Asymptotic and Numerical Approximation of a Boundary Value Problem for a Quasi-linear Differential Equation

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Abstract: In this work we are concerned about a singular quasi-linear boundary value problem involving the p-laplacian which arises in mathematical models of fluid mechanics and other fields of physics. We analyze the asymptotic behavior of the solutions of the considered ordinary differential equation near the singularities. The obtained expansions are generalizations of the previously obtained results for the classical laplacian [8]. The computational methods introduced in that work were adapted to the new problem and various numerical examples are presented.

Key–Words: Singular boundary value problem, p-laplacian, upper and lower solutions, one-parameter families of solutions, asymptotic expansions.

1 Introduction

The mathematical modelling of various phenomena of mechanics and physics leads to the following Dirichlet problem for a quasi-linear elliptic equation:

\[-\text{div} \left( |\nabla g|^{p-2} \nabla g \right) = f(|x|, g), \quad x \in B; \]
\[g(x) > 0, \quad x \in B; \]
\[g(x) = 0, \quad x \in \partial B, \quad (1)\]

where $B$ is the unit ball centered at the origin in $\mathbb{R}^N$, $N > 1$, $p > 1$.

The nonlinear differential operator in the left-hand side of (1) is known as the p-laplacian and, in the case of $p = 2$, reduces to the usual laplacian.

It is worth to remark that the laplacian operator arises in the modeling of many classical problems of fluid dynamics, as the result of the application of a "linear flow law" (see [2]). For example, in the case of heat conduction problems, if we denote by $g$ the steady state temperature, according to the Fourier’s law, the heat flow vector is proportional, in module, to $|\nabla g|$. When this law is applicable, the heat diffusion is given by $\text{div}(\nabla g)$, which yields the laplacian operator. However, in the modeling of certain processes, for example, the melting of ice at a constant temperature, the heat flow is better described by a generalized Fourier’s law, which states that the flow vector is a nonlinear function of the temperature gradient, with the form $|\nabla g|^{p-2} \nabla g$ (see, for example, [1]). Then the heat diffusion will be represented by the p-laplacian.

Similar situations, when the generalization of classical linear problems leads to nonlinear ones, involving the p-laplacian, also occur when modeling flows of non-newtonian fluids [2], and in elasticity theory, when modelling the deformation of nonlinear membranes [3].

One is often interested in radial solutions of problem (1), that is, solutions that depend only on $|x|$. In that case, problem (1) reduces to a boundary value problem for an ordinary differential equation:

\[-r^{1-N} \left( |g|^{p-1} g' \right)' = f(r, g), \quad r \in (0, 1); \]
\[g'(0) = 0, g(1) = 0, \quad (2)\]

where $N \geq 1$ is the space dimension. We shall assume that $f : (0, 1) \times (0, \infty) \to \mathbb{R}$ is continuous in the considered domain and has singularities at $y = 0$ or $r = 0$.

The existence and the properties of solutions to problem (2), under different assumptions on the function $f$, were investigated by many authors. The existence and uniqueness of solution in the case $p = 2, N = 1$, with $f(r, g) = r g^n, n < 0$ was proved by Nachman and Callegari [4], in connection with a boundary layer problem in fluid mechanics. In [5], [6],[7] and [8] the authors also considered the case $p = 2$, but with $f(r, g) = r^\sigma g^n$, $\sigma > -1$. In these works the asymptotic behavior of the solutions near the singularity at $r = 1$ has been analyzed and numerical methods were introduced, which take into account
this behavior. Moreover, upper and lower solutions were determined and, in some particular cases, an explicit formula for the exact solution was obtained.

Concerning the case \( p \neq 2 \), in [9] and [10] Wong has obtained sufficient conditions for existence and uniqueness of solutions of problem (2), for \( N = 1 \). Uniqueness results for this case were obtained also by Naito [11].

In a recent work, Jin, Yin and Wang [12], using the method of lower and upper solutions, have investigated the existence of solutions to problem (2) in the general case \( N \geq 1 \).

In the present paper, we will continue the investigation of problem (2) in the case \( N = 1 \), for \( p > 1 \) and \( f(r, g) = ar^\sigma g^n \), where \( a \) is a negative constant. The values of \( \sigma \) and \( n \) for which the existence of solution is guaranteed depend on \( p \). According to the Existence Theorem of [9], a sufficient condition for the existence of solution is \( \sigma \geq 0 \), \( n - p + 1 < 0 \), \( p \geq 2 \). On the other hand, uniqueness of solution in this case is guaranteed by Theorem 2.3 of [10]. As we shall see in the Section 4, existence of solution can be guaranteed under weaker conditions.

