Analytical-numerical investigation of a singular boundary value problem for a generalized Emden–Fowler equation

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ABSTRACT

In this paper we are concerned about a singular boundary value problem for a quasilinear second-order ordinary differential equation, involving the one-dimensional p-laplacian. Asymptotic expansions of the one-parameter families of solutions, satisfying the prescribed boundary conditions, are obtained in the neighborhood of the singular points and this enables us to compute numerical solutions using stable shooting methods.
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1. Introduction

Some mathematical models in fluid dynamics, in particular, in boundary layer theory, lead to the Emden–Fowler equation

\[ g''(u) = au^\sigma g^n(u), \quad 0 < u < u_0, \]  

(1.1)
as described, for example, in [1–3,8,12] and the references therein. In [9], a singular boundary value problem for Eq. (1.1) was treated, where \( n < 0, a < 0, u_0 > 0 \) and \( \sigma \in \mathbb{R} \). Since this equation is singular at \( u = 0 \) and \( u = u_0 \), the boundary conditions were written in the form

\[ g(0) = \lim_{u \to 0^+} u g'(u) = 0 \]  

(1.2)
\[ g(u_0) = \lim_{u \to u_0^-} [(u_0 - u) g'(u)] = 0. \]  

(1.3)

In that paper, we have described one-parameter families of solutions of the singular Cauchy problems (1.1)–(1.2) and (1.1)–(1.3) in the neighborhood of the singular points \( u = 0 \) and \( u = u_0 \). These families were then used to compute the solutions of the boundary value problems. In the present paper, we consider the following generalization of Eq. (1.1), which can be considered as the generalized Emden–Fowler equation:

\[ (|g'(u)|^{p-2} g'(u))' = au^\sigma g^n(u), \quad 0 < u < u_0, \]  

(1.4)

where \( n < 0, a < 0, p > 1, u_0 > 0 \) and \( \sigma \) are known real numbers. We are interested in positive solutions of (1.4) that satisfy the boundary conditions (1.2) and (1.3). When \( p = 2 \), Eq. (1.4) reduces to Eq. (1.1). The differential operator on
the left-hand side of (1.4) is known as the one-dimensional $p$-laplacian and has many applications in fluid mechanics and thermodynamics, when describing processes with nonlinear diffusion. In recent years many works have been devoted to the analysis of singular boundary value problems involving the one-dimensional $p$-laplacian. Boundary value problems for Eq. (1.4) with different kinds of boundary conditions have been investigated, for example, in [6,10,14].

Existence and uniqueness of positive solutions of singular boundary value problems of this kind have been studied by several authors. In [4] the authors, using a shooting argument, have established necessary and sufficient conditions for the existence of solutions, in the case $p \geq 2$. Moreover, they have proved that, under those conditions, such a solution is unique and they also derived asymptotic estimates describing the behavior of the solution and its derivative at the boundary.

Although the existence and uniqueness results of [4] do not apply to the case $p < 2$, existence and uniqueness of the positive solution is guaranteed in this case by Theorem 2 of [5], where the authors used an upper and lower solution method. All these results together allow us to conclude that the solution of our problem, when $n < 0, a < 0$, exists and is unique if $\sigma > -p$.

Our purpose is to extend the results obtained in [9] to the case $p \neq 2, p > 1$. Since $(|g'(u)|^{p-2}g'(u))' = (p-1)|g'(u)|^{p-2}g''(u)$, for $g'(u) \neq 0$ Eq. (1.4) is equivalent to

$$g''(u) = \frac{au^\sigma}{(p-1)|g'(u)|^{p-2}}g'(u),$$

and therefore it is obvious that when $p > 2$, we will have a singularity at every point where the first derivative vanishes. In this case, concerning the boundary value problem (1.4), (1.2), (1.3), besides the singularities at $u = 0$ and $u = u_0$, occurring when $p = 2$, we will have one more singularity: a point $u_M \in (0, u_0)$, such that $g'(u_M) = 0$, because a positive solution of (1.4), (1.2), (1.3) must be concave down, and must be zero at the endpoints of the interval, hence it must achieve a maximum somewhere in $(0, u_0)$.

