Computational methods for a singular boundary-value problem

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**ABSTRACT:** In this paper we apply the finite differences and the shooting methods to the numerical solution of singular boundary value problems which describe the deformation of a membrane cap.

The considered differential equation may be written in the form

\[ r^2 S''_r + 3r S'_r = \frac{\lambda^2}{2} r^{2\nu-2} + \frac{\beta \nu r^2}{S'_r} - \frac{r^2}{8S''_r}, \quad r \in [0,1] \]  

where \( S_r, r \) and \( \nu \) denote the radial stress, the membrane radius and the Poisson coefficient, respectively, \( \gamma, \lambda, \beta \) are known positive constants.

We shall look for a positive solution \( S_r \) which satisfies the following boundary conditions

\[ S_r(0) = 0 \quad \text{and} \quad S_r'(1) + (1 - \nu)S_r(1) = \Gamma \quad \text{or} \quad S_r(1) = S, \]  

where \( \Gamma \) and \( S \) are real numbers.

By using well-known iterative methods, like the Picard and Newton’s method, the nonlinear problem is reduced to a sequence of linear ones, which are then discretized by the finite differences method.

The numerical results obtained by the finite difference method are compared with those presented in other papers and the efficiency of different numerical methods is discussed.

1 **INTRODUCTION**

In this work we apply efficient numerical methods to the solution of nonlinear boundary value problems for shallow membrane caps. Consider a membrane cap in the undeformed state. Assuming that the strains and the vertical pressure are small, a rotationally symmetric deformation will be described by the exact equation

\[ \frac{d}{dr} \left( r \sqrt{2\varepsilon_r} + 1 \right) = \frac{m \sqrt{\varepsilon_r} + 1}{\Sigma_r} \]  

where \( \varepsilon_r \) and \( \varepsilon_c \) are, respectively, the radial and circumferential strains (see Dickey 1987). Moreover, we have \( m^2 = 1 + \left( 2 \varepsilon_r \right)^2 \), \( \Sigma_r = \sigma_r \sqrt{2\varepsilon_r} + 1 \) and \( \Sigma_\theta = \sigma_\theta \sqrt{2\varepsilon_\theta} + 1 \), where \( \sigma_r \) and \( \sigma_\theta \) denote, respectively, the radial and circumferential stresses of the membrane.

Assuming that the strains are small, the membrane is shallow and the vertical pressure is low, Dickey has shown, using the Hook’s law, that an approximated equation may be derived from (3) (see Dickey (1987)).
More precisely, Dickey has proved that, under the mentioned conditions, the values of \( u, \varepsilon_\theta \), \( \varepsilon_n, z' \) are small. By expanding both sides of equation (3) in Taylor series and ignoring powers of \( u \), with exponent equal or higher than 3, and powers of \( \varepsilon_\theta, \varepsilon_n, z' \), with exponent equal or higher than 2, he obtained the following equation:

\[
r^2 \frac{d^2}{dr^2} \sigma_r + 3r \frac{d}{dr} \sigma_r = \frac{E}{2} \left( \frac{d}{dr} \frac{d}{dz} \right)^2 + \nu E^2 \frac{d}{dr} \left( rG \frac{E}{\sigma_r} \right) - \frac{E^3}{2} \left( \frac{G}{\sigma_r} \right)^2
\]

(4)

where \( E \) is the elasticity modulus and \( G = F / E \), with \( F \) defined by

\[
rF = \frac{1}{h} \int_0^r \rho m P(\rho) d\rho,
\]

where \( P \) is the pressure.

Two different kinds of boundary value problems may be considered: 1) the stress problem, when the radial stress at the boundary is known \( (\sigma_r(a) = \sigma) \) and 2) the displacement problem, when the radial displacement is given at the boundary \( (u(a) = \frac{a}{E} (a\sigma_r(a) + (1 - \nu)\sigma_n(a)) - \mu) \).