When \( n < 0 \) this problem has a singularity at \( r = 1 \). For such values of \( n \), the second derivative of \( g \) is not continuous at this point. Moreover, the first derivative has also a discontinuity at this point, if \( n \leq -1 \). On the other hand, the problem has a singularity at \( r = 0 \), if \( \sigma < 0 \) or \( p > 2 \). This will be discussed in the next sections.

The outline of this paper is as follows. In Section 2 we will analyze the asymptotic behavior of the one-parameter family of solutions which satisfy the condition \( \lim_{r \to 0^+} g(r) = 0 \). In Section 3, we will investigate the asymptotic behavior of the solutions of equation (2) that satisfy \( \lim_{r \to r_0^+} g(r) = 0 \) (with \( r_0 > 0 \)). In section 4, we will obtain lower and upper solutions for the considered problem and use them to apply existence results. We will also show that in certain cases the exact solution can be found in explicit form. In Section 5 we will describe the computational method used to approximate the solution and present some numerical solutions. Finally, in Section 6, we write the main conclusions of the work.

\section{Behavior of the solution in the neighborhood of \( r = 0 \)}

Consider the Cauchy problem:

\[ (|g'(r)|^{p-2} g'(r))' = ar^\sigma g^n(r), \quad 0 < r < r_0, \quad (3) \]
\[ \lim_{r \to 0^+} g'(0) = 0. \quad (4) \]

Assume that in (3)

\[ p > 1, \sigma > -1, n < 0. \quad (5) \]

As pointed out in the introduction, equation (3) has a singularity at \( r = 0 \) when \( \sigma < 0 \) or \( p > 2 \). In this section we will analyze the asymptotic behavior of the solutions of the problem (3),(4) when \( r \to 0^+ \). Our main result is resumed in the following proposition.

\textbf{Proposition 1} Under the restrictions (5), the Cauchy problem (3)-(4) has, in the neighborhood of \( r = 0 \), a one parameter family of solutions that can be represented by:

\[ g_1(r, b) = C_1 - b r^k \left[ 1 - \frac{bn(1 + \sigma)}{2C_1(2p + 2\sigma - 1)} r^k + o(r^k) \right], \quad (6) \]

where \( k = \frac{p+\sigma}{p-1} \), \( C_1 = \left( \frac{-a(p-1)^{p-1}}{b^{p-1}(p+\sigma)^{p-1}(\sigma+1)} \right)^{-\frac{1}{n}} \) and \( b > 0 \) is the parameter.

\textbf{Proof}. In the neighborhood of \( r = 0 \), let us look for a solution of (3)-(4) in the form

\[ g(r) = C_1 - C_2 r^k [1 + o(1)], \quad g'(r) = -k C_2 r^{k-1} [1 + o(1)], \quad g''(r) = -k(k - 1) C_2 r^{k-2} [1 + o(1)], \quad r \to 0^+. \quad (7) \]

Substituting in (3) we obtain

\[ \lim_{r \to 0^+} k^{p-1} (k - 1) (p - 1) C_2^{p-1} r^{(p-1)k-p-\sigma C_1 n} \left[ 1 - C_2 r^k \right]^{-\frac{n}{n}} = a, \quad (8) \]

which implies \( k = \frac{p+\sigma}{p-1}, k - 1 = \frac{\sigma+1}{p-1} \), and with \( C_2 = b > 0 \) we have \( C_1 = \left( \frac{-a(p-1)^{p-1}}{b^{p-1}(p+\sigma)^{p-1}(\sigma+1)} \right)^{-\frac{1}{n}} \).

Defining \( y(r) \) by \( g(r) = C_1 - C_2 r^k [1 + y(r)] \), we obtain the Cauchy problem in \( y \)

\[ \left( \frac{(p-1)^2}{\sigma+1} \left[ 1 + y + \frac{r}{\sigma+1} y' \right] \right)^{p-2} \times \left[ (k - 1) k (1 + y) + 2 k r y' + r^2 y'' \right] = \left[ 1 - \frac{b}{C_2} r^k (1 + y) \right]^{-n}, \quad (9) \]
\[ y(0) = \lim_{r \to 0^+} ry'(r) = 0, \quad (10) \]

where \( r = 0 \) is a regular singular point of (9).

