Asymptotic expansions and numerical approximation of nonlinear degenerate boundary-value problems

P.M. Lima *, M.P. Carpentier

Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais, 1096 Lisboa Codex, Portugal

Abstract

In the present paper we are concerned with boundary-value problems (BVP) for the generalized Emden–Fowler equations. Asymptotic expansions of the solution are obtained near the endpoints. We use a finite-difference scheme to approximate the solution and the convergence is accelerated by means of extrapolation methods. A variable substitution is introduced to diminish the effect of the singularity at the origin. Numerical results, obtained by different methods, are presented for two particular cases. © 1999 Elsevier Science B.V. and IMACS. All rights reserved.

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1. Introduction

In the present paper we are concerned with boundary-value problems (BVP) of the form

\[ y'' = -cx^p y^q, \quad (1.1) \]
\[ y(0) = 1, \quad y(1) = 0, \quad (1.2) \]

where \( c, p \) and \( q \) are real constants, \( p > -2, q > 1 \) and \( c < 0 \). Eq. (1.1) is known as the generalized Emden–Fowler equation and many particular cases of it arise in different problems of Physics and Mechanics (see [6,12]).

The finite-difference method, combined with extrapolation procedures or deferred corrections, provides an efficient way to obtain accurate numerical approximations to nonlinear BVP [9]. However, in order to apply extrapolation techniques, the equation and the finite-difference scheme must satisfy certain conditions, in particular, an asymptotic error expansion must be known [11].

For problems of the type (1.1), (1.2), these conditions will be satisfied if \( p \) and \( q \) are nonnegative integers and we choose an adequate finite-difference scheme. When this is the case, the solution of the BVP is infinitely differentiable over the considered interval and an asymptotic error expansion is valid.

* Corresponding author. E-mail: plima@math.ist.utl.pt.
which contains only even powers of the stepsize, enabling the use of simple extrapolation procedures, such as the Richardson’s “deferred approach to the limit” [10]. However, in the present work we are interested in the case when \(-2 < p < -1\). For such values of \(p\), according to the definition given in [6], the problem is said to be degenerate of order \((1, -1 - p)\), which means that the first derivative is unbounded at \(x = 0\) and the limit
\[
\lim_{x \to 0} y'(x) x^s
\]
is finite, for \(s \geq -1 - p\). In such cases, as pointed out in [4,6], the convergence of the finite-difference method becomes slower and the derivation of an asymptotic error expansion is not straightforward. Moreover, the form of such error expansion is, in general, more complicated than in the non-degenerate case, requiring the use of more sophisticated extrapolation methods to accelerate the convergence. In [6,7], numerical schemes for degenerate BVP of the class (1.1), (1.2) were developed and numerical results were obtained for the case \(p = -0.5\), \(q = 1.5\). In [8] these numerical schemes were improved and numerical results were obtained for different values of \(p\), including cases where \(p < -1\).

In [4], accurate numerical results were obtained for particular cases of (1.1), (1.2), with (a) \(p = -0.5\), \(q = 1.5\), \(c = -1\), (b) \(p = -1\), \(q = 2\), \(c = -1\), and (c) \(p = -1.25\), \(q = 2.25\), \(c = -1\). For lower values of \(p\), the numerical approximation of the solution becomes more difficult and in order to obtain results with the same accuracy as in [4,6] a more detailed analysis of the problem is required.

In Section 2 of this paper we continue the analysis of the asymptotic behaviour of the solution of (1.1), (1.2), which was started in [4]. We use the Picard iterative method to approximate the solution of the nonlinear problem and find a relation between the first iterate and the Bessel functions. The asymptotic expansion of the Picard iterates at \(x = 0\) is obtained, which is a generalization of the expansions obtained in [4] for some particular cases of the equation. Moreover, we point out that with the given boundary conditions (1.2), the Picard iterates have another singularity at \(x = 1\) and we obtain an asymptotic expansion at this point. Asymptotic expansions for the solution of the nonlinear problem are also obtained and explicit formulæ for their coefficients are given.

Based on this analysis, in Section 3, we improve the numerical methods developed in [4], in order to obtain a good approximation in the cases where \(p\) is close to \(-2\). We present some numerical results, with \(p = -1.5\), \(q = 2.5\), \(c = -1\) and \(p = -1.7\), \(q = 2.7\), \(c = -1\), where the new techniques provide much more accurate results than those that could be obtained using the previous methods. We also present results for \(p = -1.4\), \(q = 2.0\), \(c = -0.6\), which are compared with the corresponding values obtained in [8].

2. Asymptotic expansions near the singularities

2.1. Asymptotic expansions of the Picard iterates

In order to approximate the solution of the nonlinear BVP (1.1), (1.2) we shall use the Picard sequence [6]
\[
\begin{align*}
y''_\nu(x) + cx^p y_\nu(x) &= -q y_{\nu-1}(x) \quad 0 < x < 1, \\
y_\nu(0) &= 1, \quad y_\nu(1) = 0, \quad \nu = 1, 2, \ldots,
\end{align*}
\]
with \(y_0(x) = 1 - x\) or \(y_0(x) = 0\).
The homogeneous equation

We shall begin analyzing the linear homogeneous equation associated to (2.1):

\[ y'' + cq x^p y = 0, \quad 0 < x < 1. \]  
(2.3)

The solutions of (2.3) can be expressed in term of Bessel functions. If we denote by \( C_\mu \) any solution of the Bessel equation of order \( \mu \), then according to [1, (9.1.51)] every solution \( y(x) \) of (2.3), may be represented in the form

\[ y(x) = x^{1/2} C_\mu (2\mu \sqrt{c q} x^{(p+2)/2}), \quad \mu = \frac{1}{p + 2}. \]  
(2.4)

Then Eq. (2.3) is singular at \( x = 0 \).

If \( \mu = 1/(p + 2) \) is not an integer, we have a pair of linear independent solutions of (2.3) with the form

\[ u(x) = x \sum_{k=0}^{\infty} a_k x^{k(p+2)}, \]  
(2.5)

\[ v(x) = \sum_{k=0}^{\infty} b_k x^{k(p+2)}, \]  
(2.6)

where \( a_k \) and \( b_k \) are real constants. This result can be obtained from (2.4) replacing \( C_\mu \) successively by the Bessel functions \( J_\mu \) and \( J_{-\mu} \).

If \( p + 2 = 1/n \), with \( n = 1, 2, 3, \ldots \), we have

\[ u(x) = x \sum_{k=0}^{\infty} a_k x^{k/n}, \]  
(2.7)

\[ v(x) = \frac{1}{\pi n} u(x) \ln x + \sum_{k=0}^{\infty} b_k x^{k/n}. \]  
(2.8)

In this case \( C_\mu \) is replaced by \( J_n \) and \( Y_n \) in (2.4) [1].