2. The singularity at $u = 0$

Consider the singular Cauchy problem:

$$\begin{align*}
&(|g'(u)|^{p-2}g'(u))' = au^\sigma g''(u), \quad 0 < u < u_0, \\
g(0) = \lim_{u \to 0^+} ug'(u) = 0.
\end{align*}
$$

Our main result in this section is contained in the following

**Proposition 2.1.** Assume that in (2.1) $p > 1, \sigma > -p, n < 0$ and $u_0 > 0$. The Cauchy problem (2.1)–(2.2) has, in the neighborhood of the singular point $u = 0$, a one-parameter family of solutions that can be represented by:

$$g(u, b) = \begin{cases} 
 g_1(u, b), & \sigma < -n - 1; \\
 g_2(u, b), & \sigma > -n - 1; \\
 g_3(u, b), & \sigma = -n - 1;
\end{cases}
$$

as $u \to 0^+$, where $b$ is the parameter,

$$\begin{align*}
g_1(u, b) &= \left(\frac{aA^\sigma}{(D-p-1)B(p-1)}\right)^{\frac{1}{\sigma}} u^\sigma [1 + bu^{l+1} + d_2 b^2 u^{l+1} + \cdots], \\
g_2(u, b) &= \left(\frac{a}{b(p-1)(B+1)}\right)^{\frac{1}{\sigma}} u + bu^{l+1} \left[1 + y_{0,1}(b) u^\sigma + o(u^\sigma)\right], \\
g_3(u, b) &= (-aD)^{\frac{1}{\sigma}} u \left(-\ln \frac{u}{u_0}\right)^{\frac{1}{\sigma}} \left[1 + \left(\frac{D-1}{D^2} \ln \left(-\ln \frac{u}{u_0}\right) - b\right) - \frac{1}{\ln \frac{u_0}{u}} \right] \\
&\quad + \frac{1}{2} \left(-b^2 - (p-2)b^2 - b^2 \sigma - 2 \frac{(D-1)}{D} - 2b \frac{(D-1)}{D} + \frac{2(D-1)^2}{D^3}\right) \\
&\quad + \frac{(D-1)^2}{D^3} \left(1 + 2b + (p-2)b + b\sigma\right) \ln \left(-\ln \frac{u}{u_0}\right) - \frac{(D-1)^3}{2D^2} \ln^2 \left(-\ln \frac{u}{u_0}\right) - \frac{1}{\ln^2 \frac{u_0}{u}} + \cdots,
\end{align*}
$$

where $\lambda_1 = -\frac{1}{2} \left(-1 + \frac{(p-2)b+2bD}{2A}\right) + \frac{1}{2} \left(1 + \frac{(p-2)b+2bD}{2A}\right)^2 - \frac{480}{A}$, $A = p - n - 1, B = \sigma + n + 1$ and $D = p + \sigma$; $d_2$ and $y_{0,1}$ are coefficients that can be computed by substituting (2.4) and (2.5) into (2.1), respectively.

**Proof.** The proof of this proposition uses the same techniques as in Section 3 of [9], where we considered the particular case $p = 2$. Therefore, we give here only the outline of the proof and omit the details.
1. Let us first consider the case $-p < \sigma < -n - 1$. It can be easily proved that in the neighborhood of $u = 0$, a solution of (2.1)–(2.2) may be represented by 
\[ g(u) = Cu^\sigma [1 + o(1)], \quad u \to 0^+, \]
where $C = \left(\frac{(\sigma + n - 1)\Gamma(\sigma)}{\Gamma(\sigma + n - 1)}\right)^{\frac{1}{p-1}}$ and $k = \frac{p+\sigma}{p+n-1}$.

