An extrapolation method for a Volterra integral equation with weakly singular kernel

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Abstract

In this work we consider second kind Volterra integral equations with weakly singular kernels. By introducing some appropriate function spaces we prove the existence of an asymptotic error expansion for Euler’s method. This result allows the use of certain extrapolation procedures which is illustrated by means of some numerical examples. © 1997 Elsevier Science B.V.

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1. Introduction

We consider the second kind Volterra integral equations

\[ u(t) = \int_{0}^{t} K(t, s)p(t, s)u(s) \, ds + f(t), \quad t \in (0, T], \] (1.1)

\[ p(t, s) = \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\ln(t/s)}} \frac{s^{\mu-1}}{t^\mu}, \] (1.2)

and

\[ u(t) = \int_{0}^{t} K(t, s)q(t, s)u(s) \, ds + g(t), \quad t \in (0, T], \] (1.3)

\[ q(t, s) = \frac{s^{\mu-1}}{t^\mu}. \] (1.4)

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where \( \mu > 0 \), \( K(t, s) \) is a smooth function and \( f \) and \( g \) are given functions. Equations of this type arise from certain heat conduction problems (cf. [1,12]). The special case of (1.1), with \( K(t, s) = 1 \), was studied theoretically in [12] where a somewhat complicated \( L_2 \) solution was derived, by using a method involving Watson transforms. In [8,11] some analytic expressions valid in weighted \( L_p \)-spaces were obtained, but none is useful if a tabulated solution is required. Further results were obtained in [13], where it was shown that a unique solution to (1.1) exists in \( C^m[0, T] \) if \( K(t, s) = 1 \) and \( \mu > 1 \). In [6] the general equations (1.1) and (1.3) were considered, with

\[
K(t, t) = 1, \quad K(t, s) = \text{constant}, \quad K(t, s) = h(t).
\]

There some results on existence and uniqueness of solution in the spaces \( C^m[0, T] \) were presented and some regularity estimates were derived. In particular, when \( K(t, s) = 1 \) and \( 0 < \mu \leq 1 \), Eqs. (1.1) and (1.3) were shown to have a family solutions in \( C[0, T] \) of which only one has \( C^1 \) continuity.

We note that the functions \( p(t, s) \) and \( q(t, s) \) in (1.1) and (1.3) are integrable with respect to \( s \) in \([0, T]\). However, unlike most weakly singular kernels, they do not satisfy

\[
\int_0^t p(t, s) \, ds \to 0 \quad \text{and} \quad \int_0^t q(t, s) \, ds \to 0 \quad \text{as} \quad t \to 0^+.
\]

This fact creates some difficulties in the convergence analysis of numerical methods for the above equations. Two low order product integration methods have been studied in [13], where approximations to the solution of (1.1) of orders one and two were obtained with the so called Euler and trapezoidal methods, respectively. In [5] a fourth order method was developed which employs cubic splines in the continuity class \( C^1[0, T] \). Eq. (1.1), with \( K(t, s) = 1 \), was shown to be equivalent to (1.3), whose kernel \( q(t, s) \), although unbounded, is more tractable than \( p(t, s) \). The approximate solution was then required to satisfy both Eq. (1.3) and its differentiated form at the mesh points (Hermite-type collocation). High order product integration methods have also been investigated in [4]. Although the convergence results in the above mentioned papers were obtained for the case \( K(t, s) = 1 \), they can be extended to the general equations (1.1) and (1.3).

In the present work we shall be concerned with the numerical solution of (1.3) by a low order method in conjunction with extrapolation procedures. Extrapolation methods based on the midpoint rule have been studied in [7] for nonlinear Volterra equations with smooth kernels. In [10] a comprehensive analysis of extrapolation methods and their application to various types of functional equations is presented. In particular, an extrapolation method based on the trapezoidal rule is studied in detail.

For the sake of simplicity, we shall assume that \( K(t, s) = 1 \) in (1.3). However, it is easily seen that the techniques used in this paper can be extended to the general equation. This work will be organised as follows. In Section 2, we transform Eq. (1.3) into a new equation, so that the two equations are equivalent away from the origin. Moreover, under appropriate conditions, although for a certain \( g \) the solution of the original equation may be unbounded at the origin, the corresponding solution of the new equation will be smooth. In Section 3, Euler's method is used to approximate the solution of the new equation. Following the approach developed in [10], we show that an asymptotic error expansion is valid which permits the use of certain extrapolation procedures. This allows us to improve the low accuracy of the approximations obtained by Euler's method. The theoretical results are confirmed by some numerical experiments.
2. Existence and uniqueness of solution

Let us consider the following equation:

\[ Lu(t) = g(t), \quad t \in (0, T), \]  \hspace{1cm} (2.1)

where

\[ Lu(t) := u(t) - \int_0^t \frac{s^{\mu-1}}{t^{\mu}} u(s) \, ds. \]  \hspace{1cm} (2.2)

Associated with (2.1) is the following equation:

\[ v(t) = \int_0^t \frac{s^{\mu-1-\beta}}{t^{\mu-\beta}} v(s) \, ds + \bar{g}(t), \]  \hspace{1cm} (2.3)

where \( \beta \geq 0 \) is a real number and

\[ v(t) := t^\beta u(t), \]

\[ \bar{g}(t) := t^\beta g(t). \]  \hspace{1cm} (2.4)

Introducing

\[ L_\alpha v(t) := v(t) - \int_0^t \frac{s^{\mu-1-\beta}}{t^{\mu-\beta}} v(s) \, ds, \]  \hspace{1cm} (2.6)

we can rewrite (2.3) as

\[ L_\alpha v(t) = \bar{g}(t). \]  \hspace{1cm} (2.7)

We start by introducing some definitions.

**Definition 2.1.** For \( T > 0 \) and \( m \) a non-negative integer, \( V_m[0, T] \) denotes the normed space of the real valued functions \( f \) such that \( f \in C^m[0, T] \) with

\[ \| f \|_m := \max_{0 \leq j \leq m} \max_{t \in [0, T]} \left| f^{(j)}(t) \right|. \]  \hspace{1cm} (2.8)

**Definition 2.2.** For \( T > 0, \ \beta \geq 0 \) and \( m \geq 0 \), \( V_m,\beta[0, T] \) denotes the normed space of real valued functions \( f \) such that \( t^\beta f \in V_m[0, T] \) with

\[ \| f \|_{m,\beta} := \| t^\beta f \|_m := \max_{0 \leq j \leq m} \max_{t \in [0, T]} \left| \frac{d^j}{dt^j} (t^\beta f(t)) \right|. \]  \hspace{1cm} (2.9)

**Definition 2.3.** Let \( m \geq 1 \). Then \( f \in V_m^0[0, T] \) if and only if \( f \in V_m[0, T] \) and \( f(0) = f'(0) = \cdots = f^{(m-1)}(0) = 0 \). If \( m = 0 \), we have \( V_0^0[0, T] = V_0[0, T] \).