The non-homogeneous equation

In order to analyze the properties of the Picard iterates approximating the nonlinear problem (1.1), (1.2) we must study a non-homogeneous equation with the general form

\[ y''(x) + cq x^p y(x) = f(x), \]  
(2.9)

where \( f \) is a given function (depending on the previous iterate and on \( q \)), and \( y \) satisfies the boundary conditions (2.2)

\[ y(0) = 1, \quad y(1) = 0. \]  
(2.10)

Using the Green function for the boundary-value problem (2.9), (2.10) [4], its solution may be expressed in the form

\[ y(x) = \frac{\beta(x)}{\beta(0)} \int_0^x \alpha(\xi) f(\xi) d\xi + \frac{\alpha(x)}{\beta(0)} \int_x^1 \beta(\xi) f(\xi) d\xi + \frac{\beta(x)}{\beta(0)} \int_0^1 \alpha(\xi) f(\xi) d\xi, \]  
(2.11)

where \( \alpha \) and \( \beta \) are solutions of the homogeneous equation (2.3) which satisfy the conditions.
\[ \alpha(0) = 0, \quad \alpha'(0) = -1, \quad (2.12) \]
\[ \beta(1) = 0, \quad \beta'(1) = 1. \quad (2.13) \]

The following lemma gives an expansion, in a neighborhood of \( x = 0 \), for the solution of the non-homogeneous equation (2.9), if the function \( f \) can be expanded in a certain form.

**Lemma 2.1.** Let \( p \) and \( q \) be real numbers, \( p > -2 \), \( q > 1 \), and \( s \) a function defined by

\[ s(x) = x \quad \text{if} \quad p + 2 \neq 1/n, \quad n \in \mathbb{N}, \quad (2.14) \]
\[ s(x) = x \ln x \quad \text{if} \quad p + 2 = 1/n, \quad n \in \mathbb{N}. \quad (2.15) \]

Let \( f \) denote a function which has the following expansion:

\[ f(x) = \sum_{k=1}^{\infty} \sum_{r=0}^{\infty} f_{k,r} x^{k(p+2)-2}s(x)^r, \quad 0 < x < \rho, \quad (2.16) \]

for some \( \rho \in \mathbb{R}^+ \). Then the solution of the linear BVP (2.9), (2.10) can be expanded in a neighborhood of \( x = 0 \) as follows:

\[ y(x) = 1 + y_{0,1}s(x) + \sum_{k=1}^{\infty} \sum_{r=0}^{\infty} y_{k,r} x^{k(p+2)}s(x)^r, \quad 0 < x < \rho. \quad (2.17) \]

**Proof.** From (2.5)–(2.8) we can see that \( \alpha \) and \( \beta \) (the solutions of the homogeneous equation which satisfy (2.12) and (2.13), respectively) have the following expressions:

\[ \alpha(x) = \alpha_1 x + \sum_{k=1}^{\infty} \alpha_k x^{k(p+2)+1}, \quad (2.18) \]
\[ \beta(x) = \beta_0 + \beta_1 s(x) + \sum_{k=1}^{\infty} \sum_{r=0}^{1} \beta_{k,r} x^{k(p+2)}s(x)^r. \quad (2.19) \]

Now we consider the terms on the right-hand side of (2.11):

\[ \int_{x}^{1} \beta(\xi)f(\xi) \, d\xi = \int_{x}^{\infty} \sum_{k=1}^{\infty} \sum_{r=0}^{\infty} g_{k,r} x^{k(p+2)-2}s(\xi)^r \, d\xi, \quad 0 < x < \rho. \]

Knowing that

\[ \int_{a}^{x} (\ln \xi)^r \, d\xi = x^{r+1} \sum_{i=0}^{r} c_i (\ln x)^i + \text{constant}, \]

we have

\[ \int_{x}^{1} \beta(\xi)f(\xi) \, d\xi = g_0 + \sum_{k=1}^{\infty} \sum_{r=0}^{\infty} g_{k,r} x^{k(p+2)-1}s(x)^r, \quad 0 < x < \rho. \]

Therefore the following expansion holds:

\[ \alpha(x) \int_{x}^{1} \beta(\xi)f(\xi) \, d\xi = \gamma_1 x + \sum_{k=1}^{\infty} \sum_{r=0}^{\infty} \gamma_{k,r} x^{k(p+2)}s(x)^r, \quad 0 < x < \rho. \quad (2.20) \]
Similarly we have
\[ \int_0^x \alpha(\xi) f(\xi) \, d\xi = \sum_{k=1}^{\infty} \sum_{r=0}^{\infty} c_{k,r} x^{k(p+2)} s(x)^r, \quad 0 < x < \rho. \]

Thus we obtain
\[ \int_0^x \beta(\xi) f(\xi) \, d\xi = \sum_{k=1}^{\infty} \sum_{r=0}^{\infty} \beta_{k,r} x^{k(p+2)} s(x)^r, \quad 0 < x < \rho. \] (2.21)

The expansion (2.17) is obtained by substituting (2.19)–(2.21) into (2.11).

With the help of Lemma 2.1, we can obtain the general form of the expansion, in a neighborhood of \( x = 0 \), of the Picard iterates which approximate the solution of the Emden–Fowler equation (1.1), (1.2).

This will be done in the next theorem.

**Theorem 2.2.** Let \( y_v \) be the \( v \)th Picard iterate, obtained as the solution of the linear BVP (2.1), (2.2). Let the initial approximation be \( y_0(x) = 1 - x \) or \( y_0(x) = 0 \). Then \( y_v \) has an expansion of the form
\[ y_v(x) = 1 + y_{0,1,v}(x) + \sum_{k=1}^{\infty} \sum_{r=0}^{\infty} y_{k,r,v} x^{k(p+2)} s(x)^r, \quad 0 < x < 1, \] (2.22)
where \( y_{k,r,v} \) are real coefficients independent of \( x \) and \( s \) the function defined by (2.14) and (2.15).

**Proof.** We shall prove this theorem by induction. Let us assume that \( y_v \) satisfies (2.22) for a certain \( v \geq 0 \), then we must verify that the same condition is satisfied by \( y_{v+1} \). It is known that the Picard iterate \( y_v \) satisfies \( 0 \leq y_v(x) < 1 \) for \( 0 < x < 1 \), when \( y_0(x) = 1 - x \) or \( y_0(x) = 0 \) [6]. Then
\[ y_v(x)^q = \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} \psi_{k,r,v} x^{k(p+2)} s(x)^r, \quad 0 < x < 1. \] (2.23)

Noting that \( x^p = x^{p+2} x^{-2} \) and substituting (2.22) and (2.23) into the right-hand side of (2.1), we verify that it has the same form as \( f \) in (2.16). Therefore we may apply Lemma 2.1, which assures that \( y_{v+1} \) will also satisfy condition (2.22). For \( y_0(x) = 1 - x \) the condition (2.22) is satisfied. In the case \( y_0(x) = 0 \), the first iterate \( y_1(x) \) which satisfies the boundary condition \( y_1(0) = 1 \), may be expressed, according to (2.4), in terms of Bessel functions. Therefore it has an expansion in the form (2.22). \( \Box \)

We now shall obtain an expansion, in a neighborhood of \( x = 1 \), of the Picard iterates. First we have the following result.

