Numerical Solution of Nonlinear Second Kind
Two-Dimensional Volterra Integral Equations Using
Extrapolation Methods

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Abstract. We consider the numerical solution of a class of two-dimensional nonlinear Volterra integral equations of the second kind. We start from the results of [1], where solutions of these equations were computed by the Euler and trapezoidal methods, in the linear case, and we extend these methods to nonlinear case. Then we use the Richardson extrapolation to improve the accuracy of the numerical results.

Keywords: Two-dimensional nonlinear Volterra integral equations, Euler method, Trapezoidal method, Richardson extrapolation

PACS: 02.60.Nm

INTRODUCTION

The following nonlinear two-dimensional Volterra integral equation is considered:

\[
U(X,T) = \int_0^T \int_0^X H(X,T,x',t',U(x',t'))dx'dt' + \int_0^X G(X,T,x',U(x',T))dx' \\
+ \int_0^T F(X,T,t',U(x',t'))dt' + R(X,T), \quad (X,T) \in \Omega = [0,\bar{X}] \times [0,\bar{T}].
\]

(1)

There are different motivations for studying integral equations of the form (1). First they may arise from certain hyperbolic differential equations (see [2]). On the other hand, when solving two-dimensional Volterra equations of the first kind, they are often reduced, by differentiation, to equations of the form (1), as it was done in [1]. Finally, a class of Cauchy problems for nonlinear partial differential equations, which include the so-called telegraph equation, can be also reduced to equations of the form (1).

In the present work, we will focus on the application of extrapolation techniques to improve the accuracy of results obtained by Euler and trapezoidal methods.

PRELIMINARY RESULTS

Existence and uniqueness

In [1], the existence and uniqueness of solution of (1) was proved in the case where \( F,G \) and \( H \) are linear functions of \( U \), using an argument based on the Picard iterative method. As the authors of this paper suggest, this argument can be extended to the case where \( F,G \) and \( H \) are nonlinear, provided that they satisfy the Lipschitz conditions:

\[
|G(X,T,x',U(x',T)) - G(X,T,x',U'(x',T))| \leq C_1|U(x',T) - U'(x',T)|,
\]

\[
|F(X,T,t',U(x',t')) - F(X,T,t',U'(x',t'))| \leq C_2|U(x',t') - U'(x',t')|,
\]

\[
|H(X,T,x',t',U(x',t')) - H(X,T,x',t',U'(x',t'))| \leq C_3|U(x',t') - U'(x',t')|, \quad \forall X,x' \in [0,\bar{X}], \quad \forall T,t' \in [0,\bar{T}].
\]

(2)

We can then express this result in the form of the following theorem.

Theorem 1. Let the functions \( F,G \) and \( H \) in (1) satisfy conditions (2). Moreover, let \( R \) be such that

\[
|R(X,T)| \leq R_0, \quad (X,T) \in \Omega.
\]

(3)
Then equation (1) has a unique solution in \( C(\Omega) \).

The proof of this theorem for the linear case is given in [1]. In the nonlinear case, as we noticed above, it is possible to construct a similar proof, where the conditions of boundedness of the kernels of the integral operators are replaced by the conditions (2).

### Numerical methods and their convergence

In this subsection we present two numerical methods used in [1] for solving (1) in the linear case and we discuss how they can be adapted to the case of a nonlinear equation.

Let \( N \) and \( M \) are two positive integers and define \( h = \frac{x}{N} \) and \( k = \frac{y}{M} \). Consider the net of grid points

\[
\Omega_{h,k} = \{(X_n, T_m) = (nh, mk) : n = 0, 1, \ldots, N, m = 0, 1, \ldots, M\},
\]

We shall call a grid function any function defined on \( \Omega_{h,k} \). When applying the Euler method we search for a grid function \( U_{nm} \) which satisfies the following system of nonlinear equations

\[
U_{nm} = h k \sum_{i=1}^{n} \sum_{j=1}^{m} H(X_n, T_m, X_i, T_j, U_{ij}) + h \sum_{i=1}^{n} G(X_n, T_m, X_i, U_{im})
\]

\[
+ k \sum_{j=1}^{m} F(X_n, T_m, T_j, U_{nj}) + R_{nm}, \quad n = 0, 1, \ldots, N, m = 0, 1, \ldots, M,
\]

where \( R_{nm} = R(X_n, T_m) \).

A detailed numerical analysis of the Euler method for the linear case of equation (1) was presented in [1], where it was proved that this method has first order convergence, provided that \( F, G \) and \( H \) are smooth enough functions. This result can be also extended to the nonlinear case, which will be subject of a forthcoming paper. A numerical example will be presented in the next section.