For each \( b \neq 0 \), this problem has a particular solution that can be represented by

\[ y_{par}(r, b) = \sum_{l=0, j=0, l+j \geq 1}^{+\infty} y_{l,j} (b) r^{l+j \frac{p+\sigma}{p-1}}. \]
where $0 < r \leq \delta (b), \delta (b) \geq 0$ and the coefficients $y_{0,j}$, depending on $b$, may be determined substituting $y_{par}$ in (9), which gives in the case $l = 0, j = 1$

$$y_{0,1} = - \frac{bn(1 + \sigma)}{2C'_1(2p + 2\sigma - 1)}.$$  

Writing out the leading linear homogeneous terms of (9) for solutions that satisfy (10) we obtain the equation

$$r^2 y'' + \frac{3p + p\sigma - 2}{p - 1} r y' + \frac{(p + \sigma)(\sigma + 1)}{p - 1} y = 0, \quad r \to 0^+$$

whose characteristic exponents are $\lambda_1 = -\frac{p+\sigma}{p-1} < 0$

and $\lambda_2 = -(\sigma + 1) < 0$. Therefore, following the methods of [14], for each $b \neq 0$ this problem has no other solution than $y_{par}(r,b)$.  

From Proposition 1 we can conclude that if $g$ is a solution of problem (3)-(4), then $g''$ will have a discontinuity at $r = 0$ whenever $k < 2$, that is, when $\sigma < p - 2$.

### 3 The singularity at $r = r_0$

In this section we will investigate the behavior of the solution in the neighborhood of a certain point $r_0 > 0$, where the solution vanishes. Let us consider the singular Cauchy problem:

$$\left| g'(r) \right|^{p-2} g'(r) = ar^\sigma g^n(r), \quad 0 < r < r_0, \quad g(r_0) = \lim_{r \to r_0^-} \left[ (r_0 - r) g'(r) \right] = 0. \quad (12)$$

As we shall see, the asymptotic behavior of the solutions when $r \to r_0^-$ may change significantly, depending on the value of $n$, as described in the following proposition.

**Proposition 2** Assume that in (11) $p > 1, \sigma > -1, n < 0$ and $r_0 > 0$. The Cauchy problem (11)-(12) has, in the neighborhood of the singular point $r = r_0$, a one parameter family of solutions that can be represented by:

$$g(r, c) = \begin{cases} 
g_2(r, c), & \text{if } n < -1 \text{ and } n \neq -2 - \frac{1}{p-1}; 
g_3(r, c), & \text{if } n = -2 - \frac{1}{p-1}; 
g_4(r, c), & \text{if } n > -1; 
g_5(r, c), & \text{if } n = -1; \end{cases} \quad (13)$$

as $r \to r_0^-$, where $c$ is the parameter,

$$g_2(r, c) = \left( \frac{ar_0^{(p-1)n}r}{p(p-1)(p-n)} \right)^{\frac{1}{p-1-n}} \left( r_0 - r \right)^{\frac{1}{p-1-n}} \left[ 1 + c (r_0 - r)^{\frac{1}{p-1-n}} \right], \quad (14)$$

$$\mu = \min \left\{ 1, \frac{p+1}{p-1-n} \right\};$$

$$g_3(r, c) = \left( \frac{ar_0^{(p-1)n}r}{p(p-1)(p-n)} \right)^{\frac{1}{p-1-n}} \left( r_0 - r \right)^{\frac{1}{p-1-n}} \left[ 1 + c (r_0 - r)^{\frac{1}{p-1-n}} \right][1 + c_1 (r_0 - r)^{\frac{1}{p-1-n}}]^{\frac{1}{1+c_1}} \left[ r_0 - r \right] \left[ (r_0 - r)^{\frac{1}{p-1-n}} \right]^{\frac{1}{1+c_1}}, \quad (15)$$

$$g_4(r, c) = \left( \frac{ar_0^{(p-1)n}r}{p(p-1)(p-n)} \right)^{\frac{1}{p-2-n}} \left( r_0 - r \right)^{\frac{1}{p-2-n}} \left[ 1 + c (r_0 - r)^{\frac{1}{p-2-n}} \right][1 + c_1 (r_0 - r)^{\frac{1}{p-2-n}}]^{\frac{1}{1+c_1}} \left[ r_0 - r \right] \left[ (r_0 - r)^{\frac{1}{p-2-n}} \right]^{\frac{1}{1+c_1}}, \quad (16)$$

and the coefficients $y_{0,1}, b_1, c_0, c_1, d_0, \ldots$, may be determined substituting (15) in (11).

**Proof.** First, consider the case $n < -1$. It can be easily proved that in the neighborhood of $r = r_0$, a solution of problem (11)-(12) may be given by

$$g(r) = C (r_0 - r)^k [1 + o(1)], \quad r \to r_0^-, \quad (18)$$

where $k = \frac{p}{p-1-n} > 0, k (k - 1) = \frac{p(1+n)}{(p-1-n)^2} < 0$ and $C = \left( \frac{ar_0^{(p-1)n}r}{p(p-1)(p-n)} \right)^{\frac{1}{p-1-n}}$.