**Lemma 2.3.** Let \( f \) be a function which has the following expansion:
\[ f(x) = \sum_{r=3}^{\infty} f_{0,r}(1-x)^{r-2} + \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} f_{k,r}(1-x)^{k(q+1)+r-2}, \quad \varepsilon < x < 1, \] (2.24)
where \( \varepsilon \geq 0 \). Then the solution of the linear BVP (2.9), (2.10) may be expanded in a neighborhood of \( x = 1 \) as follows:
\[ y(x) = \psi_{0,1}(1-x) + \sum_{r=3}^{\infty} \psi_{0,r}(1-x)^r + \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} \psi_{k,r}(1-x)^{k(q+1)+r}, \quad \varepsilon < x < 1. \] (2.25)
Proof. The solutions of the homogeneous equation which satisfy (2.12) and (2.13), have the following expansions:

\[ \alpha(x) = \sum_{r=0}^{\infty} \alpha_r (1 - x)^r, \quad 0 < x < 1, \]  
(2.26)

\[ \beta(x) = \beta_1 (1 - x) + \sum_{r=3}^{\infty} \beta_r (1 - x)^r, \quad 0 < x < 1. \]  
(2.27)

Now we consider the terms of the right-hand side of (2.11)

\[ \int_{0}^{x} \alpha(\xi) f(\xi) \, d\xi = g_0 + \sum_{r=3}^{\infty} g_{0,r} (1 - x)^{r-1} + \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} g_{k,r} (1 - x)^{k(q+1)+r-1}, \quad 0 < x < 1. \]  

Then we obtain

\[ \beta(x) \int_{0}^{x} \alpha(\xi) f(\xi) \, d\xi = y_{0,1} (1 - x) + \sum_{r=3}^{\infty} y_{0,r} (1 - x)^{r-1} + \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} y_{k,r} (1 - x)^{k(q+1)+r-1}, \quad 0 < x < 1. \]  
(2.28)

Similarly we have

\[ \alpha(x) \int_{0}^{1} \beta(\xi) f(\xi) \, d\xi = \sum_{r=3}^{\infty} \delta_{0,r} (1 - x)^{r-1} + \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} \delta_{k,r} (1 - x)^{k(q+1)+r}, \quad 0 < x < 1. \]  
(2.29)

The expansion (2.25) is obtained by substituting (2.27)–(2.29) into (2.11). \( \square \)

With the help of Lemma 2.3, we can prove the following theorem.

**Theorem 2.4.** Let \( y_v \) be the \( v \)th Picard iterate, obtained as the solution of the linear BVP (2.1), (2.2). Let the initial approximation be \( y_0(x) = 1 - x \) or \( y_0(x) = 0 \). Then \( y_v \) has an expansion of the form

\[ y_v(x) = \sum_{r=3}^{\infty} \gamma_{0,r} (1 - x)^{r-1} + \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} \gamma_{k,r} (1 - x)^{k(q+1)+r}, \quad 0 < x < 1, \]  
(2.30)

where \( \gamma_{k,r,v} \) are real coefficients independent of \( x \).

**Proof.** Similar to the proof of Theorem 2.2. \( \square \)

2.2. Asymptotic expansions of the solution of the nonlinear BVP

From Theorems 2.2 and 2.4 we conclude that the solution of the nonlinear BVP (1.1), (1.2) has singularities at \( x = 0 \) and \( x = 1 \). Using (2.17) we shall look for the solution of that problem in the form

\[ y(x) = 1 + y_{0,1} x + \sum_{k=1}^{\infty} \sum_{r=0}^{\infty} y_{k,r} x^{k(p+2)+r}, \quad 0 < x < 1, \]  
(2.31)
if $1/(p + 2) \notin \mathbb{N}$, and
\[
y(x) = 1 + y_{0,1} x \ln x + \sum_{k=1}^{\infty} \sum_{r=0}^{\infty} y_{k,r} x^{k/n} (x \ln x)^r, \quad 0 < x < 1,
\] (2.32)
if $1/(p + 2) \in \mathbb{N}$.

In order to obtain the coefficients $y_{k,r}$ we substitute the expansion (2.31) or (2.32) into the nonlinear equation (1.1) and we equate the coefficients of the terms of both sides that have same power of $x$ and $\ln x$. The coefficient of $x$, which is denoted by $y_{0,1}$ in (2.31) and by $y_{n,0}$ in (2.32), remains undetermined and will be denoted by $d_0$ in the following expansions. This value may be obtained from the boundary condition $y(1) = 0$.

Let us denote $\varepsilon_2$ a function such that
\[
\varepsilon_2(x) = O(x^2), \quad \text{as } x \to 0.
\]
For $-1 < p < 0$, we have
\[
y(x) = 1 + d_0 x + y_{1,0} x^{p+2} + \varepsilon_2(x).
\]
In the case $p < -1$, if $1/(p + 2) \notin \mathbb{N}$, the asymptotic expansion has one of the forms
\[
y(x) = 1 + d_0 x + \sum_{k=1}^{2n} y_{k,0} x^{k(p+2)} + \sum_{k=1}^{n} y_{k,1} x^{k(p+2)+1} + \varepsilon_2(x),
\]
if \(1/n + 1/2 \leq p + 2 < 1/n\),
\[
y(x) = 1 + d_0 x + \sum_{k=1}^{2n+1} y_{k,0} x^{k(p+2)} + \sum_{k=1}^{n} y_{k,1} x^{k(p+2)+1} + \varepsilon_2(x),
\]
if \(1/n + 1 < p + 2 < 1/n + 1/2\), \(n = 1, 2, \ldots\).
\[
(2.33)
\]
\[
(2.34)
\]
The first coefficients $y_{k,r}$ on the right-hand side of (2.33) and (2.34) have the form
\[
y_{1,0} = \frac{1}{(p+2)(p+1)}, \quad y_{2,0} = \frac{q y_{1,0}}{4(p+2)(p+\frac{3}{2})}, \quad y_{3,0} = \frac{q y_{2,0} + \frac{4}{3}(q-1)y_{1,0}^2}{q(p+2)(p+\frac{5}{2})},
\]
\[
y_{1,1} = \frac{q d_0}{(p+3)(p+2)}, \quad y_{2,1} = \frac{q y_{1,1} + q(q-1)y_{1,0}d_0}{4(p+2)(p+\frac{5}{2})}, \quad \ldots
\]
Next, we present the form of the first coefficients $y_{k,r}$, for some particular cases where $p + 2 = 1/n$, $n \in \mathbb{N}$.