**Definition 2.4.** Let \( m \geq 1 \) and \( \beta \geq 0 \). Then \( f \in V_m^\alpha[0, T] \) if and only if \( t^\beta f \in V_m^\alpha[0, T] \). If \( m = 0 \), we have \( V_0^\alpha[0, T] = V_0^\alpha[0, T] \).
We see that $V^{0}_{m}(0,T) \subset V_{m}(0,T)$ and that $V^{0}_{m,\beta}(0,T) \subset V_{m,\beta}[0,T]$, for every $\beta > 0$, $m \geq 1$. Moreover, it is easily proved that $V^{0}_{m}(0,T) \subset V^{0}_{m,\beta}(0,T)$, for every $\beta > 0$, and that $V^{0}_{m,0}(0,T) = V^{0}_{m}(0,T)$, for every $m \geq 1$. On the other hand, for every $m \geq 1$ and $\beta > 0$, there are elements of $V^{0}_{m,\beta}(0,T)$ that do not belong to $V_{m}(0,T)$.

Throughout this paper, $m$ will always be a non-negative integer and $T > 0$, $\beta \geq 0$ will be real numbers. The following theorem has been proved in [13].

**Theorem 2.1.** If $\mu > 1$ in (2.1) and the function $g$ in (2.1) belongs to $V_{m}[0,T]$ then (2.1) has a unique solution $u \in V_{m}[0,T]$.

The following theorem extends the results of Theorem 2.1 to the spaces $V_{m,\beta}[0,T]$.

**Theorem 2.2.** Let $\beta \geq 0$ and $\mu > 1 + \beta$.

(i) If the function $g$ in (2.1) belongs to $V_{m,\beta}[0,T]$ then (2.1) has a unique solution $u \in V_{m,\beta}[0,T]$.

(ii) If $g \in V^{0}_{m,\beta}(0,T)$ then (2.1) has a unique solution $u \in V^{0}_{m,\beta}[0,T]$.

**Proof.** (i) We follow the proof technique used in [13]. For any $u \in V_{m,\beta}[0,T]$, we define $w = S_{\mu}u$ such that

$$
\frac{d}{dt} w(t) = \int_{0}^{t} \frac{s^{\mu-1}}{t^{\mu}} u(s) \, ds + g(t).
$$

(2.10)

By a change of variables $s = t\sigma$, we obtain

$$
w(t) = \int_{0}^{1} \sigma^{\mu-1} u(t\sigma) \, d\sigma + g(t).
$$

(2.11)

Multiplying both sides of (2.11) by $t^{\beta}$ yields

$$
t^{\beta} w(t) = \int_{0}^{1} (t\sigma)^{\beta} u(t\sigma) \sigma^{\mu-1-\beta} \, d\sigma + t^{\beta} g(t).
$$

(2.12)

Thus, for $0 < j < m$, we have

$$
\frac{d^{j}}{dt^{j}} (t^{\beta} w(t)) - \int_{0}^{1} \frac{d^{j}}{dt^{j}} [(t\sigma)^{\beta} u(t\sigma)] \sigma^{\mu-1-\beta} \, d\sigma + \frac{d^{j}}{dt^{j}} (t^{\beta} g(t)), \quad \forall t \in [0,T].
$$

(2.13)

Setting $s = t\sigma$, we may write

$$
\frac{d^{j}}{dt^{j}} [(t\sigma)^{\beta} u(t\sigma)] = \frac{d^{j}}{ds^{j}} [s^{\beta} u(s)] \sigma^{j}.
$$

(2.14)

Now let $u_{1}, u_{2} \in V_{m,\beta}[0,T]$ and $w_{1} = S_{\mu}(u_{1})$, $w_{2} = S_{\mu}(u_{2})$. Then, from (2.14), (2.13) and an application of the mean value theorem for integrals, we obtain
From (2.15) it follows that
\[
\left| \frac{d^j}{dt^j} (t^\beta w_1(t)) - \frac{d^j}{dt^j} (t^\beta w_2(t)) \right| < \frac{1}{\mu - \beta} \left\| w_1 - w_2 \right\|_{\beta,m}.
\]
(2.15)
\[
\left| \frac{d^j}{dt^j} \left[ s^\beta (u_1(s) - u_2(s)) \right] \right| \leq \frac{1}{\sigma^{\mu-1-\beta+j}} d\sigma, \quad \forall t \in [0, T].
\]
(2.16)
From (2.15) it follows that
\[
\left\| w_1 - w_2 \right\|_{\beta,m} < \frac{1}{\mu - \beta} \left\| w_1 - w_2 \right\|_{\beta,m}.
\]
(2.16)
Since \( \mu > 1 \), (2.16) implies that \( S_\mu \) is a contraction mapping on the Banach space \( V_{m,\beta}[0, T] \). Therefore \( S_\mu \) has a unique fixed point \( u \in V_{m,\beta}[0, T] \), which is the solution of (2.1) in this space.

(ii) Let us now prove the existence of a unique solution of (2.1) in \( V_{m,\beta}[0, T] \). Since this is a subspace of \( V_{m,\beta}[0, T] \), we just have to show that
\[
S_\mu u \in V_{m,\beta}[0, T] \quad \text{whenever } u \in V_{m,\beta}[0, T].
\]
(2.17)
In order to prove (2.17), we let \( t \to 0^+ \) in (2.13) and, using the mean value theorem for integrals and the fact that \( t^\beta g \in V_0^0[0, T] \), we get
\[
\lim_{t \to 0^+} \frac{d^j}{dt^j} \left( t^\beta S_\mu u(t) \right) = \lim_{t \to 0^+} \frac{d^j}{dt^j} \left[ (t\xi)^\beta u(t\xi) \right] \int_0^1 \sigma^{\mu-1-\beta+j} d\sigma, \quad j = 0, 1, \ldots, m.
\]
(2.18)
where \( \xi \in [0, 1] \). Since \( u \in V_{m,\beta}[0, T] \) the limit on the right-hand side of (2.18) exists for \( j \leq m \), being equal to 0 when \( j \leq m - 1 \). Hence \( S_\mu u \in V_{m,\beta}[0, T] \) and this concludes the proof.