In order to improve this representation, in (2.1)–(2.2) we perform the variable substitution $g(u) = Cu^\sigma [1 + y(u)]$, and obtain a new nonlinear Cauchy problem in $y$, for which $u = 0$ is a regular singular point. For the analysis and classification of singularities of nonlinear ODEs see, for example, [13].

Writing only the linear dominant homogeneous terms of this problem, for solutions that satisfy the initial condition (2.2), we obtain a linear autonomous equation which has also a regular singularity at $u = 0$. The last equation has two real characteristic exponents, from which only one is positive; we shall denote it $\lambda_1$. Then following [11], we can assure that the above-mentioned nonlinear Cauchy problem in $y$ has a one-parameter family of solutions that may be written in the form of the Lyapunov series:
\[ y(u, b) = bu^\lambda_1 + \sum_{i=2}^{+\infty} d_i b u^{\lambda_1}, \]
where $b$ is the parameter and the coefficients $d_i$ may be determined by substituting this expression in the equation for $y$. Returning to the original variable $g$, we obtain (2.4).

2. Let us now consider the case $\sigma > -n - 1$. In the neighborhood of $u = 0$, we look for a solution of (2.1)–(2.2) in the form
\[ g(u) = C_1 u + C_2 u^\sigma [1 + o(1)]. \]  
(2.7)

Substituting (2.7) in (2.1) we can conclude that $k = \sigma + 2 + n > 0$, $k - 1 = \sigma + 1 + n > 0$, and with $C_2 = b < 0$ we have $C_1 = \left(\frac{\sigma}{(\sigma + 2 + n)\Gamma(\sigma + 1 + n)}\right)^{\frac{1}{p-1}}$.

Defining $y(u)$ by $g(u) = C_1 u + C_2 u^\sigma [1 + y(u)]$ and substituting in (2.1), we obtain a nonlinear Cauchy problem in $y$ which has a regular singular point at $u = 0$. For each $b \neq 0$, this problem has a particular solution that can be represented by
\[ y_{par}(u, b) = \sum_{i=0, j=0, 1 \leq j \geq 1}^{+\infty} y_{ij}(b) u^{\lambda_1(j+1+n)}, \quad 0 < u \leq \delta(b), \quad \delta(b) \geq 0, \]
where the coefficients $y_{ij}$, depending on $b$, may be determined by substituting $y_{par}$ in the obtained equation.

Writing out the leading linear homogeneous terms of this equation for solutions that satisfy the initial condition (2.2) we obtain a linear equation whose characteristic exponents are $\lambda_1 = 1 - (\sigma + n + 2) < 0$ and $\lambda_2 = -(\sigma + n + 2) < 0$. Therefore, following [7], for each $b \neq 0$ this problem has no other solution than $y_{par}(u, b)$. Replacing $y$ by $y_{par}$ in the expression for $g$, we finally obtain the expansion (2.5) for $g_2$.

3. Finally, consider the case $n = -1 - \sigma$. In the neighborhood of $u = 0$, we look for a solution in the form
\[ g(u) = Cu \left( -\ln \frac{u}{u_0} \right)^{\frac{1}{p-1}} [1 + o(1)], \]  
(2.8)

when $u \to 0^+$. From (2.1), (2.2), we get $C = \left(\frac{-a(\sigma + p)}{p-1}\right)^{\frac{1}{p-1}}$.