For $n = 1$ ($p = -1$) we have
\[
y(x) = 1 + y_{0,1} x \ln x + d_0 x + y_{1,1} x^2 \ln x + \varepsilon_2(x),
\] (2.35)
with
\[
y_{0,1} = 1, \quad y_{1,1} = \frac{1}{2} q.
\]
For $n = 2$ ($p = -\frac{3}{2}$) we have
\[
y(x) = 1 + y_{1,0} x^{1/2} + y_{0,1} x \ln x + d_0 x + y_{1,1} x^{3/2} \ln x + y_{3,0} x^{3/2} + y_{2,1} x^2 \ln x + \varepsilon_2(x),
\] (2.36)
with
\[
y_{1,0} = -4, \quad y_{0,1} = -4q, \quad y_{1,1} = -\frac{16q^2}{5}, \quad y_{3,0} = \frac{4q}{3}d_0 + \frac{32}{3}q(\frac{2q}{3} - 1), \quad y_{2,1} = 8q^2(\frac{2q}{3} - 1).
\]

In the same way, we may obtain asymptotic expansions of the solution of the nonlinear equation in the neighborhood of \(x = 1\). Using (2.25) we look for to the solution of (1.1), (1.2) in the form
\[
y(x) = \bar{y}_{0,0} + \sum_{k=1}^{\infty} \sum_{r=1}^{\infty} \bar{y}_{k,r}(1-x)^{k(q+1)+r}, \quad 0 < x < 1.
\]
Note that \(\bar{y}_{0,r} = 0, \forall r \geq 2\).

Substituting (2.37) into the nonlinear equation, we obtain the first coefficients \(\bar{y}_{k,r}\):
\[
\bar{y}_{1,1} = \frac{d_0^q}{(1+q)(2+q)}, \quad \bar{y}_{1,2} = \frac{-p d_0^q}{(q+2)(q+3)}, \quad \bar{y}_{1,3} = \frac{p(p-1)}{2(3+q)(4+q)}d_0^q,
\]
\[
\bar{y}_{2,1} = \frac{q d_0^{2q-1}}{2(1+q)^2(2+q)(3+2q)}.
\]

In the last formulas, \(d_0\) denotes an undetermined coefficient, whose value may be obtained from the boundary condition at \(x = 0\).

3. Numerical approximation

3.1. Outline of the numerical methods

In previous works (see [4,6]), the finite-difference method was applied to the numerical solution of some particular cases of the Emden–Fowler equation. The results obtained (with \(p = -\frac{1}{2}, -1\) and \(p = -\frac{3}{2}\)), have shown that the convergence of the method becomes slower as \(p\) approaches \(-2\), which is due to the singularity of the solution at the origin. Based on the analysis of the asymptotic behaviour of the Picard iterates, presented in [4], extrapolation methods were applied, which enabled to accelerate the convergence of the method for the considered examples. In the cases \(p = -\frac{1}{2}, q = \frac{3}{2}\) and \(p = -1, q = 2\), the results obtained by these methods had about 9–10 significant digits, and about 7 significant digits, in the case \(p = -\frac{3}{4}, q = -\frac{9}{4}\). As we shall see in this section, results with the same accuracy cannot be obtained for lower values of \(p\) using the known methods. Thus we have used the asymptotic expansions obtained in the first sections of the present work to improve the numerical approximation in such critical cases.

As we have seen in Section 2, the solution of problem (1.1), (1.2) may be approximated by the Picard iterates. Let us rewrite Eq. (2.1) in the form
\[
L y_v(x) := y_v''(x) + cq x^p y_v(x) = f(x, y_{v-1}(x)),
\]
where
\[
f(x, y(x)) = -cx^p(y(x)^q - qy(x)), \quad x \in [0, 1],
\]
with boundary conditions
\[ y_v(0) = 1, \quad y_v(1) = 0. \]  
(3.2)

According to (2.4), if \( y_0(x) \equiv 0 \), the solution of (3.1), (3.2) with \( v = 1 \) (first Picard iterate) may be expressed in terms of Bessel functions. Since the values of the last functions can be easily computed with the needed accuracy, for example, with the help of Mathematica, we have used them to check the accuracy of the numerical results.

Newton’s method may also be used to approximate the solution of (1.1), (1.2). In that case the iterative scheme has the form
\[ y''(x) + cq x^p y_{v-1}(x)^{q-1} y_v(x) = -c(1 - q)x^p y_{v-1}(x)^q, \quad v = 1, 2, \ldots, \]  
(3.3)

with the boundary conditions (3.2). This scheme was proved to converge to the solution of the nonlinear problem, if we start with \( y_0(x) = 1 - x \) [6].

Let us recall the numerical scheme which was used in [4]. Choosing an uniform grid with stepsize \( h = 1/N \) on the interval \([0, 1]\), we define the central difference operators as usual by
\[ \delta y(x) := \frac{y(x+h) - y(x-h)}{2h}, \]  
(3.4)

and
\[ \delta^2 y(x) := \frac{y(x+h) - 2y(x,h) + y(x-h)}{h^2}, \]  
(3.5)

where \( y \) is any real function on \([0, 1]\). Replacing the second derivative on the left-hand side of (3.1) by (3.5), we look for an approximate solution on the form \( y(x,h) \) which satisfies the set of equations
\[ \delta^2 y_v(x_i, h) - qx^p y_v(x_i, h) = f(x_i, y_{v-1}(x_i, h)), \quad i = 1, \ldots, N - 1, \]  
\[ y_v(0, h) = 1, \quad y_v(1, h) = 0, \]  
(3.6)

where \( x_i = ih \) are the knots of the uniform grid. In the present work we have used 6 different grids with \( h_0 = \frac{1}{100}, \quad h_1 = \frac{1}{2}h_0, \ldots, \quad h_5 = \frac{1}{200} \). To approximate the solution of the nonlinear problem, the Picard scheme was applied until the iterates satisfied the condition
\[ \sum_{i=1}^{N} \left( y_v(x_i, h) - y_{v-1}(x_i, h) \right)^2 \leq 10^{-11}. \]  
(3.7)

The first Picard iterate which satisfied (3.7) was taken as the approximate solution to the nonlinear problem.

The convergence of the finite-difference scheme (3.6) for the given class of problems was studied in [4]. There it was established that the error of the finite-difference solution allows an asymptotic expansion whose form is closely related to the asymptotic expansion of the solution near \( x = 0 \). Knowing the form of this expansion, we may apply the \( E \)-algorithm to accelerate the convergence of the numerical results.