For the sake of simplicity, we may assume without loss of generality that \( P \) is constant \( (P(\rho) = \rho) \) and the undeformed shape of the surface is given by \( z(r) = \frac{C}{1 - \left( \frac{r}{a} \right)^\gamma} \), where \( a \) and \( C \) are, respectively, the membrane radius and the height from the center to the surface, with \( \gamma > 1 \).

Suppose that \( a = 1 \) and let us introduce the following notations:

\[
S_r(a) = \frac{\sigma_r(r)}{K}; \quad K = \left( \frac{Eh^2}{h^2} \right)^{\frac{2}{3}}; \quad \lambda = CG \left( \frac{P}{Eh} \right)^{\frac{2}{3}}; \quad \beta = \frac{1}{2} \left( \frac{P}{Eh} \right)^{\frac{2}{3}}.
\]

Then (4) takes the form

\[
r^2 S_r'' + 3r S_r' = \frac{2}{2} r^{2r-2} + \frac{\beta vr^2}{S_r} - \frac{r^2}{8S_r^2}, \quad r \in [0,1].
\]

(5)

If \( S_r \) is a solution of equation (5) then the final form of the membrane may be obtained from the equations:

\[
u(r) = \frac{Kr}{E} \left[ rS_r'(r) + (1 - \nu)S_n'(r) \right] \quad w(r) = \frac{P}{2EhK} \int_0^1 \frac{t}{S_r(t)} dt,
\]

where \( u \) and \( w \) are the radial and the vertical displacement of the membrane, \( \nu \) is the Poisson coefficient \(^1\).

Since the boundary of the membrane corresponds to \( r = 1 \), the boundary condition for the stress problem is \( S_r(1) = \sigma \equiv \frac{a}{K} \) (\( \sigma > 0 \)) and the boundary condition for the displacement problem is \( S_r'(1) + (1 - \nu)S_n'(1) = \Gamma \equiv \frac{E\mu}{K}, \Gamma \in \mathbb{R} \).

In this paper, we shall consider three different cases:

**Problem A:** Analysis of large deformations of a spherical membrane (this is a particular case of problem B, which arises when \( \gamma = 2 \) and \( \beta = 0 \) in equation (5)).

**Problem B:** Analysis of all kinds of deformations of a spherical membrane, with \( \gamma = 2 \) and \( \beta > 0 \) (this is a particular case of problem C, which arises when \( \gamma = 2 \) in equation (5)).

\(^1\) Ratio between the tangential and the radial deformation in an elastic domain.
Problem C: Analysis of deformations of a membrane with the shape given by any function of the form \( z(r) = C \left( 1 - \left( \frac{r}{a} \right)^y \right) \), with \( y > 1 \).

Since \( \lambda \) depends on \( P \) as \( P^{\frac{1}{3}} \) and \( \beta \) depends on \( P \) as \( P^{\frac{1}{3}} \), we may conclude that when the pressure is low \( \lambda \) is large and \( \beta \) is small; therefore, the second term of the right-hand side of (5) may be ignored and we obtain problem A (see Baxley (1988)).

For \( y = 2 \), (problem B), Baxley & Gu (1999) have proved that the limit \( \lim_{r \to 0^+} S_0(r) \) is positive, as well as \( \lim_{r \to 0^+} r^{-1} S_0(r) \).

In the case of the stress problem, with \( y = 2 \), there exists an unique positive solution, if \( S \leq \frac{1}{4\beta v} \); and there is at least a positive solution, if \( S > \frac{1}{4\beta v} \). For the displacement problem, if \( \frac{\Gamma}{1 - \nu} \leq \frac{1}{4\beta v} \), the problem has an unique positive bounded solution; this condition is equivalent to \( 4\Gamma \beta v \leq 1 - \nu \), which is satisfied for sufficiently small \( v \). If \( \frac{\Gamma}{1 - \nu} > \frac{1}{4\beta v} \), there may exist multiple solutions.