If in (11)-(12) we perform the variable substitution

$$g(r) = C (r_0 - r)^k [1 + y(r)], \quad (19)$$

we obtain the Cauchy problem in $y$

$$\left( \frac{p(n+1)}{p+1} \right)^{\frac{p}{p-1-n}} \left[ 1 + y - \frac{p-n-1}{p} (r_0 - r) y \right]^{p-2} \left[ (r_0 - r)^2 y'' - \frac{2p}{p-1-n} (r_0 - r) y' \right] + \frac{p(n+1)}{p-1-n} (1 + y) - \left( \frac{r_0}{r_0} \right)^{\frac{1}{p-1-n}} (1 + y)^n = 0, \quad (20)$$

$$0 < r < r_0.$$
\[ y(r_0) = \lim_{r \to r_0} [(r_0 - r) y'(r)] = 0, \quad (21) \]

where \( r = r_0 \) is a regular singular point of equation (20).

Writing out the leading linear homogeneous terms of (20), for solutions that satisfy (21), we obtain the equation
\[ (r_0 - r)^2 y'' - \frac{2p+(p-2)(n+1)}{p-1-n} (r_0 - r) y' + \frac{p(1+n)}{p-1-n} y = 0, \quad u \to r_0 \quad (22) \]

whose characteristic exponents are \( \lambda_1 = -1 < 0 \) and \( \lambda_2 = -\frac{p(1+n)}{p-1-n} > 0 \). When \( n = -2 - \frac{1}{p-1} \), we obtain \( \lambda_2 = 1 \), and therefore the difference between the characteristic exponents is an integer. In this case the expansion of \( y \) has a different form and will be considered separately. For any other value of \( n \), the problem (20)-(21) has a particular solution, \( y_{par}(r) \), holomorphic in \( r = r_0 \) and if \( r \neq r_0 \),
\[ y_{par}(r) = \sum_{l=1}^{+\infty} y_l (r_0 - r)^l, \quad |r_0 - r| \leq \delta, \delta > 0, \]

where the coefficients \( y_l \) may be determined substituting \( y_{par}(r) \) in (20). For example, with \( l = 1 \), we get
\[ y_1 = -\frac{(1+n)p\sigma}{2(p-1-n)(2p+np-1-n)r_0}. \]

The one-parameter family of solutions of (20)–(21) which reduces to \( y_{par} \) when the parameter \( c \) is zero, can be represented by:
\[ y(r,c) = y_1 (r_0 - r) + O \left( (r_0 - r)^2 \right) + c (r_0 - r)^{\lambda_2} \left[ 1 + o(1) \right], \quad (23) \]

when \( r \to r_0^- \). Moreover, taking into account that (20) is an asymptotically autonomous equation, this family may be represented in the form of a convergent series:
\[ y(r,c) = y_{par}(r) + c (r_0 - r)^{\lambda_2} \left[ 1 + b_1 (r_0 - r) + b_2 (r_0 - r)^2 + \cdots \right] + c^2 (r_0 - r)^{2\lambda_2} \left[ c_0 + c_1 (r_0 - r) + c_2 (r_0 - r)^2 + \cdots \right] + \cdots, \]
\[ |r_0 - r| \leq \Delta(c), \quad \Delta(c) > 0, \quad (24) \]

and the one-parameter family of solutions of the Cauchy problem (11)–(12) that we were looking for, in the case \( n < -1 \) and \( n \neq -2 - \frac{1}{p-1} \), is given by (14).

Consider now the case \( n = -2 - \frac{1}{p-1} \). Taking into consideration that \( \lambda_2 - \lambda_1 \) is an integer we shall look for a one-parameter family of solutions of this problem in the form
\[ y(r,c) = c (r_0 - r) \left[ 1 + b_1 (r_0 - r) + \cdots \right] + \]
\[ (r_0 - r) \ln \left( \frac{r_0-x}{r_0} \right) \left[ c_0 + c_1 (r_0 - r) + \cdots \right] + \]
\[ (r_0 - r)^2 \ln^2 \left( \frac{r_0-x}{r_0} \right) \left[ d_0 + d_1 (r_0 - r) + \cdots \right] \quad (25) \]

where \( c \) is the parameter and the coefficients \( b_i, c_i \) and \( d_i \) generally depending on \( c \), may be determined substituting \( y(r,c) \) in (20).

Returning to the original variable \( g \), we obtain (15).