We shall not describe here the \( E \)-algorithm which is discussed in [2,3]. The main idea of this algorithm is to transform a sequence of approximations, obtained with different grids (which are represented by \( \{E_k^{(n)}\}_{k \in \mathbb{N}} \), into new sequences, each one converging faster than the previous one. These new sequences are represented by \( \{E_k^{(n)}\}_{k \in \mathbb{N}}, \quad n = 1, 2, \ldots \). If we know \( k_r \) terms of the initial sequence (that is, if we have used \( k_r \) different grids), we can perform \( k_r - 1 \) transforms, provided that we know the form of the first \( k_r - 1 \) terms of the asymptotic error expansion.
In the present work, this extrapolation method was used to approximate the solution in the particular cases \( p = -1.5, q = 2.5, c = -1 \), \( p = -1.7, q = 2.7, c = -1 \) and \( p = -1.4, q = 1, c = -0.6 \). As it could be expected, due to the singularity of the solution, the accuracy of the results obtained by the described method in these examples is much lower than in the examples considered before. In order to overcome the difficulties that arise for values of \( p \) close to \( -2 \), we present here two techniques that diminish the effect of the singularity and improve the accuracy of the method:

**Variable substitution**

If we introduce the variable substitution \( t = x^{p+2} \) into the asymptotic expansion (2.22) we obtain

\[
y_v(t) = 1 + y_{0,v} s(t) + \sum_{k=1}^{\infty} \sum_{r=0}^{\infty} y_{k,r,v} t^k s(t)^r, \quad 0 < t < 1,
\]

where

\[
s(t) = t^{1/(p+2)}, \quad \text{if } \frac{1}{p+2} \notin \mathbb{N},
\]

and

\[
s(t) = \frac{1}{p+2} t^{1/(p+2)} \ln t, \quad \text{if } \frac{1}{p+2} \in \mathbb{N}.
\]

In the new variable, Eq. (2.1) becomes

\[
(p + 2)^2 \left[ t^2 \frac{d^2 y_v}{dt^2} + \left( 1 - \frac{1}{p+2} \right) t \frac{dy_v}{dt} \right] - q t y_v = t^{2/(p+2)} f \left( t^{1/(p+2)}, y_{v-1}(t) \right),
\]

\[
v = 1, 2, \ldots.
\]

If \( 1/(p+2) \in \mathbb{N} \), the considered variable substitution is equivalent to \( t = x^{1/n} \), which was used in [5], giving good results. The improvement of the numerical results, obtained by means of the variable substitution, occurs because of the smoothness of the solution in the new variable. Actually, if \( 1/(p+2) = n \), with \( n = 2, 3, \ldots \), the solution of (3.11) is \( n - 1 \) times differentiable at \( t = 0 \), while the solution of (2.1) is not differentiable at \( x = 0 \).

When \( 1/(p+2) \) is not an integer, the series on the right-hand side of (3.8) will contain non-integer powers of \( t \), the first of which is \( t^{1/(p+2)} \). Therefore, the solution will be \( r \) times differentiable at the origin, where \( r \) is the integer part of \( 1/(p+2) \). The particular case which arises when \( p \) is rational, say \( p = -m/n \) has also been treated in [5]. In this case, instead of the substitution \( t = x^{p+2} \), we should use \( t = x^{1/n} \), and the solution becomes infinitely differentiable at \( t = 0 \).

These situations are illustrated by the numerical examples of Sections 3.2, 3.3 and 3.4.

The considered variable substitution may also be introduced into the Newton’s iterative scheme (3.3).

**Subtracting the singular terms**

The singularity of the solution of (3.1), (3.2) at the origin is closely related with the form of the first terms of its asymptotic expansion. According to Theorem 2.2, this asymptotic expansion is given by (2.22). Comparing (2.22) with (2.32), (2.31), we see that the singular terms of the solution of the nonlinear problem have the same form as the ones of the Picard iterates. Knowing this we can derive a method to regularize the solution of (3.1), (3.2). Suppose that all the Picard iterates can be represented in the form

\[
y_v(x) = w(x) + y_v(x), \quad v = 1, 2, \ldots,
\]
where \( w \) is a known function (containing the singular terms of \( y \)) and \( \gamma_y \in C^1[0, 1] \). Then we rewrite (3.1) in the form

\[
L \gamma_y(x) = f(x, y_{v-1})(x) - Lw(x),
\]

the boundary conditions (3.2) being substituted by

\[
\gamma_y(0) = y_v(0) - w(0), \quad \gamma_y(1) = y_v(1) - w(1), \quad v = 1, 2, \ldots.
\]

Since the solution of (3.13), (3.14) is continuously differentiable on \( T_{v0} \), the finite-difference scheme for this new BVP has a convergence order higher than the old one. Once an approximation of \( y \) is computed, the corresponding approximation of \( y \) can be obtained by (3.12).

Next, we shall analyze two specific BVPs and apply the techniques described above to their numerical solution.

As a measure of accuracy, we shall use the concept of significant digits. When we say that the approximate value \( x \) has \( s \) significant digits (\( s \in \mathbb{R}^+ \)), this means that \( s = \log_{10} \varepsilon \varepsilon \) where \( \varepsilon \) is the relative error of \( x \). In the case of the first iterate, the exact solution is known, as noted above, and therefore the evaluation of the error is straightforward. In this case, we have verified, in the examples considered below, that the best approximations are obtained after no more than 4 steps of the extrapolation process. After this the results stabilized, and the number of significant digits did not increase any more. Taking this fact into account, in the case of the nonlinear problem we have carried out 5 steps of the extrapolation process. We observed that the difference between terms of the same sequence is smaller in each sequence, up to the 4th transform. Therefore we took the term of the 5th transform as the exact value and computed the errors of the previous approximations with respect to this term.

3.2. \( p = -\frac{3}{2}, \quad q = \frac{5}{2}, \quad c = -1 \)

In this case we have \( 1/(p + 2) = 2 \in \mathbb{N} \) and, according to Theorem 2.2 the asymptotic expansion of the Picard iterates has the form

\[
y_v(x) = 1 + y_{1,0,x} x^{1/2} + y_{0,1,x} x \ln x + y_{2,0,x} x + y_{1,1,x} x^{3/2} \ln x + y_{3,0,x} x^{3/2} + y_{2,1,x} x^2 \ln x + O(x^3).
\]

From here, using the method described in [4] we can obtain the following asymptotic error expansion for the finite-difference method

\[
y_v(x, h) - y_v(x) = C_1(x) h^{1/2} + C_2(x) h \ln h + C_3(x) h + C_4(x) h^{3/2} \ln h + O(h^{3/2}),
\]

where \( C_1, \; C_2, \; C_3, \; C_4 \), are functions that do not depend on \( h \). The numerical approximation of the first Picard iterate was obtained by the finite difference scheme (3.6) with 6 different stepsizes \((N = 100, 200, \ldots, 3200)\). As it could be expected, the finite difference method in this case converges very slowly, the results obtained with the finest grid \((N = 3200)\) having about 1.2 significant digits. Using these results and the expansion (3.16) we have applied the \textit{E-algorithm} to accelerate the convergence. The numerical values were compared with the exact solution, which, according to (2.4), can be expressed in the following form:

\[
y_1(x) = B_1 x^{1/2} J_2 \left( 4 \sqrt{-\frac{2}{x}} x^{1/4} \right) + B_2 x^{1/2} Y_2 \left( 4 \sqrt{-\frac{2}{x}} x^{1/4} \right),
\]
Table 1
First and last Picard iterates in the case $p = -1.5$, $q = 2.5$. The exact values of the first iterate are also given for comparison