When \( y \neq 2 \), (problem C), Baxley & Robinson (1998) have shown that for certain values of \( \Gamma \) and \( \nu \), the solution is not monotone. The same authors have obtained sufficient conditions for existence and unicity of solution, as well as qualitative information on the solution. The problem has been divided into three different cases, depending on the value of \( \gamma : \gamma > 2.43 < \gamma < 2.1 < \gamma < 4.3 \).

In order to obtain a numerical solution, the variable substitutions \( x = r^2 \), \( u(x) = S_0(r)x \) are introduced. Then the differential equation (5) reduces to

\[
\frac{\lambda^2 x^{y-2}}{8} - \frac{x^2}{32\mu^2} + \frac{\beta v x}{4\mu} = u''(x)
\]

and the boundary conditions may be written in the form

\[
u(0) = 0, \quad a_0u(1) - a_1u'(1) = A.
\]

In the case of the stress problem, we have \( a_0 = 1, a_1 = 0 \) and \( A > 0 \); for the displacement problem, \( a_0 = 1 - \nu, a_1 = 2 \) and \( A \) may be any real number.

2 COMPUTATIONAL METHODS

Numerical results for the considered problems were obtained by the shooting method in Baxley (1988), Baxley & Gu (1999) and Baxley & Robinson (1998).

In the present work, numerical approximations for the same problems will be obtained not only by the shooting method, but also using iterative methods and finite differences schemes. The shooting method is often used to reduce a boundary value problem to an initial value one. In the case of the problem (6)-(8), this cannot be done directly, since the equation is singular at \( x = 0 \). Hence the initial conditions are given at a certain point \( x_i \), close to 0. In order to do this, we use the asymptotic behavior of the solution near the origin. For example, in the case \( y = 2 \), it has been shown in Baxley (1988) that if \( u \) is a solution of (6)-(7), then

\[
u(x_i) = Lx_i + \frac{M}{2}x_i^2, \quad (x \to 0), \quad \text{where} \quad M = -\frac{1}{32L^2} + \frac{\lambda^2}{8}, L \text{ being a certain real constant.} \]
Therefore, the initial conditions for $u$ may be written as $u(x_i) = Lx_i + \frac{1}{2}\left(-\frac{1}{32L^2} + \frac{\lambda^2}{8}\right)x_i^2$ and $u'(x_i) = L + \left(-\frac{1}{32L^2} + \frac{\lambda^2}{8}\right)x_i$, where $L$ is the shooting parameter. Then the boundary value problem reduces to finding the value of $L$, for which $u$ satisfies the boundary condition (8).

The finite differences method offers alternative algorithms for the numerical approximation of problem (6)-(7)-(8). In this case, it is convenient to reduce the nonlinear problem to a sequence of linear ones and to apply a finite difference scheme to each linear problem.

Let us consider a second order nonlinear differential equation in the general form $Lu = f(x,u)$, where $L$ is a linear differential operator. Suppose that we are looking for a solution of this equation, which satisfies the boundary conditions

$$\alpha_i u(a) + \beta_i u'(a) = \Gamma_i, \alpha_2 u(b) + \beta_2 u'(b) = \Gamma_2,$$

where $\alpha_i, \beta_i, \Gamma_i$ are given real numbers, $i=1,2$. It is well-known that a nonlinear problem of this form may be reduced to a sequence of linear ones by the Picard or the Newton method. In the case of the Picard method, a sequence of iterates $(u^{(n)}(x))$, $n \geq 0$ is constructed by solving the linear boundary value problems

$$Lu^{(n+1)}(x) = f(x,u^{(n)}(x)), x \in [a,b]$$

$$\alpha_i u^{(n+1)}(a) + \beta_i u^{(n+1)}'(a) = \Gamma_i, \alpha_2 u^{(n+1)}(b) + \beta_2 u^{(n+1)}'(b) = \Gamma_2,$$