In the case of the trapezoidal method we look for an approximate solution \( U_{nm} \) which satisfies the following system of nonlinear equations:

\[
U_{nm} = \frac{h k}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} \left[ H(X_n, T_m, X_i, T_j, U_{i-1,j}) + \frac{H(X_n, T_m, X_i, T_j, U_{i-1,j})}{2} + H(X_n, T_m, X_i, T_j, U_{ij}) \right]
\]

\[
+ \frac{h}{2} \sum_{i=1}^{n} \frac{G(X_n, T_m, X_i, U_{i-1,m}) + G(X_n, T_m, X_i, U_{im})}{2}
+ \frac{k}{2} \sum_{j=1}^{m} \left[ F(X_n, T_m, T_j, U_{n,j-1}) + F(X_n, T_m, T_j, U_{n,j-1}) \right] + R_{nm},
\]

where \( R_{nm} = R(X_n, T_m) \) and \( n = 0, 1, \ldots, N, m = 0, 1, \ldots, M \).

In [1] a numerical example was presented for the linear case of equation (1), which suggests that the trapezoidal method has convergence order 2. In the next section, we will display some numerical results, which support the conjecture that this is also valid in the nonlinear case.

When comparing the numerical methods given by equations (5) and (6) with their analogs for the linear case, the main difference is that here we have to solve a system of \( (N+1) \times (M+1) \) nonlinear equations. With this purpose we have used the Newton method, taking as initial approximation the null vector. In all the considered examples, convergence was reached after no more than 5 iterations.

### ASYMPTOTIC ERROR EXPANSIONS AND EXTRAPOLATION

For the successful application of extrapolation methods it is required that the discretization error of the obtained approximate result satisfies a certain power expansion.

An important theorem which states the conditions under which a discretization method allows an asymptotic error expansion is given in [3]. This theorem holds in a very general context, concerning differential and integral equations.
In the particular case of integral equations, the main source of such asymptotic error expansions is the Euler-Maclaurin formula. This formula was first applied to one-dimensional integration [4], but it was then extended by many authors who obtained asymptotic error expansions for the numerical solution of two-dimensional Volterra and Fredholm equations (see for instance [5] and [6]).

Based on the results of the cited papers, and taking into account that in the present work, we are dealing with problems whose solutions are infinitely differentiable, we will assume that the error of the Euler method, when applied to equation (1), with \( h = k \), allows an expansion of the form:

\[
U(x_n, t_m) - U_{\text{num}} = b_1(x_n, t_m)h + b_2(x_n, t_m)h^2 + \ldots + b_k(x_n, t_m)h^k + O(h^{k+1}),
\]

where \( b_1, b_2, \ldots, b_k \) are independent of \( h \) functions. Having second order convergence, the trapezoidal method usually allows an error expansion in even powers of \( h \). Namely, in the following examples we will assume that for this method an expansion of the following form is valid:

\[
U(x_n, t_m) - U_{\text{num}} = c_1(x_n, t_m)h^2 + c_2(x_n, t_m)h^2 + \ldots + c_k(x_n, t_m)h^{2k} + O(h^{2k+2}).
\]

Assuming that the expansion (7) (or (8) holds, the Richardson extrapolation can be applied to improve the convergence of the numerical results obtained by the Euler method (resp. trapezoidal method). In each case, we consider an initial sequence of approximations (\( E_0 \)), obtained by the considered method, using different stepizes: \( h = \frac{1}{2}, h = \frac{1}{4}, h = \frac{1}{8}, \) etc. At each step of the extrapolation process, we obtain a new sequence (\( E_i \)) with a higher convergence order. Provided that the expansion (7) or (8) is valid, and that the initial sequence of approximations has at least \( k + 1 \) terms, \( k \) steps of the extrapolation process can be performed and the convergence order is improved at each step. For formulae and details see [7].

The following numerical example illustrates this process.

**NUMERICAL EXAMPLE**

Consider a two-dimensional Volterra integral equation of the form (1), where

\[
\begin{align*}
H(X, T, x', t', U(x', t')) & = \frac{1}{4} \left\{ \sin(U(x', t')) \cos(\frac{x'-t'}{2}) \sin(\frac{x'+t'}{2}) + \cos(U(x', t')) \sin(\frac{x'-t'}{2}) \cos(\frac{x'+t'}{2}) \right\}, \\
G(X, T, x', U(x', T)) & = -\frac{1}{4} \left\{ \sin(U(x', T)) + \cos(U(x', T)) \right\}, \\
F(X, T, t', U(X, t')) & = -\frac{1}{4} \left\{ \sin(U(X, t')) - \cos(U(X, t')) \right\}, \\
R(X, T) & = 0 X X \int_0^X \int_0^X G(X, T, x', U(x', 0))dx' - \int_0^T F(X, T, t', U(0, t'))dt' + U(X, 0) + U(0, T) - U(0, 0).
\end{align*}
\]

The exact solution of this equation is \( U(x, t) = \sin(\frac{x^2}{2}) \sin(\frac{t^2}{2}) \). We will now illustrate how this equation can be approximated by the Euler and trapezoidal methods and how the obtained results can be improved by means of the Richardson extrapolation. Let \( 0 \leq X, T \leq 1 \) and consider the grid \( \Omega^{h \times h} \), as defined by (4), with \( X = T = 1 \) and \( h = k \).

