$*; the linearized flow for the last one is again a saddle, as in the case of the "ideal pendulum" described in the previous example, but for the origin the linearized flow is now a sink, as the derivative of the vector field at* $(0,0)$ *is*

$$\left(\begin{array}{cc} 0 & 1 \\ -1 & -b \end{array} \right)$$

with eigenvalues $\frac{-b \pm \sqrt{b^2-4}}{2}$ *which have both negative real part.*
So, the orbits of the linearized flow at $(0,0)$ *approach this equilibrium, as* $t \to +\infty$*, either as spirals, if* $b < 2$*, or tangent to the direction of the eigenvector associated to the largest eigenvalue (see the description of two-dimensional phase portraits in Notes V).*

The function $E(x,y)$ *is no longer a first integral for the equation, as*

$$\partial_t E(x,y) = \partial_x E(x,y)\dot{x} + \partial_y E(x,y)\dot{y} =$$

$$= \sin(x)y + y(-by - \sin(x)) = -by^2.$$

We notice however that $\partial_t E(x,y) \leq 0$*, which means that energy is decreasing (or at least not increasing) along the orbits, a very plausible conclusion in view of the physical interpretation of* b.
We may still use $E(x,y)$ *in the interpretation of the phase portrait: level curves of* E *are now crossed by the flow lines from the outside to the inside (confirm this), suggesting that most solutions must approach the origin in a similar way of the linearized flow. Because this is a two-dimensional flow, we could confirm this applying the Poincaré-Bendixon Theorem: if* $c < 2$*, the set* $\{(x,y) : E(x,y) < c\}$ *is bounded and invariant, so solutions with initial condition in this set must have a nonempty* ω*-limit; but there can be no periodic orbits because this would contradict the fact that* $E(x,y)$ *decreases along orbits. So the* ω*-limit has to be the only equilibrium point in the set,* $(0,0)$.

Notice that there are, as in the previous example, "exceptional" orbits with ω*-limit or* α*-limit equal to the other equilibrium* $(\pi,0)$*: this may be confirmed by noticing that pieces of the level set* $E(x,y) = 2$ *and the half-lines* (π,y)*, with* $y > 0$ *or* $y < 0$*, form two anti-funnels for the flow.*

1.1 Liapunov Functions

The folowing concept and results are very useful for the analysis of stability of an equilibrium.

Definition 6 *Consider the equation $\dot{x} = F(x)$ where $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 vector field, and suppose $F(x_0) = 0$.*
*A **Liapunov function** for x_0 is a real C^1 function V defined in some open subset U of Ω containing x_0 such that*

a) $V(x_0) = 0$ and $V(x) > 0$ for all $x \neq x_0$;

b) $\partial_t V(x) \leq 0$ for all $x \neq x_0$.

*If strict inequality holds in b), V is a **strict Liapunov function**.*

$\partial_t V(x)$, or $\dot{V}(x)$ denotes the time derivative along the orbit of x, ie, $\partial_t V(\Psi(t, x))$.

Theorem 7 (Liapunov Direct Method) *If there exists a Liapunov function V for the equilibrium x_0 of $\dot{x} = F(x)$, then x_0 is stable.*
If V is strict, x_0 is asymptotically stable.

Proof. Let $B_\delta(x_0)$ be some open ball, centered at the equilibrium, contained in U and let $m = \min\{V(x) : \|x - x_0\| = \delta\}$; by hypothesis, $m > 0$; there exists, by continuity, there exists $U_1 \subset B_\delta(x_0)$ such that $V(x) < m/2$ for all $x \in U_1$. As V is nonincreasing along the orbits of the flow, we have $\Psi(t, x) \in B_\delta(x_0)$ for all $t > 0$. As δ is arbitrary, x_0 is stable.

Suppose now that V is stric. Let $x \in U_1$ as before; let $y \in \omega(x)$ and t_n an increasing sequence of time values such that

$$\lim_{n \to +\infty} \Psi(t_n, x) = y;$$

because V is strictly decreasing along orbits, we must have

$$V(y) < V(\Psi(t_n, x)$$

for all n, and so, in fact,

$$V(y) < V(\Psi(t, x), \ \forall t > 0.$$

Suppose $y \neq x_0$; then, by the same reasoning, $V(\Psi(1, y)) < V(y)$; by continuity, for any z sufficiently close to y we must also have $V(\Psi(1, z)) < V(y)$; but then, in particular, we would have $V(\Psi(t_n + 1, x)) < V(y)$, for sufficiently big n, which is a contradiction.
We conclude that $\omega(x) = x_0$ for any $x \in U_1$, ie, the equilibrium is asymptotically stable. ■

Example 8 *Consider*

$$\begin{cases} \dot{x} = x(x - 2y - 1) \\ \dot{y} = y(-5x + y - 2) \end{cases}$$

and $V(x, y) = x^2 + y^2$ *(a standard choice). We have*

$$\partial_t V(x, y) = x^2(x - 2y - 1) + y^2(-5x + y - 2)$$

which is strictly negative for every $(x, y) \neq (0, 0)$ *in an open neighborhood of the origin.*
We conclude that this equilibrium is asymptotically stable.