$n=0,1,...$, where $u^{(0)}(x)$ is a certain function, defined on $[a,b]$. If we use the Newton method, the sequence of linear boundary value problems to be solved has the form

$$Lu^{(n+1)}(x) = f(x,u^{(n)}(x)) + \frac{\partial f}{\partial u}(x,u^{(n)}(x))|_{u=u^{(n)}}(u^{(n+1)}(x) - u^{(n)}(x)), x \in [a,b]$$

$$\alpha_i u^{(n+1)}(a) + \beta_i u^{(n+1)}'(a) = \Gamma_i, \alpha_2 u^{(n+1)}(b) + \beta_2 u^{(n+1)}'(b) = \Gamma_2.$$

In general, the Newton method converges faster than the Picard method, since its convergence is quadratic. These iterative methods were analysed in Mooney (1979), where sufficient conditions were imposed on $f$, such that the iterative scheme converges monotonically to the solution of the nonlinear problem. In order to apply the iterative methods, proposed in the cited works, it is essential to know a suitable initial approximation, which is usually a subsolution or a supersolution of the considered problem.

A function $u(x)$ is said to be a subsolution of the considered nonlinear problem if it satisfies the inequalities $Lu(x) - f(x,u(x)) \leq 0, x \in [a,b]$, $\alpha_i u(a) + \beta_i u'(a) \leq \Gamma_i$, $\alpha_2 u(b) + \beta_2 u'(b) \leq \Gamma_2$.

If we reverse the signs of these inequalities, we shall obtain the definition of a supersolution. According to the results of Mooney (1979), if we start the Picard method or the Newton method with a subsolution, the method will converge monotonically upwards to the exact solution. If we start the Picard method with a supersolution, it will converge downwards to the true solution. In the case of problem (6)-(7)-(8), these results may be applied if $\beta = 0$, $\gamma = 2$ (problem A). For other values of $\beta$ and $\gamma$ the convergence of the iterative methods was not proved theoretically, but the numerical experiments have shown that the methods are applicable if $\beta$ is not too large (see section 4).

For the particular case of the problem (6)-(7)-(8), we have used subsolutions of the form, suggested by the authors in Baxley (1988), Baxley & Gu (1999), and Baxley & Robinson...
(1998). The form of these subsolutions is shown in Table 1, where \(a\) and \(C\) are adjustable parameters, and the constants, satisfy \(1 \leq \delta \leq 1 + 2/(1 - v), r = 3/2 - 9/8, q = 3/2 - 15/8\), respectively.

<table>
<thead>
<tr>
<th>Problems A and B</th>
<th>Problem C</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\gamma = 2)</td>
<td>(4/3 &lt; \gamma &lt; 2)</td>
</tr>
<tr>
<td>(u^{(0)}(x) = Cx(\delta - x))</td>
<td>(u^{(0)}(x) = (1 - a)^{-v} x^{3/2})</td>
</tr>
</tbody>
</table>

Table 1: subsolutions of the boundary value problems

Once a subsolution of the considered problem is found, the iterative schemes of the Picard and Newton methods may be written for the boundary value problem (6)-(7)-(8). In the case of the stress problem, the equations of the Newton method (see (12)-(13)) will have the following form:

\[
\begin{align*}
\left( \frac{\beta \nu x}{4u^{(n)}(x)^2} - \frac{x^2}{16u^{(n)}(x)^3} \right) u^{(n+1)}(x) = &\ \frac{\lambda^2}{8} x^{-2} - \frac{x^2}{32u^{(n)}(x)^2} + \frac{\beta \nu x}{4u^{(n)}(x)} + \\
+ \left( \frac{\beta \nu x}{4u^{(n)}(x)^2} - \frac{x^2}{16u^{(n)}(x)^3} \right) u^{(n)}(x), \\
u^{(n+1)}(0) = &\ 0, \ u^{(n+1)}(1) = S
\end{align*}
\]

(15)