**Remark.** An alternative way of proving part (i) of Theorem 2.2 could be the following. If \( g \) is a function of \( V_{m,\beta}[0, T] \) then \( \tilde{g} \), defined by (2.5), belongs to \( V_{m}[0, T] \). Now, since \( \mu > 1 + \beta \), Theorem 2.1 can be applied to (2.7) and we conclude that this equation has a unique solution \( u \in V_{m}[0, T] \). But there exists only one \( u \in V_{m,\beta}[0, T] \) such that \( u(t) := v(t)/t^\beta \) and, of course, \( u \) is the solution of (2.1).

**Example 2.1.** Let us consider Eq. (2.1) with
\[
g(t) = \frac{1 - \cos t}{\sqrt{t}}.
\]
Since \( t^{1/2} g(t) = 1 - \cos t \in V_2^0[0, T] \), then \( g \in V_2[0, T] \). If \( \mu > 3/2 \), then it follows from Theorem 2.2 that (2.1) has a unique solution \( u \in V_2[0, T] \).

We now give two lemmas which will be used in the next section.

**Lemma 2.1.** If \( g \in V_{m,\beta}[0, T] \), with \( m \geq 1 \), then \( g' \in V_{m-1,\beta}[0, T] \). In general, if \( m \geq k \) then the \( k \)th order derivative of \( g \) belongs to \( V_{m-k,\beta}[0, T] \).

**Proof.** Since \( g \in V_{m,\beta}[0, T] \), we have that \( \tilde{g}(t) := t^\beta g \in V_{m}[0, T] \). Therefore, using Taylor’s theorem, we may write
\[
t^\beta g(t) = \frac{t^m}{m!} \tilde{g}^{(m)}(0) + r_m(t), \quad \forall t \in [0, T],
\]
(2.19)
where \( r_m = o(t^m) \). Dividing both sides of (2.19) by \( t^\beta \), we obtain
\[
g(t) = \frac{t^{m-\beta}}{m!} g^{(m)}(0) + t^\beta r_m(t), \quad \forall t \in (0, T].
\] (2.20)

From (2.20) it follows that
\[
g'(t) = \frac{(m-\beta)t^{m-\beta-1}}{m!} g^{(m)}(0) - \beta t^{\beta-1} r_m(t) + t^\beta r'_m(t), \quad \forall t \in (0, T].
\] (2.21)

Finally, multiplying both sides of (2.21) by \( t^\beta \), we obtain
\[
t^\beta g'(t) = \frac{(m-\beta)t^{m-\beta-1}}{m!} g^{(m)}(0) - \beta t^{\beta-1} r_m(t) + t^\beta r'_m(t), \quad \forall t \in [0, T].
\] (2.22)

Since \( r_m(t) = o(t^m) \), by applying the l'Hospital's rule we may conclude that
\[
\beta t^{\beta-1} r_m(t) + r'_m(t) = o(t^{m-1}).
\]

Therefore, from (2.22) it follows that \( t^\beta g'(t) \in V^0_{m-1}[0, T] \). Thus, \( g' \in V^0_{m-1, \beta}[0, T] \).

Using this result repeatedly, we may conclude that if \( g \in V^0_{m, \beta}[0, T] \) and \( m \geq k \) then \( g^{(k)} \) belongs to \( V^0_{m-k, \beta}[0, T] \). 

**Lemma 2.2.** Let \( g \in V^0_{k, \beta}[0, T] \) and let \( \nu \) be a real number. If \( k + [\nu] \geq 0 \), then \( t^\nu g \in V^0_{k+\nu, \beta}[0, T] \), where \([\nu]\) stands for the integral part of \( \nu \).

**Proof.** We have, with \( \bar{g}(t) := t^\beta g(t) \),
\[
t^\beta g(t) = \frac{t^k}{k!} \bar{g}^{(k)}(0) + O(t^{k+1}).
\] (2.23)

Multiplying both sides of (2.23) by \( t^\nu \), we obtain
\[
t^{\beta+\nu} g(t) = \frac{t^{k+\nu}}{k!} \bar{g}^{(k)}(0) + O(t^{k+\nu+1}).
\] (2.24)

Differentiating \( r \) times both sides of (2.24), we may write
\[
d^r dt^r(t^{\beta+\nu} g(t)) = (k + \nu) \cdots (k + \nu - r + 1) \frac{t^{k+\nu-r}}{k!} \bar{g}^{(k)}(0) + O(t^{k+\nu-r+1}),
\] (2.25)

where \( r = 1, 2, \ldots, k + [\nu] \). From (2.25), it follows that \( t^\nu g \in V^0_{k+\nu, \beta}[0, T] \). 

3. **Euler’s method and asymptotic error expansion**

3.1. **Euler’s method**

In this section, we show that an approximation to the solution \( u \) of the original equation (2.1) can be obtained by applying Euler’s method to the transformed equation (2.7). Let us assume that the function \( g \) in (2.1) belongs to \( V^0_{m, \beta}[0, T] \) for some \( \beta \geq 0 \) and \( m > 0 \), and let \( \mu > 1 + \beta \). Then the function \( \bar{g} \) in (2.7) is in \( V^0_{m}[0, T] \). From Theorem 2.2 it follows that (2.1) and (2.7) are uniquely
solvable in the spaces $V^0_{m,\beta}[0,T]$ and $V^0_m[0,T]$, respectively. Moreover, the solutions $u$ and $v$ of these equations can be related through (2.4) for $t > 0$.

In order to approximate $v$ on $[0,T]$, let us define an uniform grid $X_h$, with stepsize $h := T/N$,

$$X_h := \{ t_i = ih, \ 0 \leq i \leq N \}. \quad (3.1)$$

Setting $t = t_k$ in (2.3) gives

$$v(t_k) = t_k^{\alpha-\mu} \int_0^{t_k} s^{\mu-1-\beta} v(s) \, ds + \bar{g}(t_k), \quad 1 \leq k \leq N. \quad (3.2)$$

We first consider the case when $u$ is bounded at $t = 0$. Then we approximate the integral on the right-hand side of (3.2) by a product integration rule

$$\int_0^{t_k} s^{\mu-1-\beta} v(s) \, ds \approx \sum_{i=0}^{k-1} D_i v(t_i), \quad (3.3)$$

where

$$D_i := \int_{t_i}^{t_{i+1}} s^{\mu-1-\beta} \, ds = h^{\mu-\beta} \frac{(i+1)^{\mu-\beta} - i^{\mu-\beta}}{\mu - \beta}, \quad 1 \leq k \leq N-1. \quad (3.4)$$