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>First iterate approximate</th>
<th>First iterate exact</th>
<th>Last iterate approximate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.189832</td>
<td>0.1898353</td>
<td>0.57756</td>
</tr>
<tr>
<td>0.2</td>
<td>0.116591</td>
<td>0.1165948</td>
<td>0.48542</td>
</tr>
<tr>
<td>0.3</td>
<td>0.081913</td>
<td>0.0819163</td>
<td>0.41478</td>
</tr>
<tr>
<td>0.4</td>
<td>0.060700</td>
<td>0.0607028</td>
<td>0.35146</td>
</tr>
<tr>
<td>0.5</td>
<td>0.045773</td>
<td>0.0457752</td>
<td>0.29121</td>
</tr>
<tr>
<td>0.6</td>
<td>0.034191</td>
<td>0.0341929</td>
<td>0.23233</td>
</tr>
<tr>
<td>0.7</td>
<td>0.024497</td>
<td>0.0244986</td>
<td>0.17403</td>
</tr>
<tr>
<td>0.8</td>
<td>0.015874</td>
<td>0.0158747</td>
<td>0.11597</td>
</tr>
<tr>
<td>0.9</td>
<td>0.007819</td>
<td>0.0078189</td>
<td>0.057978</td>
</tr>
</tbody>
</table>

where $B_1$ and $B_2$ are known constants, $J_2$ and $Y_2$ are Bessel functions of the first and second kind, respectively. From this comparison, it follows that the best values obtained by extrapolation (which are displayed in Table 1) have 4.5–4.7 significant digits.

The approximate solution of the nonlinear problem which satisfies (3.7) is obtained after 31 iterations of the Picard scheme, if we start with $y_0(x) = 1 - x$ and take $N = 1000$ (the number of iterations increases for greater values of $N$). If we use the Newton’s iterative scheme (3.3), starting with the same initial approximation, 6 iterations are enough to obtain the same result.

The convergence of the finite-difference method may be accelerated using the E-algorithm, based on the error expansion (3.16). The numerical results obtained by extrapolation have about 5 significant digits and are displayed in Table 1.

More accurate results for the first Picard iterate and the solution of the nonlinear problem may be obtained if we apply the variable substitution $x = t^2$ and replace the original equation by (3.11), with $1/(p + 2) = 2$. Introducing the variable substitution into (3.15), we obtain the asymptotic expansion in the form

$$y_\nu(t) = 1 + y_{1,0,v} t + 2 y_{1,1,v} t^2 \ln t + y_{2,0,v} t^2 + 2 y_{1,1,v} t^3 \ln t + y_{3,0,v} t^3 + 2 y_{2,1,v} t^4 \ln t + O(t^4).$$  (3.18)

Therefore, when we apply the finite-difference method to approximate the solution, we can obtain the following asymptotic error expansion:

$$y_\nu(t, h) - y_\nu(t) = D_1(t) h^2 \ln h + D_2(t) h^2 + D_3(t) h^3 \ln h + D_4(t) h^3 + D_5(t) h^4 \ln h + O(h^4).$$  (3.19)

Using the E-algorithm, based on the asymptotic expansion (3.19), we obtain results with up to 11.5 significant digits. These results are displayed in Table 2, as well as the numerical solution of the nonlinear problem.
Table 2
First and last Picard iterates in the case $p = -1.5$, $q = 2.5$, using the variable substitution $t = x^{1/2}$

<table>
<thead>
<tr>
<th>$t_i$</th>
<th>First iterate</th>
<th>Last iterate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.5074943039</td>
<td>0.78512839300</td>
</tr>
<tr>
<td>0.2</td>
<td>0.3090986551</td>
<td>0.67239888065</td>
</tr>
<tr>
<td>0.3</td>
<td>0.2024185585</td>
<td>0.58963428398</td>
</tr>
<tr>
<td>0.4</td>
<td>0.1382715785</td>
<td>0.51805403317</td>
</tr>
<tr>
<td>0.5</td>
<td>0.09677295155</td>
<td>0.44871595438</td>
</tr>
<tr>
<td>0.6</td>
<td>0.06818040865</td>
<td>0.3762040895</td>
</tr>
<tr>
<td>0.7</td>
<td>0.04708484529</td>
<td>0.29715752345</td>
</tr>
<tr>
<td>0.8</td>
<td>0.03014400556</td>
<td>0.20896404161</td>
</tr>
<tr>
<td>0.9</td>
<td>0.01505002183</td>
<td>0.11016818138</td>
</tr>
</tbody>
</table>

Fig. 1. The improvement of the accuracy obtained by means of the E-algorithm in the case $p = -1.5$, $q = 2.5$, when no variable substitution is applied. Each line represents the number of significant digits of the entries of a sequence.

Fig. 2. The improvement of the accuracy obtained by means of the E-algorithm in the case $p = -1.5$, $q = 2.5$, when the variable substitution $t = x^{1/2}$ is applied. Each line represents the number of the significant digits of the entries of a sequence.
Table 3
Comparison of the relative error (as a function of $N$) using different methods, in the case $p = -1.5$, $q = -2.5$.
Method 1—simple finite-difference scheme; method 2—removing the singularity; method 3—variable substitution

<table>
<thead>
<tr>
<th>Method</th>
<th>$N = 100$</th>
<th>$N = 200$</th>
<th>$N = 400$</th>
<th>$N = 800$</th>
<th>$N = 1600$</th>
<th>$N = 3200$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.831E−01</td>
<td>0.584E−01</td>
<td>0.411E−01</td>
<td>0.290E−01</td>
<td>0.205E−01</td>
<td>0.145E−01</td>
</tr>
<tr>
<td>2</td>
<td>0.139E−01</td>
<td>0.547E−02</td>
<td>0.213E−02</td>
<td>0.826E−03</td>
<td>0.321E−03</td>
<td>0.127E−03</td>
</tr>
<tr>
<td>3</td>
<td>0.985E−03</td>
<td>0.309E−03</td>
<td>0.932E−04</td>
<td>0.273E−04</td>
<td>0.780E−05</td>
<td>0.218E−05</td>
</tr>
</tbody>
</table>

Figs. 1 and 2 show how the convergence is accelerated at each step of the extrapolation process, in the case of no variable substitution and with the variable substitution $t = x^{1/2}$, respectively. Each line in the graphic corresponds to a sequence of the E-algorithm, and the steeper the line is, the faster the corresponding sequence converges. Thus these graphics confirm that the extrapolation process indeed accelerates the convergence.

The solution of (3.1), (3.2) with $p = -3/2$ may be also approximated by using (3.12), where

$$y(x) = y_{2,0}x + y_{1,0}x^{3/2} \ln x + y_{3,0}x^{3/2} + y_{2,1}x^2 \ln x + O(x^2).$$

With this purpose let us prove the following lemma.

**Lemma 3.1.** Let $y_0(x) = 1 - x$ and $y_v(x)$ be the solution of (3.1), (3.2) with $p = -3/2$. Then we can write

$$y_v(x) = 1 - 4x^{1/2} - 4q x \ln x + \overline{y}(x), \quad v = 1, 2, \ldots,$$

where $\overline{y}$ has the form (3.20).