If we choose the Picard method to solve the problem (6)-(8), then, in order to guarantee the convergence, we must first transform the nonlinear equation, by adding to each side of equation (6) the term

\[
\left( \frac{\beta \nu x}{4u^{(0)}(x)^2} - \frac{x^2}{16u^{(0)}(x)^3} \right) u(x),
\]

where \(u^{(0)}(x)\) is a subsolution. Then the equations of the iterative scheme (10)-(11) take the form:

\[
\begin{align*}
\left( \frac{\beta \nu x}{4u^{(0)}(x)^2} - \frac{x^2}{16u^{(0)}(x)^3} \right) u^{(n+1)}(x) = &\ \frac{\lambda^2}{8} x^{-2} - \frac{x^2}{32u^{(n)}(x)^2} + \frac{\beta \nu x}{4u^{(n)}(x)} + \\
+ \left( \frac{\beta \nu x}{4u^{(0)}(x)^2} - \frac{x^2}{16u^{(0)}(x)^3} \right) u^{(n)}(x), \\
u^{(n+1)}(0) = &\ 0, \ u^{(n+1)}(1) = S
\end{align*}
\]

(16)
The iterative schemes for the case of the displacement problem are obtained just by changing the boundary conditions.

Given a linear boundary value problem of the form (15) or (16), a finite difference method is used to discretize the equation. By applying a three-point scheme with central differences in an uniform grid, we reduce the problem to a system of linear equations of the form $AU=R$, where

$$
A = \begin{bmatrix}
    b_{1,1} & c_{1,2} & & & \\
    a_{2,1} & b_{2,2} & c_{2,3} & & \\
    & a_{3,2} & b_{3,3} & c_{3,4} & \\
    & & & \ddots & \ddots \\
    & & & & a_{N-1,N-2} & c_{N-2,N-1} \\
    & & & & & b_{N-1,N-1}
\end{bmatrix}.
$$

In the case of the Newton method, we have:

$$
\begin{align*}
    a_{i,i+1} &= 1/h^2, & i = 1, \ldots, N-2 \\
    b_{i,i} &= \frac{\beta v \chi_i}{4 u_i^{(n)}}, & i = 1, \ldots, N-1 \\
    c_{i,i+1} &= 1/h^2, & i = 1, \ldots, N-2 \\
    R &= \begin{bmatrix} r_1, r_2, \ldots, r_{N-1}, r_N \end{bmatrix}^T,
\end{align*}
$$

where

$$
\begin{align*}
    r_i &= \frac{\beta v \chi_i}{4 u_i^{(n)}}, & i = 1, \ldots, N
\end{align*}
$$

According to our notations, the approximate solution of the $(n+1)$-th linear boundary value problem is given by the vector

$$
U^{(n+1)} = \begin{bmatrix} U_1^{(n+1)}, U_2^{(n+1)}, \ldots, U_{N-1}^{(n+1)}, U_N^{(n+1)} \end{bmatrix}^T,
$$

where $U_i^{(n+1)}$ is the approximate value of $u^{(n+1)}(x_i)$, $x_i$ being the knots of a uniform grid of stepsize $h$ ($x_i = ih$).

The schemes (15) and (16) are iterated until the approximate solution satisfies the condition

$$
\left\| U^{(n+1)} - U^{(n)} \right\| \leq \varepsilon,
$$

where the Euclidean norm is considered and $\varepsilon$ is the needed precision (in our computations we have used $\varepsilon = 0.5 \cdot 10^{-6}$).

Since the discretization error allows an expansion in powers of the stepsize, the accuracy of the numerical results may be further improved by means of the Richardson extrapolation (see, for example, (Brezinski & Zaglia 1991)).

3 NUMERICAL RESULTS

In this section, we shall present numerical results for some examples of problems B and C. We have considered some different sets of values of the equation parameters. Since the numerical
results obtained by the Picard method do not differ significantly from those obtained by the Newton method, we present only the last ones.

In Table 2, we display some numerical results for problem B with displacement boundary conditions, in the cases of $\lambda = 0.233$ and $\lambda = 0.991$, $\beta = 1.0$, $\nu = 0.3$, $\Gamma = 0.2$. The results obtained by using Richardson extrapolation with two grids are also shown.