The case when $g$ and $u$ are unbounded at the origin can be dealt with in a similar way, by noting that the limits

$$\bar{g}(0) = \lim_{t \to 0^+} t^{\beta} g(t) \quad (3.5)$$

and

$$v(0) = \lim_{t \to 0^+} t^{\beta} u(t) \quad (3.6)$$

are finite. So even if $u$ becomes infinite as $t \to 0^+$, we can still use (3.3). In order to obtain $v(0)$, let $t \to 0^+$ in (2.3):

$$v(0) = \lim_{t \to 0^+} t^{\beta-\mu} \int_0^{t} s^{\mu-1-\beta} v(s) \, ds = \bar{g}(0). \quad (3.7)$$

Since $u \in V^0_{m,\beta}[0,T]$ and $\mu > 1 + \beta$, it may be shown that

$$\lim_{t \to 0^+} t^{\beta-\mu} \int_0^{t} s^{\mu-1-\beta} v(s) \, ds = \frac{v(0)}{\mu - \beta}. \quad (3.8)$$

Substituting (3.8) into (3.7) yields

$$v(0) = \bar{g}(0) \frac{\mu - 1 - \beta}{\mu - \beta}. \quad (3.9)$$
From (3.2) and (3.3), we obtain the following equation:

\[ v^h_k = t^\beta v^h_{k-1} + \sum_{i=0}^{k-1} D_i v^h_i + g(t_k), \quad 1 \leq k \leq N, \]

(3.10)

where \( v^h_0 := v(0) \) and \( v^h_k \) is an approximate value of \( t^\beta u(t_k) \), \( 1 \leq k \leq N \). Then we set

\[ u^h_k = v^h_k / t^\beta_k, \quad 1 \leq k \leq N, \]

(3.11)

where \( u^h_k \) is an approximation of \( u(t_k) \).

### 3.2. Convergence and asymptotic error expansion

In this section, we shall prove that Euler’s method given by (3.10) is convergent when \( g \in V_m[0, T] \). Moreover, we shall show that, under appropriate conditions, the error allows a certain asymptotic expansion. We first introduce some notations. Consider the vector space

\[ \mathbb{R}^{N+1} = \{ v^h = (v^h_0, v^h_1, \ldots, v^h_N)^T, \quad v^h_i \in \mathbb{R}, \quad 0 \leq i \leq N \}, \]

with the maximum norm

\[ \| v^h \| := \max_{0 \leq i \leq N} | v^h_i |. \]

(3.12)

Define the linear restriction operator \( r^h : V_m[0, T] \to \mathbb{R}^{N+1} \) by

\[ (r^h f(t))_i := f(t_i), \quad 0 \leq i \leq N, \]

(3.13)

and let \( L^h_\beta : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1} \) be the discrete linear operator given by

\[ (L^h_\beta v^h)_k := \begin{cases} v^h_0, & k = 0, \\ v^h_k - t^\beta_k \sum_{i=0}^{k-1} D_i v^h_i, & 1 \leq k \leq N, \end{cases} \]

(3.14)

so that the approximate solution \( v^h \), defined by (3.10), satisfies

\[ L^h_\beta v^h = r^h g. \]

(3.15)

Using the results contained in \([10]\), the method (3.15) is convergent and an asymptotic error expansion for the error \( v^h - r^h v \) is valid if the following conditions are satisfied:

1. For every function \( \bar{g} \in V^0_k[0, T] \), (2.7) has a unique solution in \( V^0_k[0, T] \), \( k = 0, 1, \ldots, m \).
2. The method (3.15) is stable in the following sense: there exists a constant \( M \), independent of \( h \), such that for every \( \bar{g} \in V^0_m[0, T] \), the approximate solution \( v^h \) to the exact solution \( v \) of (2.7) satisfies

\[ \| v^h \| \leq M \| r^h g \|. \]

(3.16)

3. There exist functions \( C_q \in V^0_{m-q}[0, T], \quad 1 \leq q \leq m - 1, \) such that

\[ (L^h_\beta (r^h v))_k - r^h (L_\beta v)_k = \sum_{q=1}^{m-1} C_q(t_k) h^q + r_m(t_k), \quad 1 \leq k \leq N, \]

(3.17)

where \( r_m(t) = O(h^m) \).
From Theorem 2.2 it follows that (2.7) satisfies the condition (1). We now give a sufficient condition for (3.16) to hold.

**Theorem 3.1.** If $\bar{g} \in V_0[0,T]$ and $\mu > 1 + \beta$, then $v^h$ satisfies (3.16).

**Proof.** Taking modulus in (3.10) gives

$$|v^h_k| \leq |\bar{g}(t_k)| + \max_{0 \leq i \leq N} |v^h_i| t_k^{\beta - \mu} \sum_{i=0}^{k-1} D_i, \quad 1 \leq k \leq N. \tag{3.18}$$

Since $\mu > 1 + \beta$, we may write

$$\sum_{i=0}^{k-1} D_i = \frac{t_k^{\mu - \beta}}{\mu - \beta}, \quad 1 \leq k \leq N. \tag{3.19}$$

Substituting (3.19) into (3.18), yields

$$\|v^h\| \leq \|r^h \bar{g}\| + \rho \|v^h\|, \tag{3.20}$$

where $\rho := 1/(\mu - \beta) < 1$. Thus

$$\|v^h\| \leq \frac{1}{1 - \rho} \|r^h \bar{g}\|. \tag{3.21}$$

Hence, (3.16) is satisfied with $M = 1/(1 - \rho)$. $\square$

Before examining (3.17), we prove a convergence result.

**Theorem 3.2.** If $\bar{g} \in V_0[0,T]$ and $\mu > 1 + \beta$ then

$$\|L^h_{\beta}(r^h v) - r^h(L_{\beta}v)\| \to 0, \quad as \ h \to 0^+, \tag{3.22}$$

$$\|v^h - r^h v\| \to 0, \quad as \ h \to 0^+. \tag{3.23}$$

**Proof.** We first prove (3.22). From (2.7) and (3.15)

$$(L^h_{\beta}(r^h v) - r^h(L_{\beta}v))_k = t_k^{\beta - \mu} \sum_{i=0}^{k-1} \left( \int_{t_i}^{t_{i+1}} s^{\mu - 1 - \beta} v(s) ds - D_i(v(t_i)) \right), \quad 1 \leq k \leq N. \tag{3.24}$$

Using the mean value theorem for integrals yields

$$\int_{t_i}^{t_{i+1}} s^{\mu - 1 - \beta} v(s) ds = v(\xi_i) \int_{t_i}^{t_{i+1}} s^{\mu - 1 - \beta} ds, \tag{3.25}$$

with $\xi_i \in (t_i, t_{i+1})$, $0 \leq i \leq k - 1$. Substituting (3.25) into (3.24), we obtain

$$(L^h_{\beta}(r^h v) - r^h(L_{\beta}v))_k = t_k^{\beta - \mu} \sum_{i=0}^{k-1} D_i(v(\xi_i) - v(t_i)), \quad 1 \leq k \leq N. \tag{3.26}$$
which yields, after taking modulus,

\[
|\left( L^h_{\beta}(r^h v) - r^h (L^h_{\beta} v) \right)_k | \leq t_k^{\beta-\mu} \max_{0 \leq i \leq k-1} |v(\xi_i) - v(t_i)| \sum_{i=0}^{k-1} D_i, \quad 1 \leq k \leq N. \tag{3.27}
\]

Since \( v \in V_0[0,T] \) then

\[
\max_{0 \leq i \leq N} |v(\xi_i) - v(t_i)| \to 0 \quad \text{as} \quad h \to 0^+.
\]

This gives (3.22), which means that the method (3.15) is consistent.