Introducing
\[ g(u) = Cu \left( -\ln \frac{u}{u_0} \right)^{\frac{1}{p-1}} [1 + y(u)], \]
we obtain a new nonlinear Cauchy problem in $y$. For this problem, the singular point $u = 0$ is irregular. Hence we make the change of variable $\tau = \left( -\ln \frac{u}{u_0} \right)^{\frac{1}{p-1}}$ so that in the new problem the irregular singularity is at infinity. The initial conditions in the new variable become:
\[ \lim_{\tau \to \infty} y(\tau) = 0, \quad \lim_{\tau \to \infty} \frac{dy(\tau)}{d\tau} = 0. \]  
(2.9)

We then search for a particular solution of the obtained problem in the form
\[ y_{par}(\tau) = \frac{b_1}{\tau^{2(\sigma+p)}} + \frac{b_2}{\tau^{3(\sigma+p)}} + \cdots + \ln \tau \left[ \frac{c_1}{\tau^{1(\sigma+p)}} + \frac{c_2}{\tau^{2(\sigma+p)}} + \cdots \right] + \ln^2 \tau \left[ \frac{d_1}{\tau^{2(\sigma+p)}} + \frac{d_2}{\tau^{3(\sigma+p)}} + \cdots \right] + \cdots. \]  
(2.10)

Substituting (2.10) in the obtained equation for $y$, we can compute the coefficients of this series. Using arguments similar to those of Section 3 of [9], it is possible to show that the considered nonlinear equation for $y$ has a one-parameter family of solutions of the form
\[ y(\tau, b) = y_{par}(\tau) + \frac{b}{\tau^{\sigma+p}} + o\left(\frac{1}{\tau^{\sigma+p}}\right), \quad \text{when } \tau \to \infty. \]  
(2.11)
where \( b \) is the parameter and \( y_{\text{par}}(\tau) \) is given by (2.10). Returning to the original variable \( u \), we conclude that when \( n = -1 - \sigma \), the problem (2.1)–(2.2) has a one-parameter family of solutions that can be represented by (2.6).

**Remark 2.1.** From the formulas for \( g_1, g_2 \) and \( g_3 \) it follows that these functions are positive (at least, in a neighborhood of \( u = 0 \)). Therefore, the positive solution we are looking for can be approximated by these functions for \( u \) sufficiently small.

3. The singularity at \( u = u_0 \)

Let us now consider the singular Cauchy problem:

\[
\begin{align*}
&\left| g(u) \right|^{p-2} g'(u) = au^n g^\sigma(u), \quad 0 < u < u_0, \\
g(u_0) &= \lim_{u \to u_0^-} (u - u_0) g(u) = 0.
\end{align*}
\]

This problem was analyzed in [10], where we have proved the following

**Proposition 3.1.** Assume that in (3.1) \( p > 1, \sigma > -p, n < 0 \) and \( u_0 < 0 \). The Cauchy problem (3.1)–(3.2) has, in the neighborhood of the singular point \( u = u_0 \), a one-parameter family of solutions that can be represented by:

\[
g(u, c) = \begin{cases} 
  g_4(u, c), & n < -1 \text{ and } n \neq 2 - \frac{1}{p-1}; \\
  g_5(u, c), & n = -2 - \frac{1}{p-1}; \\
  g_6(u, c), & n > -1; \\
  g_7(u, c), & n = -1;
\end{cases}
\]

as \( u \to u_0 \), where \( c \) is the parameter,

\[
g_4(u, c) = \left( \frac{au^n_0(p+1-n)^p}{p^n-1(1+n)(p-1)} \right)^{\frac{1}{p^n-1}} (u_0 - u)^p \left[ 1 + \frac{(1+n)p\sigma}{(p-1)(2p+np-n-1)u_0} \right] \times (u_0 - u) + c (u_0 - u) \frac{n}{p^n-1} + O((u_0 - u)^{1+n}).
\]

\[
\mu = \min \left\{ 1, 1 - \frac{p(n+1)}{p^n-1} \right\};
\]