Exercise 9 *Complete the computations of the previous example.*

Exercise 10 *Let*

$$\begin{cases} \dot{x} = -x + 2x(x + y)^2 \\ \dot{y} = -y^3 + 2y^3(x + y)^2 \end{cases}$$

Confirm that $V(x, y) = x^2 + y^2$ *is a strict Liapunov function for the equilibrium* $(0, 0)$*, determining the largest set where* $\dot{V} < 0$*,*

We have already seen an application of Liapunov function in the example of the pendulum with friction. There the same $V(x, y)$ was not a strict Liapunov function. However, we were able to prove the asymptotic stability of the equilibrium at $(0, 0)$, because the orbits merely crossed the set where $\dot{V} = 0$ (the points $(x, 0)$), and so it was clear that V was still strictly decreasing along the orbits.
This observation may be generalized and we have the following improvement on the theorem:

Proposition 11 (La Salle's Invariance Principle) *Let* V *be a Liapunov function for the equilibrium* x_0 *of* $\dot{x} = F(x)$*. Suppose* S *is a closed neighborhood of* x_0 *positively invariant under the flow (ie,* $\Psi(t, x) \in S$*, for all* $x \in S$ *and* $t > 0$*). Let* $Z = \{x \in U : \dot{V}(x) = 0\}$*. If* $Z \cap S$ *does not contain any orbit other than* x_0*, then* x_0 *is asymptotically stable and*

$$\lim_{t \to +\infty} \Psi(t, x) = x_0, \ \forall x \in S.$$

Remark 12 *Given an equilibrium, the set of points that have that equilibrium as its* ω*-limit set is called the* ***basin of attraction*** *of the equilibrium. This notion generalizes to other* ω*-limit sets of the flow. The identification of the basin of attraction of invariant sets is of great practical and theoretical importance.*

Proof. Suppose $x \in S$ and $z \in \omega(x)$; so $\Psi(t,z) \in S$ for all $t \geq 0$, because S is closed and positively invariant; in fact, $\Psi(t,z) \in S$ for all t, as seen in Proposition 6 (iii) and v)) in Notes VII.
As in the proof of Liapunov's Theorem, we find that

$$V(z) \leq V(\Psi(t,x))$$

for all $t \geq 0$. We conclude, again by the same reasoning, that then $V(\Psi(t,z)) = V(z)$ for all $t \geq 0$: if for some positive s we had $V(\Psi(t,z)) < V(z)$, by continuity we would also have $V(\Psi(s,\Psi(t,x))) < V(z)$ for some large t, but $V(\Psi(s,\Psi(t,x))) = V(\Psi(s+t,x)) \geq V(z)$, a contradiction. As the same reasoning applies to any $\Psi(t,z)$ instead of z, we get that V is constant along the orbit of z.
This implies $\partial_t V(\Psi(t,z)) = 0$, for all t, ie, the orbit of z is contained in $Z \cap S$ and so we must have $z = x_0$. ■

Similar ideas give also a criterion of instability.

Proposition 13 *Suppose V is a C^1 function defined in an open neighborhood U of the equilibrium point x_0 of $\dot{x} = F(x)$, such that $V(x_0) = 0$ and $\partial_t V(x) > 0$ for all $x \in U \setminus \{x_0\}$. If, for every open set U' containing x_0 there is a point $xinU'$ such that $V(x) > 0$, then x_0 is not stable.*

Proof. Suppose that x_0 is stable. Let ε be such that the closed ball $\overline{B_\varepsilon(x_0)}$ is contained in U and also in the domain of F. There is then some δ such that

$$\|x - x_0\| < \delta \implies \Psi(t,x) \in B_\varepsilon(x_0)\, \forall t \geq 0;$$

also, by the continuity of V, there is some $\alpha > 0$ such that $V(x) \leq \alpha$ for every $x \in \overline{B_\varepsilon(x_0)}$.
By the hypothesis on V, there is $x \in B_\delta(x_0)$ such that $V(x) > 0$. The previous conditions on the definition of δ imply that $V(\Psi(t,x)) \leq \alpha$ for all $t \geq 0$; we will show that this does not happen, by finding a positive lower bound on the time derivative of V along the orbit of x. To be precise, we will now show that

$$\beta = \inf\{\partial_t V(\Psi(t,x)) : t \geq 0\} > 0;$$

suppose that $\beta = 0$; then there is an increasing and unbounded sequence of time values t_i such that