In order to prove (3.23), we write

\[
L^h_{\beta}(v^h - r^h v) = L^h_{\beta} v^h - L^h_{\beta}(r^h v) = r^h q - L^h_{\beta}(r^h v). \tag{3.28}
\]

Now an application of Theorem 3.1 yields

\[
\|v^h - r^h v\| \leq M \|r^h (L^h_{\beta} v) - L^h_{\beta}(r^h v)\| \tag{3.29}
\]

and (3.23) follows from (3.29) and (3.22). In other words, we have proved that stability together with consistency imply convergence. \( \square \)

**Corollary.** If the function \( g \) in (2.1) belongs to \( V^0_{0,\beta}[0,T] \) for some \( \beta \) satisfying \( \mu > 1 + \beta \), then Euler's method by given by (3.15) is convergent.

**Proof.** We have that \( q = t^\beta g \) belongs to \( V^0_{0,\beta}[0,T] \) and convergence follows from Theorem 3.2. \( \square \)

**Theorem 3.3.** Let \( v \) and \( v^h \) be the solutions of (2.7) and (3.15), respectively, and let \( \mu > 1 + \beta \), with \( \beta \geq 0 \). If \( v \) belongs to \( V^0_{m}[0,T] \) for some \( m \geq 1 \), then the following asymptotic expansion is valid:

\[
(L^h_{\beta}(r^h v) - r^h (L^h_{\beta} v))_k = \sum_{q=1}^{m} C_q(t_k) h^q + O(h^m), \quad 1 \leq k \leq N, \tag{3.30}
\]

with \( C_q \in V^0_{m-1}[0,T] \).

**Proof.** From (2.7) and (3.14) it follows that

\[
(L^h_{\beta}(r^h v) - r^h (L^h_{\beta} v))_k = t_k^{\beta-\mu} \left( \int_0^{t_k} s^{\mu-1-\beta} v(s) \, ds - \sum_{i=0}^{k-1} D_i v(t_i) \right), \quad 1 \leq k \leq N. \tag{3.31}
\]

Using the variable substitution \( s = t_i + \sigma h \), we may write

\[
\int_0^{t_k} s^{\mu-1-\beta} v(s) \, ds = h \sum_{i=0}^{k-1} \int_0^1 (t_i + \sigma h)^{\mu-1-\beta} v(t_i + \sigma h) \, d\sigma. \tag{3.32}
\]

On the other hand, from (3.4), we obtain

\[
\sum_{i=0}^{k-1} D_i v(t_i) = h \sum_{i=0}^{k-1} \int_0^1 (t_i + \sigma h)^{\mu-1-\beta} v(t_i) \, d\sigma. \tag{3.33}
\]
Combining (3.32) and (3.33) gives
\[
\int_{0}^{t_k} s^\mu \phi(s) \, ds - \sum_{i=0}^{k-1} D_i \phi(t_i) = h \sum_{i=0}^{k-1} \int_{0}^{1} \left( \phi(t_i + \sigma h) - \phi(t_i) \right) \left( t_i + \sigma h \right)^\mu \, d\sigma.
\] (3.34)

Using a Taylor’s expansion, we may rewrite (3.34) in the form
\[
\int_{0}^{t_k} s^\mu \phi(s) \, ds - \sum_{i=0}^{k-1} D_i \phi(t_i)
\]
\[
- \sum_{r=0}^{m-1} h^r \int_{0}^{1} \theta_r(\sigma) h \sum_{i=0}^{k-1} \phi^{(r+1)}(t_i + \sigma h)(t_i + \sigma h)^\mu \, d\sigma + O(h^{m+1}),
\]
where
\[
\theta_r(\sigma) := \frac{\sigma^{r+1}}{(r+1)!} (-1)^{r+1}.
\] (3.36)

In order to obtain (3.30), we shall need to analyse the sum
\[
\sum_{i=0}^{k-1} (t_i + \sigma h)^\mu \phi^{(r+1)}(t_i + \sigma h),
\]
(3.37)

taking into account the smoothness of the functions involved. We start by defining
\[
\Phi(s) := \phi(t_k s),
\]
(3.38)

so that we may write
\[
\int_{0}^{t_k} t^\mu \phi^{(r+1)}(t) \, dt = \frac{t_k^\mu}{t_k^{r+1}} \int_{0}^{1} s^\mu \phi^{(r+1)}(s) \, ds.
\] (3.39)

We now consider two cases.

(i) We first assume that \( \mu - \beta \) is an integer and apply the Euler–MacLaurin summation formula to the integral on the right hand side of (3.39) (see, e.g., [3]). If \( \bar{h}_k := h/t_k = 1/k \), then we have for each \( \sigma \in [0, 1] \),
\[
\int_{0}^{1} s^\mu \phi^{(r+1)}(s) \, ds = \bar{h}_k \sum_{i=0}^{k-1} ((i + \sigma)\bar{h}_k)^\mu \phi^{(r+1)}((i + \sigma)\bar{h}_k)
\]
\[
- \sum_{q=0}^{m-r-2} \bar{h}_k^{q+1} \frac{D_{q+1}(\sigma)}{(q+1)!} \left. \frac{d^q}{ds^q} (s^\mu \phi^{(r+1)}(s)) \right|_{s=1} - \left. \frac{d^q}{ds^q} (s^\mu \phi^{(r+1)}(s)) \right|_{s=0}
\]
\[+ O(\bar{h}_k^{m-r}), \quad 0 \leq r \leq m - 1,
\] (3.40)
\[
\frac{h}{t_k^{\mu-\beta}} \sum_{i=0}^{k-1} (t_i + \sigma h)^{-1-\beta v(r+1)} (t_i + \sigma h) t_k^{r+1}
\]
\[
= \int_0^1 s^{\mu-1-\beta} \Phi^{(r+1)}(s) \, ds + t_k^{r+1} \sum_{q=0}^{m-r-2} \frac{h^{q+1}}{q!(q+1)!} B_{q+1}(\sigma)
\]
\[
\times \left( \left. \frac{d^q}{ds^q} (s^{\mu-1-\beta v(r+1)}(st_k)) \right|_{s=1} - \left. \frac{d^q}{ds^q} (s^{\mu-1-\beta v(r+1)}(st_k)) \right|_{s=0} \right)
\]
\[
+ O \left( \frac{h^{m-r}}{t_k} \right), \quad 0 \leq r \leq m-1.
\]
(3.41)