Let us now consider the case \( n > -1 \). In the neighborhood of \( r = r_0 \), we shall look for a solution of this problem in the form:
\[ g(r) = C_1 (r_0 - r) + C_2 (r_0 - r)^k \left[ 1 + o(1) \right], \]
\[ g'(r) = -C_1 - kC_2 (r_0 - r)^{k-1} \left[ 1 + o(1) \right], \]
\[ g''(r) = k(k-1)C_2 (r_0 - r)^{k-2} \left[ 1 + o(1) \right], \quad r \to r_0^- \]

From (11)-(12),
\[ (p+1)k(k-1)C_2 (r_0 - r)^{k-2-n} C_1^{p-n}|_{r=r_0^+} = a r_0^\sigma, \]

which implies \( k = 2 + n > 0 \), \( k - 1 = n + 1 > 0 \) and with \( C_2 = c < 0 \) we get \( C_1 = \left( \frac{a r_0^{\sigma}}{(p+1)k(k-1+n)c} \right)^{\frac{1}{p-\sigma}} \).

Next, we define the function \( g(r) \) by
\[ g(r) = C_1 (r_0 - r) + c (r_0 - r)^k \left[ 1 + y(r) \right]. \]

Substituting in (11)-(12), we obtain the Cauchy problem in \( y \)
\[ \left[ 1 + \frac{y}{C_1} (r_0 - r)^{k-1} - \frac{y'}{C_1} (r_0 - r)^{k-1} y' \right]^p \times \]
\[ \left[ (r_0 - r)^2 y'' - 2k (r_0 - r) y' + k(k-1)(1+y) \right] = k(k-1) \left( \frac{r}{r_0} \right)^{\sigma} \left[ 1 + \frac{y}{C_1} (r_0 - r)^{k-1} (1+y) \right]^n, \]
\[ 0 < r < r_0, \quad (26) \]

where \( r = r_0 \) is a regular singular point of (26)–(27).

For any \( c \neq 0 \), this problem has a particular solution that can be represented by the convergent series
\[ y_{par}(r,c) = \sum_{l=0,j=0}^{+\infty} y_{l,j} (c) (r_0 - r)^{l+j(1+n)}, \]

where \( 0 \leq r_0 - r \leq \delta(c), \delta(c) > 0 \) and the coefficients \( y_{l,j} \) may be determined substituting \( y_{par} \) in (26).
If we write the leading linear homogeneous terms of (26) we obtain
\begin{equation}
(r_0 - r)^2 y'' - 2(2 + n)(r_0 - r)y' + (2 + n)(n + 1)y = 0,
\end{equation}
(28)
as \( r \to r_0^- \). This equation has the characteristic exponents \( \lambda_1 = 1 - n < 0 \) and \( \lambda_2 = -2 - n < 0 \). Then (26)–(27) has no other solution than \( y_{par}(r, c) \). Summarizing, we conclude that, for any \(-1 < n < 0\), the Cauchy problem (11)–(12) has a one-parameter family of solutions that can be represented by (16).

Finally, consider the case \( n = -1 \). If in the neighborhood of \( r = r_0 \), we seek a solution in the form
\begin{align*}
g(r) &= C (r_0 - r) \left( -\ln \left( \frac{r_0 - r}{r_0} \right) \right) \frac{1}{p} (1 + o(1)), \\
g'(r) &= -C \left[ -\ln \left( \frac{r_0 - r}{r_0} \right) \right] \frac{1}{p} - \\
g''(r) &= \frac{C}{p(r_0 - r)} \left[ (-1 + \frac{1}{p}) - \ln \left( \frac{r_0 - r}{r_0} \right) \right] \left( 1 + o(1) \right),
\end{align*}
(29)
as \( r \to r_0^- \). From (11)-(12) we conclude that \( C = \left( -\frac{ar_0^{-1}}{p-1} \right)^{\frac{1}{p}} \).

Next, we define \( y(r) \) by
\begin{equation}
g(r) = C (r_0 - r) \left( -\ln \left( \frac{r_0 - r}{r_0} \right) \right) \frac{1}{p} [1 + y(r)].
\end{equation}
Substituting in (11)-(12), we obtain the Cauchy problem
\begin{align*}
p \left[ 1 + y - \frac{1}{p} \left( \ln \left( \frac{r_0 - r}{r_0} \right) \right)^{\frac{1}{p}} (1 + y) - (r_0 - r)y' \right]^{p-2} \\
+ \frac{1}{p \ln \left( \frac{r_0 - r}{r_0} \right)} \left[ -1 - \left( -\frac{1}{p} \ln \left( \frac{r_0 - r}{r_0} \right) \right)^{-1} \right] (1 + y)
\end{align*}
(30)
where the singular point \( r = r_0 \) is irregular. So, we perform the change of variable \( \tau = \left( -\ln \left( \frac{r - r_0}{r_0} \right) \right)^{\frac{1}{p}} \).