**Proof.** This lemma can be proved by induction. First we verify that $y_1$ satisfies (3.21). This may be done by considering Eq. (3.1) with $v = 1$ and equating coefficients of the same powers of $x$ in both sides of the equation. Then we assume that $y_{v-1}$ satisfies (3.21) and, substituting again into the equation, we obtain that $y_v$ also satisfies (3.21). □

Hence, the singular terms have the same form in all the Picard iterates. Note that the coefficients on the right-hand side of (3.21) are the same as in expansion (2.36). Defining

$$w(x) = 1 - 4x^{1/2} - 4q x \ln x + 3x,$$

we can replace the original problem by (3.13), (3.14) and solve numerically this last problem. Note that the term $3x$ is not singular and was added to $w(x)$ only to obtain $w(1) = 0$. The relative errors of the numerical solutions obtained by the considered method with different stepsizes (without extrapolation) are displayed in Table 3. Comparing these results with the ones obtained by other methods, we observe that this technique improves significantly the accuracy of the finite-difference scheme but it is not as efficient as the variable substitution. The asymptotic expansion of the solution in this last case is given by (3.20). From this formula, we obtain a new asymptotic error expansion:

$$\overline{y}_v(x, h) - \overline{y}(x) = C'_1(x)h^{1/2} \ln h + C'_2(x)h^{3/2} + C'_3(x)h^2 \ln h + O(h^2).$$

After extrapolation, based on this asymptotic expansion, the results have 6.7–6.8 significant digits.
Table 4
First and last Picard iterates in the case $p = -1.7, q = 2.7$

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>First iterate approximate</th>
<th>exact</th>
<th>Last iterate approximate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.02447</td>
<td>0.024400</td>
<td>0.4263</td>
</tr>
<tr>
<td>0.2</td>
<td>0.01311</td>
<td>0.013110</td>
<td>0.3638</td>
</tr>
<tr>
<td>0.3</td>
<td>0.00865</td>
<td>0.008653</td>
<td>0.3135</td>
</tr>
<tr>
<td>0.4</td>
<td>0.006193</td>
<td>0.0061951</td>
<td>0.2669</td>
</tr>
<tr>
<td>0.5</td>
<td>0.004576</td>
<td>0.0045782</td>
<td>0.2217</td>
</tr>
<tr>
<td>0.6</td>
<td>0.003377</td>
<td>0.0033794</td>
<td>0.1771</td>
</tr>
<tr>
<td>0.7</td>
<td>0.001552</td>
<td>0.00155269</td>
<td>0.08848</td>
</tr>
<tr>
<td>0.8</td>
<td>0.000763</td>
<td>0.00076353</td>
<td>0.04424</td>
</tr>
</tbody>
</table>

3.3. $p = -1.7, q = 2.7, c = -1$

Let us consider a new case of Eq. (3.1), (3.2), with $p = -1.7, q = 2.7$. In this case, we have $1/(p + 2) = \frac{10}{7} \notin \mathbb{N}$ and therefore the asymptotic expansion of the Picard iterates, according to (2.22), has the form

$$y_0(x) = 1 + y_{1,0} x^{0.3} + y_{2,0} x^{0.6} + y_{3,0} x^{0.9} + y_{0,1,2} + y_{4,0} x^{1.2} + y_{1,1} x^{1.3} + O(x^{1.5}). \quad (3.23)$$

The numerical results show that the convergence of the finite-difference method is even slower than in the previous example. The first Picard iterate in this case may be represented, according to (2.4), in the form

$$y_1(x) = B_1' x^{1/2} J_{10/3} \left( \frac{20}{7} \sqrt{-2.7} x^{0.15} \right) + B_2' x^{1/2} J_{-10/3} \left( \frac{20}{7} \sqrt{-2.7} x^{0.15} \right), \quad (3.24)$$

where $B_1'$ and $B_2'$ are known constants, and $J_{-10/3}$ and $J_{10/3}$ are Bessel functions of the first kind. Comparing the numerical results, obtained with $N = 3200$, with the exact values, given by (3.24), we see that the relative error is about 0.8–0.9, which means that these results would be meaningless without extrapolation. The asymptotic error expansion used for extrapolation is obtained, in this case, from (3.23) and has the form

$$y_e(x, h) - y_e(x) = F_1(x) h^{0.3} + F_2(x) h^{0.6} + F_3(x) h^{0.9} + F_4(x) h^{1.2} + F_5(x) h^{1.3} + O(h^{1.5}), \quad (3.25)$$

where the $F_i$s are functions that do not depend on $h$. The relative error of the best results obtained by extrapolation (displayed in Table 4), is smaller than $3 \times 10^{-3}$.

In order to approximate the solution of the nonlinear problem with the given values of $p$ and $q$, we have carried out 45 iterations of the Picard scheme (in the case $N = 1000$, starting with $y_0(x) = 1 - x$). When the Newton’s method was used, only 6–7 iterations were needed (depending on the step size).
Table 5
First and last Picard iterates in the case $p = -1.7$, $q = 2.7$, using the variable substitution $t = x^{1/10}$

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>First iterate</th>
<th>Last iterate</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.9872843028774</td>
<td>0.995290115645</td>
</tr>
<tr>
<td>0.2</td>
<td>0.9052524376328</td>
<td>0.964804333031</td>
</tr>
<tr>
<td>0.3</td>
<td>0.7282382925916</td>
<td>0.897378124064</td>
</tr>
<tr>
<td>0.4</td>
<td>0.499551512140</td>
<td>0.802786545870</td>
</tr>
<tr>
<td>0.5</td>
<td>0.290560395405</td>
<td>0.699275915573</td>
</tr>
<tr>
<td>0.6</td>
<td>0.144189688434</td>
<td>0.600390614680</td>
</tr>
<tr>
<td>0.7</td>
<td>0.061692162516</td>
<td>0.510468132039</td>
</tr>
<tr>
<td>0.8</td>
<td>0.022987835691</td>
<td>0.420626430652</td>
</tr>
<tr>
<td>0.9</td>
<td>0.007307532243</td>
<td>0.2904705454</td>
</tr>
</tbody>
</table>

Table 6
Comparison of the relative error (as a function of $N$) using different methods. Method 1—simple finite-difference scheme; method 2—variable substitution

<table>
<thead>
<tr>
<th>Method</th>
<th>$N = 100$</th>
<th>$N = 200$</th>
<th>$N = 400$</th>
<th>$N = 800$</th>
<th>$N = 1600$</th>
<th>$N = 3200$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.265E+00</td>
<td>0.208E+00</td>
<td>0.165E+00</td>
<td>0.131E+00</td>
<td>0.105E+00</td>
<td>0.845E−01</td>
</tr>
<tr>
<td>2</td>
<td>0.336E−03</td>
<td>0.840E−04</td>
<td>0.210E−04</td>
<td>0.525E−05</td>
<td>0.131E−05</td>
<td>0.324E−06</td>
</tr>
</tbody>
</table>

In order to accelerate the convergence, we applied the E-algorithm, based on the asymptotic expansion (3.25), since it is valid for all the Picard iterates. The results obtained after extrapolation have approximately 3 significant digits and are displayed in Table 4.