Combining (3.41) and (3.39) gives
\[
\frac{h}{t_k^{\mu-\beta}} \sum_{i=0}^{k-1} (t_i + \sigma h)^{-1-\beta v(r+1)} (t_i + \sigma h)
\]
\[
= \int_0^t s^{\mu-1-\beta v(r+1)}(t) \, dt + t_k^{r+1} \sum_{q=0}^{m-r-2} \frac{h^{q+1}}{q!(q+1)!} B_{q+1}(\sigma)
\]
\[
\times \left( \left. \frac{d^q}{ds^q} (s^{\mu-1-\beta v(r+1)}(st_k)) \right|_{s=1} - \left. \frac{d^q}{ds^q} (s^{\mu-1-\beta v(r+1)}(st_k)) \right|_{s=0} \right)
\]
\[
+ t_k^{r+1} O \left( \frac{h^{m-r}}{t_k} \right), \quad 0 \leq r \leq m-1.
\]
(3.42)

Substituting (3.42) into (3.35) and using (3.31), we obtain an expansion which is of the type of (3.30).

To prove this, we derive an expression for the coefficients \( C_q, \quad 1 \leq q \leq m-1 \). Let us define
\[
\alpha_r := \int_0^1 \theta_r(\alpha) \, d\sigma, \quad 0 \leq r \leq m.
\]

Collecting the terms in \( h \), we easily obtain that
\[
C_1(t_k) = t_k^{\beta-\mu} \alpha_0 \int_0^{t_k} t^{\mu-\beta-1} v'(t) \, dt.
\]
(3.43)

We may define
\[
C_1(t_k) := t^{\beta-\mu} \alpha_0 \int_0^t s^{\mu-\beta-1} v'(s) \, ds,
\]
(3.44)

for \( t \in [0, T] \). Since \( v \in V_{m-1}^0[0, T] \), then \( C_1 \in V_{m-1}^0[0, T] \).

An expression for \( C_j(t) \), with \( j \geq 2 \), can be obtained in a similar way. Assuming that \( m \geq j \) we define
with $t \in [0, T]$. Since $v \in V^0_m[0, T]$, then the first term of (3.45) belongs to $V^0_{m-j}[0, T]$. In order to prove that $C_j \in V^0_{m-j}[0, T]$, we need to analyse the following expression:

$$
\int_0^T \left( \frac{d^{j-r-2}}{ds^{j-r-2}} (s^{\mu-1}v^{(r+1)}(s)) \right) \left. \left|_{s=0} \right|_{s=t} - \frac{d^{j-r-2}}{ds^{j-r-2}} (s^{\mu-1}v^{(r+1)}(s)) \right|_{s=0},
$$

(3.46)

for each $r$ with $0 \leq r \leq j - 2$, $j \geq 2$. Applying Lemma 2.1, we have that, for each $t \in [0, T]$,

$$
\frac{d^{j-r-2}}{ds^{j-r-2}} (s^{\mu-1}v^{(r+1)}(st)) \in V^0_{m-r+\mu-\beta - 2}[0, T],
$$

(3.47)

Making again use of Lemma 2.1, we see that

$$
\frac{d^{j-r-2}}{ds^{j-r-2}} (s^{\mu-1}v^{(r+1)}(st)) \in V^0_{m+\mu-\beta-j}[0, T],
$$

as a function of the variable $s$. From Definition 2.3 it follows that the second term of (3.46) vanishes. Then, using the product differentiation rule, (3.46) reduces to a sum of terms in the variable $t$, all being in the class $V^0_{m+\mu-\beta-j}[0, T]$. Finally, from (3.45) and an application of Lemma 2.2 yields that $C_j \in V^0_{m-j}[0, T]$.

(ii) We now consider the case when $\mu - \beta$ is not an integer. By using the generalized Euler-Maclaurin summation formula for functions with algebraic singularities (see, e.g., [3,9]) an expansion for (3.37) can be obtained as follows. We have, with $\bar{h}_k := h/\tau_k$ and $\Phi$ as in (3.38),

$$
\int_0^1 s^{\mu-1-\beta} \Phi^{(r+1)}(s) \, ds = \bar{h}_k \sum_{i=0}^{k-1} \left( (i + \sigma) \bar{h}_k \right)^{\mu-1-\beta} \Phi^{(r+1)}((i + \sigma) \bar{h}_k)
$$

$$
- \sum_{q=0}^{m-r-2} \frac{\bar{h}_k^{\mu-1-\beta}}{q!} \gamma(-\mu + \beta + 1 - q, \sigma) \Phi^{(r+1+q)}(0)
$$

$$
- (-1)^q \gamma(-q, 1 - \sigma) \frac{d^q}{ds^q} \left( s^{\mu-1-\beta} \Phi^{(r+1)}(s) \right) \bigg|_{s=1} + O(\bar{h}_k^{m-r}).
$$

(3.48)

Here $\gamma(a, s)$ is the periodic generalized zeta function defined by

$$
\gamma(a, s) := \zeta(a, \bar{s}), \quad s - \bar{s} = \text{integer,} \quad 0 < \bar{s} \leq 1,
$$

where $\zeta(a, s)$ is the generalized Riemann zeta function given by

$$
\zeta(a, s) = \sum_{k=0}^{\infty} \frac{1}{(s + k)^a},
$$

$$
\zeta(-a, s) = \frac{2 \alpha \Gamma(a+1)}{(2\pi)^a} \sum_{k=1}^{\infty} \frac{\sin[2\pi sk - \pi a/2]}{k^a + \alpha - 1}, \quad \Re a > 0.
$$
Now $\Phi^{(r+1)}(0) = v^{(r+1)}(0)t_k^{r+1}$ and, since $v \in V_m^0[0,T]$, then $\Phi^{(r+1+q)}(0) = 0$, $0 \leq q \leq m - 2$. Finally, after expressing $h_k$ in terms of $h$ we arrive at the following expansion:

$$
\sum_{i=0}^{k-1} (t_i + \sigma h)^{\mu-1-\beta} v^{(r+1)}(t_i + \sigma h)
$$

$$
= \int_0^{t_k} t^{\mu-1-\beta} v^{(r+1)}(t) \, dt + t_k^{\mu-\beta} \sum_{q=0}^{m-2} \frac{h^{q+1}}{q!} \gamma(-q,1-\sigma) \frac{d^q}{ds^q} \left( s^{\mu-\beta-1} v^{(r+1)}(st_k) \right) \bigg|_{s=1}
$$

$$
+ \frac{t_k^{\mu-\beta}}{t_k^{r+1}} O\left( \frac{h}{t_k} \right)^{m-r} .
$$

(3.49)

After substituting (3.49) into (3.35), it can be shown that (3.31) possesses an expansion which is of the type of (3.30). The proof is similar to the one used for the case when $\mu - \beta$ is an integer, therefore we omit it. \[ \square \]

We are now in a position to state the following theorem which provides the basis for Richardson extrapolation.