\[
g_5(u, c) = \left( \frac{au^n_0(p+1-n)^p}{p^n-1(1+n)(p-1)} \right)^{\frac{1}{p^n-1}} (u_0 - u)^p \left[ 1 + c (u_0 - u) \left[ 1 + \frac{b_1 (u_0 - u) + \ldots}{u_0 - u} \right] \right] + (u_0 - u) \ln \left( \frac{u_0 - u}{u_0} \right) \left[ c_0 (u_0 - u) + c_1 (u_0 - u)^2 + \ldots \right] + (u_0 - u)^2 \ln^2 \left( \frac{u_0 - u}{u_0} \right) [a_0 + \ldots] + \ldots.
\]

\[
g_6(u, c) = \left( \frac{au^n_0}{(p-1)(2+n)(1+n)c} \right)^{\frac{1}{p^n-1}} (u_0 - u) + c (u_0 - u)^{2+n} \left[ 1 + y_{0,1} (c) (u_0 - u)^{1+n} + o \left( (u_0 - u)^{1+n} \right) \right],
\]

with \( c < 0; \)

\[
g_7(u, c) = \left( \frac{au^n_0 p}{p^n-1} \right)^{\frac{1}{p^n-1}} (u_0 - u) \left( -\ln \left( \frac{u_0 - u}{u_0} \right) \right)^{\frac{1}{p^n-1}} \left[ 1 + \frac{(p-1)}{p^n-1} \ln \left( -\ln \left( \frac{u_0 - u}{u_0} \right) \right) - c \right] \frac{1}{\ln \left( \frac{u_0 - u}{u_0} \right)}
\]

\[
+ \left\{ \frac{1}{2} \left( -c^2 - c^2 (p-2) + 2 \frac{(p-1)}{p} + 2 \frac{(p-1)^3}{p^3} - 2c (p-1) \right) \right\} \left( u_0 - u \right) \ln \left( -\ln \left( \frac{u_0 - u}{u_0} \right) \right) \left( -\ln \left( \frac{u_0 - u}{u_0} \right) \right) \left( u_0 - u \right) \ln^2 \left( -\ln \left( \frac{u_0 - u}{u_0} \right) \right) \left( u_0 - u \right) \ln \left( \frac{u_0 - u}{u_0} \right) + \ldots
\]

and the coefficients \( b_1, c_0, c_1, a_0, \ldots \), may be determined by substituting (3.5) in (3.1).

4. The singularity at the solution maximum

Consider the singular Cauchy problem:

\[
\begin{align*}
&\left| g(u) \right|^{p-2} g'(u) = au^n g^\sigma(u), \quad 0 < u < u_m, \\
&\lim_{u \to u_m^-} g'(u) = 0,
\end{align*}
\]

where \( u_m \in (0, u_0) \) is such that \( g'(u_m) = 0 \). Note that \( u_m \) is a singular point when \( p > 2 \).
Proposition 4.1. Assume that $p > 1$, $\sigma > -p$, $n < 0$, $u_M \in (0, u_0)$ and $u_0 > 0$. The Cauchy problem (4.1)–(4.2) has, in the left neighborhood of the point $u = u_M$, a one-parameter family of solutions that can be represented by

$$g(u, d) = \left( -\frac{a u_M^\sigma (p - 1)^{p - 1}}{p^{p - 1} |d|^{p - 1}} \right)^{-\frac{1}{p}} - d (u_M - u)^\frac{p}{p - 1} \left[ 1 + y_{1,0} (d) (u_M - u) + o ((u_M - u)) \right], \quad u \to u_M,$$

where $d > 0$ is the parameter.

Proof. In the left neighborhood of $u = u_M$, we seek a solution of problem (4.1)–(4.2) in the form:

$$g(u) = C_1 - C_2 (u_M - u)^k \left[ 1 + o(1) \right].$$

From (4.1)–(4.2) we have

$$\lim_{u \to u_M} \left[ (|g'(u)|^{p-2} g'(u))^\sigma g^n(u) \right] = au_M^\sigma.$$

Substituting (4.4) in (4.5) we obtain

$$-(p - 1)C_2^{-1}k^{-1} (k - 1) C_1^n (u_M - u)^{k-p-k} \bigg|_{u = u_M} = au_M^\sigma,$$

which implies that $k = \frac{p}{p - 1} > 0$, $k - 1 = \frac{1}{p - 1} > 0$ and with $C_2 = d > 0$, $C_1 = \left( -\frac{a u_M^\sigma (p - 1)^{p - 1}}{p^{p - 1} |d|^{p - 1}} \right)^{-\frac{1}{p}}$.