$$\lim_i \partial_t V(\Psi(t_i,x)) = 0;$$

because, by the stability of x_0, $\psi(t_i,x) \in \overline{B_\varepsilon(x_0)}$, a compact set, there is a subsequence, which we denote t_i again, such that $\Psi(t_i,x)$ converges to some y. By the continuity of $\partial_t V$, $\partial_t V(y) = 0$. The hypothesis made on V imply that we must have $y = x_0$. But, because $\partial_t V(\Psi(t,x)) > 0$,

$$V(\Psi(t_i,x)) > V(x) > 0$$

and so, passing to the limit, $V(x_0) \geq V(x) > 0$ which is impossible.
So $\beta > 0$; But then

$$V(\psi(t,x)) = V(x) + \int_0^t \partial_s V(\psi(s,x))ds \geq V(x) + \beta t,$$

for all positive t; taking t sufficiently large, we would get $V(\Psi(t,x)) > \alpha$, a contradiction. ■

1.2 Stability of equilibrium points by linearization

Since the application of the previous results depends on finding an appropriate function V, it is useful to have other results that detect the stability, asymptotic stability, or instability, of an equilibrium. We present a result in this direction:

Proposition 14 *Suppose x_0 is an equilibrium of $\dot{x} = F(x)$ and that all eigenvalues of $A = DF(x_0)$ have negative real part. Then x_0 is asymptotically stable.*

Proof. It is enough to consider $x_0 = 0$: for the general case, a translation $y = x - x_0$ gives $\dot{y} = G(y) = F(y + x_0)$, $G(0) = F(x_0) = 0$, and $DG(0) = DF(x_0)$. Using the Taylor formula for F at the origin, we have then

$$\dot{x} = F(x) = F(x) - F(0) = DF(0)x + g(x)$$

where g satisfies

$$\lim_{x\to 0} \frac{g(x)}{\|x\|} = 0.$$

From the study of linear equations with constant coefficients, we know that there exist $K \geq 1$ and $a > 0$ such that

$$\|e^{tA}x\| \leq Ke^{-at}\|x\|, \forall t \geq 0;$$

there is also a $\delta > 0$ such that

$$\|x\| < \delta \implies \|g(x)\| \leq \frac{a}{2K}\|x\|.$$

We will prove that any x with $\|x\| < \delta/K$ is in the basin of attraction of 0. For this we start by noticing that $\Psi(t,x)$ is a solution of the linear, non-homogeneous equation

$$\partial_t u(t) = Au(t) + g(u(t)),$$

and so we may apply the general formula, obtained by the method of "variation of constants" (see Proposition 13 in Notes IV), to get

$$\Psi(t,x) = e^{tA}x + \int_0^t e^{(t-s)A} g(\Psi(s,x))\, ds.$$

We would now want to use this to obtain an estimate of $\|\Psi(t,x)\|$, but, in order to use the estimate established for g, we must assume that $\|\Psi(s,x)\| < \delta$. So let

$$t_1 = sup\{t : \|\Psi(t,x)\| < \delta\},$$

and take $t \leq t_1$; we have

$$\|\Psi(t,x)\| \leq Ke^{-at}\|x\| + \int_0^t Ke^{-(t-s)a}\|g(\Psi(s,x))\|ds \leq$$

$$\leq Ke^{-at}\|x\| + \int_0^t Ke^{-(t-s)a}\frac{a}{2K}\|\Psi(s,x)\|ds.$$

Multiplying by e^{at},

$$e^{at}\|\Psi(t,x)\| \leq K\|x\| + \int_0^t e^{sa}\frac{a}{2}\|\Psi(s,x)\|ds,$$

ie, the function $v(t) = e^{at}\|\Psi(t,x)\|$ satisfies

$$v(t) \leq K\|x\| + \int_0^t \frac{a}{2}v(s)ds,$$

and we are in the position of applying Gronwall's inequality and obtain, after a few simple computations

$$v(t) \leq K\|x\|e^{\frac{a}{2}t}.$$

Then, multiplying by e^{-at},

$$\|\Psi(t,x)\| \leq K\|x\|e^{-\frac{a}{2}t} \leq \delta e^{-\frac{a}{2}t} < \delta;$$

taking the limit we find that, if $t_1 < +\infty$, we would have $\|\Psi(t_1,x)\| \leq \delta e^{-\frac{a}{2}t_1} < \delta$ and so, by continuity, the same would happen for some $t > t_1$, contradicting the definition of t_1. So $t_1 = +\infty$ and we have that, for every x with $\|x\| < \delta/K$ and $t > 0$,

$$\|\Psi(t,x)\| \leq \delta e^{-\frac{a}{2}t}$$

which implies asymptotic stability. ■

This proposition illustrates a general idea, that we confirmed in several examples, that may be stated as