The Cauchy problem (30) becomes
\begin{align*}
&\left[ 1 + y - \frac{1}{p} \tau^{-2-p}(1 + y) - \frac{1}{p} \tau^{-1-p}y \right]^{p-2} \\
&\left[ \tau^2 \dot{y} + \tau (-p + 3 - p\tau^p) \dot{y} \\
&\quad - (p - 1 + p\tau^p) (1 + y) \right] = \\
&-p\tau^p \left( 1 - \frac{\sigma}{\tau^p} \right) (1 + y)^{-1}, \\
&0 < \tau < +\infty,
\end{align*}
(31)
\[ \lim_{\tau \to +\infty} y(\tau) = 0. \]
(32)
Let us search for a solution of this problem in the form:
\begin{align*}
y_{par}(\tau) &= \frac{b_1}{\tau^p} + \frac{b_2}{\tau^{2p}} + \cdots + \ln \tau \left[ \frac{c_1}{\tau^p} + \frac{c_2}{\tau^{2p}} + \cdots \right] \\
&\quad + \ln^2 \tau \left[ \frac{d_1}{\tau^p} + \frac{d_2}{\tau^{2p}} + \cdots \right] + \cdots,
\end{align*}
(33)
where the coefficients \( b_i, c_i, d_i \) may be determined substituting this expression in (31). If in (31), we only write the leading linear homogeneous terms, (neglecting the term with the factor \( e^{-\tau^p} \)), for solutions that satisfy (32), we obtain the equation
\begin{equation}
\tau^2 \ddot{y} + \tau (-p + 3 - p\tau^p) \dot{y} - (p - 1 + p\tau^p) y = 0, \quad \tau \to \infty.
\end{equation}
(34)
Here, the singular point (at infinity) is irregular. Introducing the new variables \( z_1 = \tau^{-p+2} y \) and \( z_2 = \tau^{-3-p} \dot{y} \), (34) becomes
\begin{equation}
\dot{z} = \left( \tau^{p-1} A_0 + \frac{1}{\tau} A_1 \right) z, \quad \tau_0 \leq \tau < \infty,
\end{equation}
(35)
where \( z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \), \( A_0 = \begin{bmatrix} 0 & 0 \\ p^2 & p \end{bmatrix} \) and \( A_1 = \begin{bmatrix} -p + 2 & 1 \\ p - 1 & 0 \end{bmatrix} \). Our purpose now is to obtain the behavior of solutions at infinity. Following the methods described, for example, in [13], we perform the change of variables \( z = T_0 (I + \frac{1}{\tau^p} T_2) v \), where \( I \) is the second order identity matrix, and \( T_0 = \begin{bmatrix} 1 & 0 \\ -p & 1 \end{bmatrix} \). (\( T_0 \) diagonalizes the matrix \( A_0 \), this is, \( T_0^{-1} A_0 T_0 = \begin{bmatrix} 0 & 0 \\ 0 & p \end{bmatrix} \).). Introducing \( z = T_0 \omega \),
\begin{equation}
\omega = T_0^{-1} z, \quad \text{where} \quad T_0^{-1} = \begin{bmatrix} 1 & 0 \\ p & 1 \end{bmatrix},
\end{equation}
(35) may be rewritten as
\begin{equation}
\dot{\omega} = \left( \tau^{p-1} \widetilde{A}_0 + \frac{1}{\tau} \widetilde{A}_1 \right) \omega, \quad \tau_0 \leq \tau < \infty,
\end{equation}
(36)
where \( \widetilde{A}_0 = \begin{bmatrix} 0 & 0 \\ 0 & p \end{bmatrix} \) and \( \widetilde{A}_1 = \begin{bmatrix} -2(p - 1) & 1 \\ (p - 1)(2p - 1) & p \end{bmatrix} \). Next, we choose \( T_2 \) in
\[ \omega = (I + \frac{1}{\tau} T_2) v \] to get an asymptotic diagonalization of the matrix of system (36) up to \( O \left( \frac{1}{\tau^p} \right) \) terms as \( \tau \to \infty \). Choosing \( T_2 = \begin{bmatrix} 0 & \frac{1}{p} \\ (p-1)(2p-1) & 0 \end{bmatrix} \), we obtain the equation for \( v \):

\[
\dot{v} = \left( \tau^{p-1} \overline{A}_0 + \frac{1}{\tau} \overline{A}_1 + B(\tau) \right) v, \quad r_0 \leq \tau < \infty,
\]

where \( \overline{A}_1 = \begin{bmatrix} -2(p-1) & 0 \\ 0 & p \end{bmatrix} \) and \( B(\tau) = O \left( \frac{1}{\tau^p} \right) \) when \( \tau \to \infty \). As it is easily seen, (37) has a one-parameter family of solutions that tend to zero as fast as \( \frac{1}{\tau^p} \), when \( \tau \to \infty \). Then we conclude that the one-parameter family of solutions of problem (31)–(32) can be represented as

\[
y(\tau, c) = y_{\text{par}}(\tau) + \frac{c}{\tau^p} + o \left( \frac{1}{\tau^p} \right), \quad \tau \to \infty,
\]

where \( c \) is the parameter and \( y_{\text{par}}(\tau) \) is given by (33).