If in (4.1)–(4.2) we perform the variable substitution

$$g(u) = C_1 - d (u_M - u)^k \left[ 1 + y(u) \right],$$

we obtain a Cauchy problem in $y$. For any $d \neq 0$, we look for a particular solution of this problem in the form

$$y_{par}(u, d) = \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} \frac{y_{ij}(d) (u_M - u)^{i+j}(p-1)}{(p-1)^{i+j}}, \quad 0 \leq u_M - u \leq \delta(d), \delta(d) \geq 0.$$

The $y_{ij}$ coefficients can be obtained by substituting (4.7) into the nonlinear equation for $y$.

On the other hand, if we write the leading linear homogeneous terms of the equation for $y$ (for solutions that satisfy (4.2), we obtain a linear equation whose characteristic exponents are $\lambda_1 = -1 < 0$ and $\lambda_2 = -1 - \frac{1}{p - 1} < 0$. Then the equation for $y$ has no other solution than $y_{par}(u, b)$. Finally, substituting $y$ by $y_{par}$ in (4.6), we obtain (4.3). □

The expansion for the one-parameter family of solutions satisfying the condition $\lim_{u \to u_M^+} g(u) = 0$ can be obtained in the same way. Considering, at last, the singular Cauchy problem:

$$(|g'(u)|^{p-2} g'(u))' = au^n g^n(u), \quad u_M < u < u_0,$$

$$\lim_{u \to u_M^+} g'(u) = 0,$$

we will omit the proof of the following proposition.

Proposition 4.2. Assume that $p > 1$, $\sigma > -p$, $n < 0$, $u_M \in (0, u_0)$ and $u_0 > 0$. The Cauchy problem (4.8)–(4.9) has, in the right neighborhood of the point $u = u_M$, a one-parameter family of solutions that can be represented by

$$g(u, d) = \left( -\frac{a u_M^\sigma (p - 1)^{p - 1}}{p^{p - 1} |d|^{p - 1}} \right)^{-\frac{1}{p}} + d (u - u_M)^\frac{p}{p - 1} \left[ 1 + y_{1,0}^+(d) (u - u_M) + o ((u - u_M)) \right],$$

as $u \to u_M^+$, where $d < 0$ is the parameter.

5. Upper and lower solutions

Definition 5.1. We say that $h(u)$ is a lower (resp. upper) solution of the problem (1.4), (1.3), (1.2) if $h(u) \in C[0, u_0] \cap C^2(0, u_0)$ and satisfies

$$(|h'(u)|^{p-2} h'(u))' + au^n h^n(u) \leq 0 \quad \text{(resp.} \geq 0), \quad 0 < u < u_0,$$

$h(0) = h(u_0) = 0$. 

We follow the ideas of [9,10] to find upper and lower solutions for problem (1.4), (1.3), (1.2), and in some particular cases, a closed formula for its exact solution.

If we split problem (1.4), (1.3), (1.2) into two auxiliary boundary value problems:

\[
\begin{align*}
\text{(A)} & \quad \begin{cases} 
(\|g'(u)\|^p - 2 g'(u))' = au^n g^n(u), & 0 < u < u_M \\
g(0) = \lim_{u \to 0^+} u g'(u) = 0 \\
\lim_{u \to u_0^+} g'(u) = 0
\end{cases} \\
\text{(B)} & \quad \begin{cases} 
(\|g'(u)\|^p - 2 g'(u))' = au^n g^n(u), & u_M < u < u_0 \\
g(u_0) = \lim_{u \to 0^-} [(u_0 - u) g'(u)] = 0 \\
\lim_{u \to u_0^-} g'(u) = 0
\end{cases}
\end{align*}
\]