<table>
<thead>
<tr>
<th>Stepsize</th>
<th>$\lambda = 0.233$</th>
<th>Richardson extrapolation</th>
<th>$\lambda = 0.991$</th>
<th>Richardson extrapolation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/20</td>
<td>0.3397907604</td>
<td>0.3397816272</td>
<td>0.2847918027</td>
<td>0.2847920085</td>
</tr>
<tr>
<td>1/40</td>
<td>0.3397839105</td>
<td>0.3397816266</td>
<td>0.2847919571</td>
<td>0.2847920085</td>
</tr>
<tr>
<td>1/80</td>
<td>0.3397821975</td>
<td>0.3397816265</td>
<td>0.2847919957</td>
<td>0.2847920086</td>
</tr>
<tr>
<td>1/160</td>
<td>0.3397817693</td>
<td>0.3397816265</td>
<td>0.2847920053</td>
<td>0.2847920086</td>
</tr>
<tr>
<td>1/320</td>
<td>0.3397816222</td>
<td>0.3397816265</td>
<td>0.2847920078</td>
<td>0.2847920086</td>
</tr>
</tbody>
</table>

The graphic of the numerical solution corresponding to this case is displayed in Figure 1, where the corresponding initial approximation and the first Newton iterate are also shown.

Figure 1: Graphics of the Newton iterates in the case of problem B with $\lambda = 0.233$, $\beta = 1.0$, $\nu = 0.3$, $\Gamma = 0.2$

![Figure 1](image)

Table 3 illustrates the improvement of the accuracy obtained by the Richardson extrapolation, when 2 or 3 different grids are used.

The convergence speed of the iterative methods is compared in Table 4. This table refers to the numerical solution of problem C with stress boundary conditions and different values of $\gamma$. The computations were performed in a PC with a Pentium III processor (665 MHz). The algorithms were implemented in the language Mathematica.

<table>
<thead>
<tr>
<th>Stepsize</th>
<th>Approximation by the finite difference method</th>
<th>Richardson extrapolation with 2 grids</th>
<th>Richardson extrapolation with 3 grids</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/20</td>
<td>0.1743120026</td>
<td>0.1743083851</td>
<td>0.1743083851</td>
</tr>
<tr>
<td>1/40</td>
<td>0.1743092897</td>
<td>0.1743083852</td>
<td>0.1743083851</td>
</tr>
<tr>
<td>1/80</td>
<td>0.1743086113</td>
<td>0.1743083851</td>
<td>0.1743083851</td>
</tr>
<tr>
<td>1/160</td>
<td>0.1743084417</td>
<td>0.1743083851</td>
<td>0.1743083851</td>
</tr>
<tr>
<td>1/320</td>
<td>0.1743083993</td>
<td>0.1743083851</td>
<td>0.1743083851</td>
</tr>
</tbody>
</table>
Table 4: Number of iterations and computing time for different methods and different values of \( \gamma \ (\lambda = 1.0, \beta = 1.0, \nu = 0.3, S = 0.38, h = 1/80) \)

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>Shooting</th>
<th>Picard</th>
<th>Newton</th>
<th>Shooting</th>
<th>Picard</th>
<th>Newton</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.6</td>
<td>7</td>
<td>33</td>
<td>5</td>
<td>0.03</td>
<td>0.972</td>
<td>0.3</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>27</td>
<td>5</td>
<td>0.03</td>
<td>0.761</td>
<td>0.28</td>
</tr>
</tbody>
</table>

4 CONCLUSIONS

The finite differences method, combined with the Picard and Newton iterative methods, enabled us to reduce the original problem to the solution of a sequence of linear systems of N equations, where N+1 is the number of gridpoints of the considered mesh.

The analysis of the numerical results has shown that the finite difference method has second order convergence, as it should be expected. Moreover, the Richardson extrapolation, using up to 3 different grids enabled us to improve the accuracy and obtain results with 10 significant digits.

Concerning the iterative methods, as it may be seen in Table 3, the fastest convergence is obtained with the Newton method, for all the considered cases.

The shooting method is also very efficient, but its application requires that the initial value of the shooting parameter is sufficiently close to the exact one, and this condition may be not easy to satisfy.

REFERENCES