**Theorem 3.4.** Let $v$ and $v^h$ be the solutions of (2.7) and (3.19), respectively, and let $\mu > 1 + \beta$, with $\beta \geq 0$. If $v$ belongs to $V_m^0[0,T]$, for some $m \geq 1$, then the following asymptotic error expansion is valid:

$$
(u - u^h)_k = \sum_{q=0}^{m-2} C_q'(t_k) h^q + O(h^m), \quad 1 \leq k \leq N,
$$

(3.50)

with $C_q' \in V_{m-q}^0[0,T]$.

**Proof.** The above expansion follows from conditions (1)–(3), after an application of [10, Theorem 2.1, p. 18]. \[ \square \]

**Remark.** We note that Eqs. (2.1) and (2.7) are equivalent away from the origin. Therefore the convergence results for (2.7) are still valid for (2.1) in the above sense.

### 4. Numerical results

In this section the theoretical results of the previous sections are illustrated by means of some numerical examples. Following the remark of last section, we shall be interested here in verifying the convergence results for (2.7). The algorithm (3.10) was applied to the auxiliary equation (2.7), with

$$
\tilde{q}(t) := t^\beta g(t), \quad g(t) := t^\alpha \left( 1 - \frac{1}{\mu + \alpha} \right), \quad t \in [0,1],
$$

(4.1)

for several choices of the constants $\alpha, \beta$ and $\mu$. The corresponding exact solutions of Eqs. (2.1) and (2.7) are $u(t) = t^\alpha$ and $v(t) = t^{\alpha+\beta}$, respectively. The norms of the errors in the approximate solution...
Table 1

<table>
<thead>
<tr>
<th>Example 4.1</th>
<th>Example 4.2</th>
<th>Example 4.3</th>
<th>Example 4.4</th>
<th>Example 4.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>$|e|$</td>
<td>$|e|$</td>
<td>$|e|$</td>
<td>$|e|$</td>
</tr>
<tr>
<td>$h = 1/40$</td>
<td>0.14D-0</td>
<td>0.16D-1</td>
<td>0.22D-1</td>
<td>0.14D-1</td>
</tr>
<tr>
<td>$h = 1/80$</td>
<td>0.10D-0</td>
<td>0.22D-2</td>
<td>0.12D-1</td>
<td>0.69D-2</td>
</tr>
<tr>
<td>$h = 1/160$</td>
<td>0.73D-1</td>
<td>0.41D-2</td>
<td>0.59D-2</td>
<td>0.35D-2</td>
</tr>
<tr>
<td>$h = 1/320$</td>
<td>0.52D-1</td>
<td>0.21D-2</td>
<td>0.30D-2</td>
<td>0.18D-2</td>
</tr>
<tr>
<td>rate</td>
<td>$k \approx 0.49$</td>
<td>$k \approx 0.98$</td>
<td>$k \approx 0.97$</td>
<td>$k \approx 0.98$</td>
</tr>
</tbody>
</table>

Table 2

Using the Richardson's extrapolation to improve the numerical approximation of $v(1)$, in Example 4.2

<table>
<thead>
<tr>
<th>$i$</th>
<th>$S_i = T_0(i)$</th>
<th>$T_1(i)$</th>
<th>$T_2(i)$</th>
<th>$T_3(i)$</th>
<th>$T_4(i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.16D-1</td>
<td>0.23D-3</td>
<td>0.30D-5</td>
<td>0.17D-6</td>
<td>0.20D-7</td>
</tr>
<tr>
<td>1</td>
<td>0.82D-2</td>
<td>0.60D-4</td>
<td>0.52D-6</td>
<td>0.29D-7</td>
<td>0.35D-8</td>
</tr>
<tr>
<td>2</td>
<td>0.41D-2</td>
<td>0.15D-4</td>
<td>0.90D-7</td>
<td>0.51D-8</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.21D-2</td>
<td>0.39D-5</td>
<td>0.16D-7</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.10D-2</td>
<td>0.99D-6</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.52D-3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$v^h$ to $v$ are displayed in Table 1 for several values of the stepsize $h$. The following quantity has been used as an estimate of the convergence order:

$$k := \log_2 \left( \frac{\|r^h v - v^h\|}{\|r^{2h} v - v^{2h}\|} \right).$$

In the cases when an asymptotic error expansion of the type (3.50) was valid, Richardson’s extrapolation was used to improve the results at $t = 1$. Following [2], we have considered the sequence $\{S_i\}_{i \in \mathbb{N}}$, where $S_i = v^{h_i}(1)$, with $h_0 = 1/40$, $h_{k+1} = h_k/2$, $k = 0, 1, 2, \ldots$. Successive transforms of this sequence were then obtained by using the formulae

$$T_0^{(i)} = S_i,$$

$$T_k^{(i)} = \frac{h_{i+k} T_{k-1}^{(i)} - h_i T_{k-1}^{(i+1)}}{h_{i+k} - h_i}, \quad k = 1, 2, \ldots, \quad i = 0, 1, \ldots.$$

The results illustrating this extrapolation process are shown in Tables 2–5. Using the relation (3.11), approximate values to $u(t)$, with $t > 0$, can be obtained. We finish this section by giving an example which illustrates the convergence results for a more general equation (1.3).