A significant improvement of the accuracy can be obtained by means of a variable substitution. Since $1/(p + 2) = \frac{10}{7}$ is a rational number, instead of the variable substitution $x = t^{10/3}$, we may use $x = t^{10}$. When we introduce this variable substitution into (3.23), we see that $y_\nu(t)$ may be expanded in a series of integral powers of $t$, which means that $y_\nu(t)$ is infinitely differentiable at $t = 0$. When the solution has this property, it is known that the error of the finite difference scheme has an expansion in even powers of $h$. Then the convergence can be accelerated by the Richardson’s extrapolation, which is a particular case of the E-algorithm. The results obtained by extrapolation in this case are displayed in Table 5 and their relative error is about $10^{-12}$.

We have applied the variable substitution $x = t^{10}$ to each Picard iterate and obtained a new approximation of the solution of the nonlinear equation. The Richardson extrapolation was used again to accelerate the convergence, giving about 12 significant digits. These results are displayed in Table 5. The relative errors of the results obtained with variable substitution and without it are displayed in Table 6.

One interesting property of the obtained numerical results is that the relative error, in many cases, is constant or almost constant with respect to the independent variable ($x$ or $t$). This can be seen in Fig. 3, where the graphic of the relative error, as a function of $x$, is plotted for two different cases.
Fig. 3. The relative error of the approximations obtained by the finite-difference method with \( h = 0.01 \) when no variable substitution is applied, in the cases \( p = -1.5, q = 2.5 \) and \( p = -1.7, q = 2.7 \).

### Table 7

Numerical results after extrapolation in the case \( p = -1.4, q = 2, c = -0.6 \), and the corresponding results obtained by Mooney [8]

<table>
<thead>
<tr>
<th>( x_i )</th>
<th>( y(x_i) )</th>
<th>Mooney</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.69545</td>
<td>0.69544</td>
</tr>
<tr>
<td>0.2</td>
<td>0.58707</td>
<td>0.58706</td>
</tr>
<tr>
<td>0.3</td>
<td>0.50107</td>
<td>0.50106</td>
</tr>
<tr>
<td>0.4</td>
<td>0.42374</td>
<td>0.42374</td>
</tr>
<tr>
<td>0.5</td>
<td>0.35048</td>
<td>0.35047</td>
</tr>
<tr>
<td>0.6</td>
<td>0.27924</td>
<td>0.27924</td>
</tr>
<tr>
<td>0.7</td>
<td>0.20899</td>
<td>0.20899</td>
</tr>
<tr>
<td>0.8</td>
<td>0.13920</td>
<td>0.13920</td>
</tr>
<tr>
<td>0.9</td>
<td>0.06958</td>
<td>0.06958</td>
</tr>
</tbody>
</table>

3.4. \( p = -1.4, q = 2.0, c = -0.6 \)

Finally, we have carried out a numerical experiment for a case which was considered before by Mooney in [8]. As in the previous examples we have computed the numerical solution by two methods. The first method is the application of the finite-difference scheme (3.6) and the extrapolation of the results by the E-algorithm, based on the asymptotic error expansion

\[
y(x, h) - y_h(x) = C_1(x)h^{0.6} + C_2(x)h + C_3(x)h^{1.2} + O(h^{1.6}). \tag{3.26}
\]

The results obtained by this method are displayed in Table 7 and are very close to those obtained by Mooney [8], with about 5 significant digits. The second method uses the variable substitution \( t = x^{1/5} \). Since \( p = -\frac{4}{5} \), with this substitution the solution becomes infinitely differentiable near the origin and the Richardson extrapolation may be successfully applied. In this case the accuracy is drastically improved, up to 12 significant digits. An example of the result obtained by this method may be seen in Table 8,
Table 8
Extrapolation process for the approximation of $y(0.9)$, with $p = -1.4$, $q = 2$, $c = -0.6$ when the variable substitution $t = x^{1/5}$ is used. The corresponding value in the old variable, obtained from Table 7 by interpolation, is $y(0.9^5) \approx 0.28596$

<table>
<thead>
<tr>
<th>$E_k^{(0)}$</th>
<th>$E_k^{(1)}$</th>
<th>$E_k^{(2)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.285968949063</td>
<td>0.285961438483</td>
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<tr>
<td>0.285963316128</td>
<td>0.285961438555</td>
<td>0.2859614385604</td>
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<tr>
<td>0.285961907948</td>
<td>0.285961438560</td>
<td>0.2859614385609</td>
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<tr>
<td>0.285961555907</td>
<td>0.285961438560</td>
<td>0.2859614385609</td>
</tr>
</tbody>
</table>

Fig. 4. Graphics of some Picard iterates, in the case $p = -1.5$, $q = 2.5$, when no variable substitution is used.

where the original sequence of approximations of $y(0.9)$ is displayed, as well as the transforms obtained by the Richardson’s extrapolation.

3.5. Concluding remarks

The codes of the algorithms were written in FORTRAN. The extrapolation was carried out with the help of the programs given in [3]. All the computations were carried out in a Pentium at 166 MHz and the set of results with the considered 6 grids are obtained in about 3–4 seconds.

In the examples above, we have seen that the asymptotic expansions of the Picard iterates and the solution, near the origin, give us useful information that can be employed to improve the convergence of the numerical approximation. However, the direct use of these series is not a good way to obtain approximations of the solution near the origin, due to the existence of the singular terms and the growth of the $y_{k,r}$ coefficients. Moreover, some of the coefficients of the series (for example, $d_0$ in (2.33) and (2.34)), depend on the boundary condition at $x = 1$ and can be obtained only by the numerical solution of the equation.

On the other hand, the asymptotic expansion of the solution at $x = 1$, given by (2.37), has fast convergence and can be used to obtain a reasonable approximation of the solution, for values of $x$ not close to 0. However, this requires the use of $d_0$, a constant that can be evaluated only numerically. It should also be noted that, for the considered range of values of $q$, the second derivative of the solution is small when $x \approx 1$ and therefore the solution has a “linear” behaviour near this endpoint.

This may be observed in the graphics of Fig. 4, which represent some of the Picard iterates in the case $p = -1.5$, $q = 2.5$. When the variable substitution $t = x^{1/2}$ is applied, these iterates have the graphics
Fig. 5. Graphics of some Picard iterates, in the case $p = -1.5$, $q = 2.5$, when the variable substitution $t = x^{1/2}$ is applied. Comparing with the previous figure, we see that in this case the functions are much smoother near the origin.

displayed in Fig. 5. We can see that in this case the graphics are not so steep as $t$ tends to 0, which results from the boundedness of the derivative at this point.

Although the examples considered in this paper do not come from any specific physical problem, they have the the same kind of singularities that arise in many models of physics and engineering. Therefore, we think that the present study may find application in the solution of practical problems of these fields.

References