Looking for an expansion of this family similar to (33) and returning to the original variable \( u \), we obtain (17).

From Proposition 2, some conclusions can be taken concerning the singularity of the problem (11),(12) at \( r = r_0 \). When \(-1 < n < 0 \), \( g' \) is continuous at \( r = r_0 \), but \( g'' \) isn’t. When \( n \leq -1 \), the first derivative is also discontinuous at this point.

## 4 Lower and Upper Solutions

Let us consider again the boundary value problem

\[
(g'(r)|^{p-2} g'(r))' = a r^\sigma g^n(r), \quad 0 < r < r_0
\]

\[
\lim_{r \to 0^+} g'(r) = 0,
\]

\[
\lim_{r \to r_0^-} g'(r) = 0,
\]

with \( p > 1 \), \( \sigma > -1 \) and \( n < -1 \).

**Definition 3** We shall say that \( h(r) \) is a lower (resp. upper) solution of the problem (38)-(40) if \( h(r) \in C [0, r_0] \cap C^2 (0, r_0) \) and satisfies

\[-(g'(r)|^{p-2} g'(r))' + a r^\sigma h^n(r) \leq 0 \quad (\text{resp.} \geq 0),
\]

\[0 < r < r_0, \quad h'(0) \leq 0 \quad (\text{resp.} \geq 0), \quad h(r_0) = 0.
\]

We want upper and lower solutions to verify

\[
\lim_{r \to 0^+} h'(r) = 0,
\]

\[
\lim_{r \to r_0^-} h(r) = 0,
\]

\[
\lim_{r \to r_0^-} \frac{h(r)}{g(r)} = \alpha_1 \neq 0,
\]

where the last condition means that the upper and lower solutions are asymptotically equivalent to the exact solution in the neighborhood of the singular point \( r = r_0 \). Taking into account the asymptotic behavior of the solutions at \( r = 0 \) and \( r = r_0 \), studied in the previous sections, we will look for lower and upper solutions in the form

\[
h(r) = B \left( r_0^k - r^k \right)^{\frac{p}{p-1-n}},
\]

where \( k = \frac{\sigma + p}{p-1} \) and \( B \) is a positive constant. Note that in the case \( p = 2 \) we obtain the upper and lower solutions, introduced in [8]. We have \( k > 1 \), which follows from \( \sigma > -1 \) and \( p > 1 \). We want to define \( B \) in such a way that \( h \) satisfies equation (38), that is, \( h \) must verify

\[
(|h'(r)|^{p-2} h'(r))' = ah(r)^n r^\sigma
\]

With this purpose, we note that

\[
(|h'(r)|^{p-2} h'(r))' = \frac{1}{Bkp}(-1 + p) r^\sigma (r_0^k - r^k)^{\frac{p}{p-1-n}} \left( \frac{Bkp}{p-1-n} \right)^p q(r)
\]

where \( q(r) = -(p - 1 - n)(k - 1)r_0^k + (1 + n + (k - 1)p)r_0^k, \quad r \in [0, r_0] \).

On the other hand, from (41) we have

\[
h(r)^n = B^n \left( r_0^k - r^k \right)^{\frac{p}{p-1-n}}
\]

By substituting (43) and (44) into (42) we obtain

\[
-\frac{1}{Bkp} (-1 + p) \left( \frac{Bkp}{p-1-n} \right)^p q(r) + aB^n = 0.
\]

Note that if

\[
1 + n + (k - 1)p = 0
\]

then \( q(r) \) reduces to a constant:

\[
q(r) \equiv r_0^k(k - 1)(1 + n - p).
\]

In this case, that is, if \( n = (1 - k)p - 1 \), equation (45) takes the form

\[
-\frac{1}{Bkp} (-1 + p) \left( \frac{Bkp}{p-1-n} \right)^p r_0^k(k - 1)(1 + n - p) + aB^n = 0
\]

Solving (48) with respect to \( B \), we obtain

\[
B = \left( \frac{-a(p - 1 - n)(p-1)}{r_0^k(p - 1)(kp)^{p-1}(k - 1)} \right)^{\frac{1}{p-1-n}}.
\]
Hence we conclude that when the condition (46) is satisfied the exact solution of the problem (38)-(40) has the form (41), with $B$ given by (49). For values of $n, p, \sigma$ which do not satisfy that condition we cannot find an explicit formula for the exact solution. However, by analyzing the function $q(r)$, we can obtain conditions on $B$, under which the function $h$ can be an upper or a lower solution, according to Definition 3. This is summarized in the next proposition.