The Euler and the trapezoidal methods, given by (5) and (6), respectively, were applied to the numerical solution of this problem.

Let the \( L_{\infty} \) error norm and the corresponding convergence rate be defined by

\[
\| e(h) \|_{\infty} := \max \{ \| U(x_i, T_j) - U_{\text{ref}} \|_{\infty}, 0 \leq i \leq N, 0 \leq j \leq M \}, \quad \alpha := \frac{\log(\| e(h) \|_{\infty}/\| e(h/2) \|_{\infty})}{\log(2)}.
\]

The Newton iteration method is applied to solve the nonlinear systems of equations (5) and (6) in the case \( h = k \). Tables 1 and 2 show the \( L_{\infty} \) errors in the case of the Euler and trapezoidal methods, respectively. In these tables \( \alpha \)’s, \( i = 0, 1, \ldots, 4 \), denote the \( L_{\infty} \) rates of convergence.

All the computations were performed in a personal computer using MATLAB codes.

**CONCLUSIONS**

When using the Euler method, we expect that the initial sequence of approximations \( E_0 \) has first order convergence, and at each step of the extrapolation process the convergence order is improved by 1 unit. After 3 steps, the sequence \( E_3 \) is obtained, which is expected to have 4-th order convergence.
TABLE 1. $L_{\infty}$ error norms for the Euler method with Richardson extrapolation. $E_0$ denotes the initial sequence, $E_i$ is the i-th transform in the extrapolation process.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$E_0$</th>
<th>$\alpha_0$</th>
<th>$E_1$</th>
<th>$\alpha_1$</th>
<th>$E_2$</th>
<th>$\alpha_2$</th>
<th>$E_3$</th>
<th>$\alpha_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$2.69e-2$</td>
<td>1.10</td>
<td>$1.80e-3$</td>
<td>2.05</td>
<td>$2.21e-5$</td>
<td>1.43</td>
<td>$5.96e-6$</td>
<td>3.86</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>$1.26e-2$</td>
<td>1.05</td>
<td>$4.33e-4$</td>
<td>1.87</td>
<td>$8.26e-6$</td>
<td>2.66</td>
<td>$4.09e-7$</td>
<td>3.79</td>
</tr>
<tr>
<td>$\frac{1}{4}$</td>
<td>$6.06e-3$</td>
<td>1.03</td>
<td>$1.18e-4$</td>
<td>1.96</td>
<td>$1.31e-6$</td>
<td>2.81</td>
<td>$2.95e-8$</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{8}$</td>
<td>$2.97e-3$</td>
<td>1.01</td>
<td>$3.04e-5$</td>
<td>1.98</td>
<td>$1.87e-7$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{16}$</td>
<td>$1.47e-3$</td>
<td>1.01</td>
<td>$7.74e-6$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{32}$</td>
<td>$7.33e-4$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

TABLE 2. $L_{\infty}$ error norms for the trapezoidal method with extrapolation method. $E_0$ denotes the initial sequence, $E_i$ is the i-th transform in the extrapolation process.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$E_0$</th>
<th>$\alpha_0$</th>
<th>$E_1$</th>
<th>$\alpha_1$</th>
<th>$E_2$</th>
<th>$\alpha_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$6.25e-3$</td>
<td>2.10</td>
<td>$1.46e-4$</td>
<td>4.16</td>
<td>$1.03e-6$</td>
<td>6.33</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>$1.45e-3$</td>
<td>2.02</td>
<td>$8.15e-6$</td>
<td>4.03</td>
<td>$1.28e-8$</td>
<td>2.84</td>
</tr>
<tr>
<td>$\frac{1}{4}$</td>
<td>$3.57e-4$</td>
<td>2.00</td>
<td>$4.98e-7$</td>
<td>3.93</td>
<td>$1.78e-9$</td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{8}$</td>
<td>$8.95e-5$</td>
<td>2.00</td>
<td>$3.27e-8$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\frac{1}{16}$</td>
<td>$2.24e-5$</td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

In the case of the trapezoidal method, the initial sequence is expected to have quadratic convergence, while each subsequent transform $E_i$ must be convergent with order $2i+2$.

Though the numerical results, in general, agree with this description, in some cases, specially in the last two transforms, there is some discrepancy between the experimental rates of convergence and the theoretical ones. This can be explained by the fact that the results of these steps of the extrapolation process are strongly affected by the error of the Newton method (which is of the order of $10^{-9}$).

Nevertheless, the error norms of the results obtained by the extrapolation process show that this algorithm, being very cheap from the computational point of view, provides a very significant improvement of the accuracy.

REFERENCES