and if in problem (A) we perform the variable substitution \( v = u_M - u \) and in problem (B) \( v = u - u_M \), we easily see that these two problems become of the same kind as the problems studied in [10] when \( \sigma = 0 \). Therefore, proceeding as in [10], when \( \sigma = 0 \) and \( n < -1 \), we will look for upper and lower solutions of problem (A) and (B) in the form (see [10] for details and definition of upper and lower solutions of problems (A) and (B)):

\[
h_A(u) = \gamma_A \left( u_M^{\frac{p}{p+1}} - (u_M - u)^{\frac{p}{p+1}} \right)^{\frac{p}{p+1}},
\]
\[
h_B(u) = \gamma_B \left( (u_0 - u_M)^{\frac{p}{p+1}} - (u - u_M)^{\frac{p}{p+1}} \right)^{\frac{p}{p+1}},
\]

respectively, where \( \gamma_A \) and \( \gamma_B \) are positive constants. Since \( \sigma = 0 \), it follows from the formulas in [10] that \( \gamma_A = \gamma_B \) in the above formulas (for fixed values of \( p, a \) and \( n \)). Therefore, the continuity condition \( h_A(u_M) = h_B(u_M) \) will be satisfied if and only if \( u_0 - u_M = u_M \), that is, when \( u_M = \frac{u_0}{2} \). In this case we seek upper and lower solutions of problem (1.4), (1.3), (1.2) of the form

\[
h(u) = \gamma \left( \frac{u_0}{2} \right)^{\frac{p}{p+1}} - \left| \frac{u_0}{2} - u \right|^{\frac{p}{p+1}},
\]

and we easily obtain the following result.

**Proposition 5.1.** Assuming that \( \sigma = 0 \) and \( n < -1 \), the function \( h \) defined by (5.1) will be

- a lower solution of the problem (1.4), (1.3), (1.2) if \( \gamma \) satisfies

\[
\gamma \leq \gamma_1 = \left( \frac{-a(p-1)^{p-1}(p - n - 1)^{p-1}}{p^{2(p-1)} \left( \frac{u_0}{2} \right)^{p+1}} \right)^{\frac{1}{p+1}}, \quad 1 + n - (2 + n)p \leq 0,
\]

or

\[
\gamma \leq \gamma_2 = \left( \frac{a(p-1)^{p-1}(p - n - 1)^{p}}{p^{2(p-1)+1}(n + 1) \left( \frac{u_0}{2} \right)^{p+1}} \right)^{\frac{1}{p+1}}, \quad 1 + n - (2 + n)p \geq 0.
\]

- an upper solution of the problem (1.4), (1.3), (1.2) if \( \gamma \) satisfies

\[
\gamma \geq \gamma_2, \quad 1 + n - (2 + n)p \leq 0,
\]

or

\[
\gamma \geq \gamma_1, \quad 1 + n - (2 + n)p \geq 0.
\]

When \( 1 + n - (2 + n)p = 0 \) \( \Leftrightarrow n = -2 - \frac{1}{p+1} \), \( \gamma = \gamma_1 = \gamma_2 = (-a)^{\frac{p}{p+1}} \left( \frac{u_0}{2} \right)^{-\frac{1}{p+1}} \) and with this value of \( \gamma \) the function \( h \) defined by (5.1) is the exact solution of problem (1.4), (1.3), (1.2).