**Proposition 4** Let $\sigma > -1$, $n < -1$, $p > 1$ and let $B > 0$ be a constant. Then the function defined by (41) will be

- **a lower solution** of the problem (38)-(40) if $B$ satisfies

  \[ B \leq B_1 = \left( \frac{(p-1-n)^p(p-1)^{p-1}}{p^{p-1}(1+n)(\sigma+p)p!r_0^p} \right)^{1/(p-1-n)}, \]

  when $n \leq (1 - k)p - 1$,

  and

  \[ B \leq B_2 = \left( \frac{-a(p-1-n)^p(p-1)^{p-1}}{p^{p-1}(1+n)(\sigma+p)p!(\sigma+1)r_0^p} \right)^{1/(p-1-n)}, \]

  when $-1 > n \geq (1 - k)p - 1$.

- **an upper solution** of the problem (38)-(40) if $B$ satisfies

  \[ B \geq B_1, \quad \text{when } -1 > n \geq (1 - k)p - 1, \]

  and

  \[ B \geq B_2, \quad \text{when } n \leq (1 - k)p - 1. \]

We see that when $n = (1 - k)p - 1$, $B_1 = B_2 = B$, where $B$ is given by (49). As we have remarked above, in this case we have an exact solution in the form (41).

The upper and lower solutions can be used to obtain suitable initial approximations for the application of computational methods. On the other hand, we can use them to apply the existence results obtained in [12]. Actually, choosing an upper solution $g_1$ and a lower solution $g_2$ of the form (41) the conditions of proposition 2.1 of that work are satisfied and therefore the problem has at least one solution $g$, such that $g_1 \leq g \leq g_2$. Hence, we conclude that the problem (38)-(40) is solvable for $p > 1$, $\sigma > -1$, $n < -1$. Note that the problem is also solvable in the case $-1 \leq n < 0$, but in this case the upper and lower solutions must be sought with a different form.

## 5 Numerical results

Let us briefly describe the numerical algorithm we used in order to obtain the approximate solutions of problem (38)-(40). When $p > 2 + \sigma$, for given values of $a, n, r_0, \sigma$, we have considered two regular Cauchy problems

\[
\begin{align*}
\{ & (|g'(r)|^{p-2}g'(r))' = ar^n g^n(r), \quad 0 < r < \frac{r_0}{2}, \\
& g(\delta) = \overline{g}_i(\delta, b), \quad g'(\delta) = \overline{g}'_i(\delta, b) \\
\}\quad (50)
\end{align*}
\]

which we denote Problem A and Problem B, respectively. Here $i = 2, 3, 4$ or 5, depending on $n$, $\delta$ is a small constant. In our calculations we have used $\delta = 0.001$. The functions $\overline{g}_1, \overline{g}_2, \overline{g}_3, \overline{g}_4$ and $\overline{g}_5$ are computed according to the formulae (6), (14) and (16), respectively, ignoring the remainders of the series. We solve problems A and B for certain values of the parameters $b$ and $c$. With this purpose we use the program NDSolve of Mathematica [15]. Then, following the idea of the shooting method, we determine the values of $b$ and $c$ so that $g$ and $g'$ are continuous at $r = r_0/2$. This gives a system of two nonlinear equations, which is solved by the Newton’s method.

When $p \leq 2$ and $\sigma \geq 0$, since we have no singularity at $r = 0$, we solve the auxiliary problem (B) starting from $r_0 - \delta$, for a certain value of the parameter $c$. Then we compute $c$ by the shooting method, requiring that the boundary condition at $r = 0$ is satisfied.

In figures 1,2,3 approximate solutions of problem (38)-(40) are displayed, for different values of $\sigma, n, r_0, a$ and $p$.

![Figure 1: Solution of the boundary value problem with $r_0 = 5, \sigma = 0.5, a = -1, p = 4$, for different values of $n$.](image-url)

In table 1 we compare the computed values of the solutions at $r = r_0/2$ with the exact ones, in some cases where the exact solution is known. The obtained numerical results show that the used method has an accuracy of about 8 digits.
6 Conclusions and future work

In the present paper, previous results about asymptotic behavior of solutions of singular boundary value problems, involving the classical one-dimensional Laplacian, were extended to the case of the one-dimensional p-Laplacian. This extension enabled us to obtain accurate numerical approximations of a new class of problems. In the future, we are planning to generalize the results to the multidimensional case ($N > 1$).

Acknowledgements: The authors acknowledge financial support from FCT, through project POCTI/MAT/45700/2002.

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