**Example 4.1.** Let $\alpha = -0.5$ and $\mu = 2$. We have set $\beta = 0.7$ which implies that $v(t) = t^{0.2} \in V_0[0, 1]$ and, consequently, $u(t) = t^{-0.5} \in V_{0.07}[0, 1]$. On the other hand, we have $\mu > 1 + \beta$ and convergence
Table 3
Using the Richardson's extrapolation to improve the numerical approximation of \(v(1)\), in Example 4.3

<table>
<thead>
<tr>
<th>(i)</th>
<th>(S_i = T_0^{(i)})</th>
<th>(T_1^{(i)})</th>
<th>(T_2^{(i)})</th>
<th>(T_3^{(i)})</th>
<th>(T_4^{(i)})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.22D-1</td>
<td>0.80D-3</td>
<td>0.11D-3</td>
<td>0.28D-4</td>
<td>0.88D-5</td>
</tr>
<tr>
<td>1</td>
<td>0.12D-1</td>
<td>0.28D-3</td>
<td>0.38D-4</td>
<td>0.10D-4</td>
<td>0.31D-5</td>
</tr>
<tr>
<td>2</td>
<td>0.59D-2</td>
<td>0.10D-3</td>
<td>0.14D-4</td>
<td>0.35D-5</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.30D-2</td>
<td>0.35D-4</td>
<td>0.48D-5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.15D-2</td>
<td>0.13D-4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.77D-3</td>
<td></td>
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</tr>
</tbody>
</table>

Table 4
Using the Richardson's extrapolation to improve the numerical approximation of \(v(1)\), in Example 4.4

<table>
<thead>
<tr>
<th>(i)</th>
<th>(S_i = T_0^{(i)})</th>
<th>(T_1^{(i)})</th>
<th>(T_2^{(i)})</th>
<th>(T_3^{(i)})</th>
<th>(T_4^{(i)})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.14D-1</td>
<td>0.22D-3</td>
<td>0.84D-6</td>
<td>0.11D-8</td>
<td>0.54D-12</td>
</tr>
<tr>
<td>1</td>
<td>0.69D-2</td>
<td>0.54D-4</td>
<td>0.11D-6</td>
<td>0.63D-10</td>
<td>0.26D-13</td>
</tr>
<tr>
<td>2</td>
<td>0.35D-2</td>
<td>0.14D-4</td>
<td>0.1D-7</td>
<td>0.39D-11</td>
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</tr>
<tr>
<td>3</td>
<td>0.18D-2</td>
<td>0.34D-4</td>
<td>0.17D-8</td>
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<td></td>
</tr>
<tr>
<td>4</td>
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<td>0.86D-6</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.44D-3</td>
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<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 5
Using the Richardson's extrapolation to improve the numerical approximation of \(v(1)\), in Example 4.5

<table>
<thead>
<tr>
<th>(i)</th>
<th>(S_i = T_0^{(i)})</th>
<th>(T_1^{(i)})</th>
<th>(T_2^{(i)})</th>
<th>(T_3^{(i)})</th>
<th>(T_4^{(i)})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.27D-1</td>
<td>0.10D-2</td>
<td>0.81D-4</td>
<td>0.11D-4</td>
<td>0.25D-5</td>
</tr>
<tr>
<td>1</td>
<td>0.14D-1</td>
<td>0.32D-4</td>
<td>0.21D-4</td>
<td>0.32D-5</td>
<td>0.6D-6</td>
</tr>
<tr>
<td>2</td>
<td>0.72D-2</td>
<td>0.95D-4</td>
<td>0.54D-5</td>
<td>0.8D-6</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.36D-2</td>
<td>0.28D-4</td>
<td>0.14D-5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.18D-2</td>
<td>0.8D-5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.92D-3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Follows from Theorem 3.2. We have computed the errors in the numerical solution \(v^h\) to (2.7), for several values of \(h\). This is shown in Table 1, indicating a low order of convergence. This low order could be expected, because in this case the conditions of Theorem 3.4 are not satisfied for any integer \(m\). Therefore there might not exist an error expansion of the type of (3.50) and the Richardson's extrapolation method is not expected to improve the accuracy of the results.
Example 4.2. Let $\alpha = -0.5$, $\mu = 4$ and $\beta = 2.5$. Again we have $u(t) = t^{-0.5}$ but, since $v(t) = t^2 \in V_{2,2.5}^0[0,T]$, the convergence condition $\mu > 1 + \beta$ is satisfied and an application of Theorem 3.4, with $m = 2$, gives first order of convergence. This is confirmed by the results displayed in Table 1. The absolute errors of the approximations obtained by the process (4.3)-(4.4), are shown in Table 2. Each column of this table corresponds to a new transform. We see that the convergence is accelerated in the first two steps of the extrapolation process, while in the subsequent steps there is no significant improvement. This is in agreement with the remainder of the asymptotic error expansion (3.50) being of order $O(h^2)$.

Example 4.3. Let $\alpha = 1$ and $\mu = 1.5$. Imposing the convergence condition $\mu > 1 + \beta$ gives $\beta < 0.5$ and we chose $\beta = 0$. Note that in this case $u(t) = t$ is an infinitely differentiable function. However $v(t) = t \in V_{1}^0[0,1]$. Therefore, an asymptotic expansion of the type of (3.50) is valid with $m = 1$, which is confirmed by the results displayed in Table 1. An application of the Richardson’s extrapolation process gives the results contained in Table 3. We observe that the convergence is accelerated only from the first to the second column, but not in the subsequent columns.

Example 4.4. Let $\alpha = 4.5$ and $\mu = 1.5$. As in the previous example, we chose $\beta = 0$. However in this case $u \in V_{4.5}^0[0,1]$ and $v \in V_{2}^0[0,1]$, due to the higher value of $\alpha$. Therefore, according to Theorem 3.4, the numerical solution $v_h$ has an asymptotic error expansion with an $O(h^4)$ remainder term. The absolute errors obtained in this case are displayed in Table 1, indicating that the convergence order is three. In Table 4 we show the extrapolated results. We would expect the convergence to be accelerated at least until the third transform. The errors indicate that the order of convergence is approximately equal to 1 for the first column (initial sequence), 2 for the second column (first transform), 3 for the third column (second transform), 4 for the fourth column (third transform) and 4.4 for the fifth column. The continuation of the extrapolation process has shown that the convergence cannot be accelerated any further.

Example 4.5. Let $K(t,s) := e^{t-s}$ and

$$g(t) := e^t - \frac{e^t}{t^\mu} \left(2 - e^t \left(2 + 2t + t^2\right)\right)$$

in (1.3), with $\alpha = 1.8$ and $\mu = 1.2$. Choosing $\beta = 0$, then $u(t) = t^{1.8} \in V_{1,1.2}^0[0,1]$ and $v(t) = t^{1.8} \in V_{1}^0[0,1]$. The values in Table 1 indicate first order of convergence as predicted by Theorem 3.4. Moreover we would expect the error in the numerical solution $v_h$ to be $O(h)$. This is in agreement with the results obtained by Richardson’s extrapolation which are shown in Table 5.

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References