6. Numerical results

In order to numerically solve problem (1.4), (1.3), (1.2) we must remember that when \( p > 2 \) we have three singular points: two of them are the endpoints of the interval and the other one, \( u_M \), is the point where the solution attains its maximum. Let us briefly describe the numerical algorithm we have used for the computation of the approximate solutions. Since we do not know a priori the location of \( u_M \), we must begin with some initial approximation for this point and then...
compute it iteratively. With this purpose, let us split problem (1.4), (1.3), (1.2) into two auxiliary boundary value problems, problem (A) and problem (B), defined in the previous section. For a given value of \( u_M \), these problems are of the same kind as the problems considered in [9, 10], since they only have singularities at the endpoints. Therefore they can be solved by the shooting method, as described there. Namely, the asymptotic expansions of Sections 2–4 are used to obtain approximate values of the solutions in the neighborhood of the singularities and in this way the original boundary value problems are reduced to regular Cauchy problems that can be solved by standard programs.

Finally, suppose that we know two values \( u_{M_0} \) and \( u_{M_1} \) such that \( g_A(u_{M_0}) > g_B(u_{M_0}) \) and \( g_A(u_{M_1}) < g_B(u_{M_1}) \), where \( g_A(u_{M_i}) \) and \( g_B(u_{M_i}) \) are the solutions of Problem (A) and Problem (B), respectively, at \( u_M = u_{M_i}, i = 0, 1 \). Then, by continuity, there must be some \( \hat{u}_M \in [u_{M_0}, u_{M_1}] \) such that \( g_A(\hat{u}_M) = g_B(\hat{u}_M) \). This value of \( \hat{u}_M \), can be determined, for example, by the bisection method.

In Table 1 we present some numerical results for four cases: in case 1, we consider \( \sigma = 3, n = -3, a = -1, p = 5 \) and \( u_0 = 1 \); in case 2, \( \sigma = -0.9, n = -0.1, a = -1, p = 4 \) and \( u_0 = 2 \); in case 3, \( \sigma = -0.9, n = -0.1, a = -1, p = 1.5 \) and \( u_0 = 1 \); and finally, in case 4, \( \sigma = 0, n = -2.5, a = -1, p = 3 \) and \( u_0 = 1 \). In case 4, the exact solution is known, and given by \( g(u) = 2^{\frac{1}{2}} \left( \frac{1}{u^{\frac{1}{2}} - \left| u - \frac{1}{2} \right|} \right)^{\frac{3}{2}} \) (see (5.1)). The exact values of the solution, obtained by this formula, are compared in Table 2 with the numerical results. In Figs. 1–4 the graphics of the solutions and their derivatives are plotted for cases 1–4, respectively.

### Table 1

<table>
<thead>
<tr>
<th>Case</th>
<th>( u_M )</th>
<th>( g(u_M) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.60425</td>
<td>0.47855</td>
</tr>
<tr>
<td>2</td>
<td>0.91129</td>
<td>0.81135</td>
</tr>
<tr>
<td>3</td>
<td>0.30274</td>
<td>0.45211</td>
</tr>
<tr>
<td>4</td>
<td>0.50000</td>
<td>0.62996</td>
</tr>
</tbody>
</table>

### Table 2

<table>
<thead>
<tr>
<th>( u )</th>
<th>Approximate ( g(u) )</th>
<th>Exact ( g(u) )</th>
<th>Absolute error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.27247440</td>
<td>0.27247439</td>
<td>( 1 \times 10^{-8} )</td>
</tr>
<tr>
<td>0.2</td>
<td>0.41528555</td>
<td>0.41528552</td>
<td>( 3 \times 10^{-8} )</td>
</tr>
<tr>
<td>0.3</td>
<td>0.51864159</td>
<td>0.51864154</td>
<td>( 5 \times 10^{-8} )</td>
</tr>
<tr>
<td>0.4</td>
<td>0.59181356</td>
<td>0.59181348</td>
<td>( 8 \times 10^{-8} )</td>
</tr>
<tr>
<td>0.5</td>
<td>0.62996186</td>
<td>0.62996052</td>
<td>( 1.3 \times 10^{-6} )</td>
</tr>
</tbody>
</table>

Acknowledgement

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References