15 (Linearization Principle) *In "most" cases, the phase portrait of the flow of $\dot{x} = F(x)$, in the neighborhood of an equilibrium point x_0, is qualitatively the same as the one of the linear flow $e^{tA}x$ near the origin, where $A = DF(x_0)$.*

This principle is realized in the following important theorem, which largely supersedes the previous proposition, and that will be discussed later:

Theorem 16 (Grobman-Hartman Theorem) *Let x_0 be an equilibrium point of the equation $\dot{x} = F(x)$, and $\Psi(t,x)$ denote its flow. If the eigenvalues of $A = DF(x_0)$ all have non-zero real part (A is an hyperbolic linear vector field), then there exist open neighborhoods U of x_0 and V of 0, and an homeomorphism $h : U \to V$ mapping x_0 to 0, that conjugates Ψ with the linear flow:*

$$h(\Psi(t,x)) = e^{tA}h(x),$$

for all (t,x) such that x and $\Psi(t,x)$ are in U.

Grobman-Hartman theorem is complemented by an equally fundamental result, of which we state the most basic version:

Theorem 17 (Stable Manifold for an hyperbolic equilibrium) *Let x_0 be an equilibrium point for $\dot{x} = F(x)$, where $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 vector field, and suppose that $DF(x_0)$ is an hyperbolic matrix. Suppose E^s, the invariant stable space of $DF(x_0)$ generated by the generalized eigenvectors associated to the eigenvalues with negative real part, has dimension k.*
Then the set

$$W^s(x_0) = \{x \in \Omega : \omega(x) = x_0\}$$

is a k-dimensional C^1 manifold and the tangent space of $W^s(x_0)$ at x_0 is E^s. Furthermore, there exists a neighborhood U of x_0, and constants $K \geq 1$, $a > 0$ such that

$$\|\Psi(t,x) - x_0\| \leq Ke^{-at}\|x\|, \ \forall x \in W^s(x_0) \cap U, \ t \geq 0.$$

A similar result gives the existence of an unstable manifold, of dimension $n-k$, tangent at x_0 to the unstable space E^u.

Example 18 *Consider the two dimensional equation*

$$\dot{x} = -x \qquad \dot{y} = 2y + x^2.$$

The origin is the only equilibrium point, and the linearization of the vector field there is

$$\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$$

so the stable space for the linear equation is the 1-dimensional space generated by $(1,0)$ and the unstable space is generated by $(0,1)$.
Obviously, this last subspace is invariant under the nonlinear flow too and consists of the union of the equilibrium and two orbits with α-limit equal to $(0,0)$. The analysis of the phase portrait shows the existence of two orbits that have the origin as ω-limit: it suffices, for instance, to notice that the lines $y = 0$ and

$y = -\frac{x^2}{2}$ (the horizontal isocline) constitute two anti-funnels for the flow. The fact that the divergence of F is everywhere equal to 1 shows that these are the only solutions with that behaviour.
These facts are, however, also a consequence of the general theorems stated before. And they may be confirmed in this example, by solving explicitly the equation: the solution with initial condition $x(0) = a$ and $y(0) = b$ is given by

$$x(t) = ae^{-t} \qquad y(t) = (b + \frac{a^2}{4})e^{2t} - \frac{a^2}{4}e^{-2t};$$

so the ω-limit is $(0,0)$ exactly when $b + \frac{a^2}{4} = 0$, ie, $\{(x, -\frac{x^2}{4}) : x \in \mathbb{R}\}$ is the unstable manifold of $(0,0)$ for the flow.

When the linearization of the vector field at an equilibrium is not hyperbolic, the behaviour of nearby orbits depends on the nonlinear terms. We consider here only the two-dimensional case: suppose that (x_0, y_0) is an equilibrium for

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = F \begin{pmatrix} x \\ y \end{pmatrix}$$

and that $DF((x_0, y_0))$ has purely imaginary eigenvalues.
It may happen that the orbits of the flow near the equilibrium are periodic, as in the linear equation defined by $DF((x_0, y_0))$, or they may have the origin as its ω-limit, or as its α-limit. These possibilities are, for example, illustrated by the equation

$$\begin{cases} \dot{x} = -y + ax(x^2 + y^2) \\ \dot{y} = x + ay(x^2 + y^2) \end{cases}$$

for different values of a: this may be verified studying $\partial_t V(x, y)$ for $V(x, y) = x^2 + y^2$.
We recall nonetheless that the eigenvalues of $DF((x_0, y_0))$ still give some information on the geometry of the orbits, namely, they can not approach the equilibrium, as $t \to \pm\infty$, along any definite direction, and must wind around it infinitely many times (see the discussion at the end of section 1.1 of Notes VI).